Palestine Journal of Mathematics

Vol. 12(Special Issue III) (2023) , 19–23 © Palestine Polytechnic University-PPU 2023

PROJECTIVE MODULES OVER COMMUTATIVE NOETHERIAN GRADED RINGS

Ratnesh Kumar Mishra, Pratibha and Rakesh Mohan

MSC 2010 Classifications:13A02; 13C10 Keywords and phrases: Projective module, Graded ring

Abstract R be a commutative graded Noetherian ring. It is well-known that Artinian modules have secondary representations. We prove that the same holds also for any projective R-module. Then the class graded modules over R which possess a secondary representation is more extensive than the class of Artinian graded modules over R by proving that every Projective module over R has a secondary representation. The other objective of this paper is to show indecomposability of projective R-module.

1 Introduction

The concept of Projective module over various rings has been studying in ([1], [2], [5], [6], [8], [9], [10], [11], [12], [14], [15]and [16]). A projective module observes the apprehension of the independent module. If direct summand of a free module is a projective module over $\mathcal{R}$, some direct sum $\oplus_I\mathcal{R}$. $\mathcal{A}$ itself is the direct sum of some facsimiles of $\mathcal{R}$ is not intended axiomatically. An example is given by $\mathcal{A}$ is a ring of integer, this is a module over the ring $R = \mathbb{Z}\oplus\mathbb{Z}$ with respect to the multiplication explained by $(b_1 \oplus b_2)y = b_1y$. Hereinafter, as long as an independent module is noticeably projective, globally it is not detained by antithetical. It is true, if $\mathcal{R}$ is a principal ideal domain or a polynomial ring over a field (Quillen and Suslin 1976). This explain that, for example, $\mathcal{Q}$ is a nonprojective $\mathbb{Z}$-module, since this is not independent.

Projective modules is always a direct sum of projective module, but direct products are not assigned by this stuff. For instance, the uncountable direct product $\mathbb{Z}\oplus\mathbb{Z}\oplus$........ is not a projective $\mathbb{Z}$-module. Endowing to this prime definition, a module $\mathcal{A}$ is projective if, at any moment $\mathcal{A}$ is a quotient of a module $\mathcal{B}$, there stands a module X alike the direct sum $\mathcal{A}\oplus X$ is isomorphic to $\mathcal{B}$. The purpose of this paper to show the secondary representation of projective module over the commutative Noetherian graded ring and to study the premises of projective module over graded ring. And, the class of $\mathcal{R}$ modules which acquire a secondary representation is more comprising than the category of Artinian $\mathcal{R}$-modules by affirming that each projective $\mathcal{R}$-module has a secondary representation.

Let $\mathcal{R} = \oplus_{s>0}\mathcal{R}_s$ be a commutative graded ring. Then it is well known, $\mathcal{R}$ is a Noetherian ring if and only if the ring $\mathcal{R}_0$ is Noetherian and the $\mathcal{R}_0$-algebra $\mathcal{R}$ is finitely generated. In this paper, we would like to enlarge this outcome to the case of more general graded rings. Suppose H is sum of Abelian groups and $\mathcal{R}$ a commutative ring. Then $\mathcal{R}$ is called H-graded if there is given a family $\{\mathcal{R}_h\}_{h\in H}$ of subgroups of R such that $\mathcal{R} = \oplus_{h\in H}\mathcal{R}_h$, $\mathcal{R}_g\mathcal{R}_h \subset \mathcal{R}_{g+h}$ for all $h, g \in H$. Notice that in case $\mathcal{R}$ is H-graded, $\mathcal{R}_0$ (resp. $\mathcal{R}_H$ ($h \in H$) is a subring (resp. a R_0-submodule) of $\mathcal{R}$. We shall convenience the following symbols and terminology. Assume that r is a graded ideal of $\mathcal{R}$ and $\mathcal{B}$ is submodule of the $\mathcal{R}$-module.

Definition 1.1. [2] The $\mathcal{P}$ is called a projective module over R if it is fulfilled one of the given statements:

(a) In $\mathcal{R}$- modules $\mathcal{A}, \mathcal{B}$ and an $\mathcal{R}$-linear onto map $\alpha : \mathcal{A} \to \mathcal{B}$, the canonical map from $\beta : Hom_{\mathcal{R}}(\mathcal{P}, \mathcal{A}) \to Hom_{\mathcal{R}}(\mathcal{P}, \mathcal{B})$ sending β to $\alpha\beta$ is onto.

(b) In $\mathcal{R}$-module $\mathcal{A}$ and a onto $\mathcal{R}$-linear map $\alpha : \mathcal{A} \to \mathcal{P}$, there is the existence of a $\mathcal{R}$-linear map $\gamma : \mathcal{P} \to \mathcal{A}$ s.t. $\alpha\gamma = 1_{\mathcal{P}}$.

(c) There is the existence of an $\mathcal{R}$-module $\mathcal{Q}$ s.t. $\mathcal{P} \oplus \mathcal{Q} \cong \mathcal{R}^m$ for few positive integer m, therefore, $\mathcal{P} \oplus \mathcal{Q} \cong \mathcal{R}^m$ is free.

2 Secondary representation for projective module over a graded ring

Lemma 2.1. *Assume e is a p-primary graded ideal of $\mathcal{R}$, and $\mathcal{P}$ be a projective $\mathcal{R}$- module. Then $(0:_{\mathcal{P}} e)$, if non zero, is p-secondary.*

Proof. Consider $r \in R$. If $r \in p$, then $\oplus r \in e$ for non negative integer n, so that $\oplus r$ annihilates $(0:_{\mathcal{P}} e)$. Differently, if $r \notin p$, then we identify that

$$(0:_{\mathcal{P}} e) = a(0:_{\mathcal{P}} e)$$

as follows. Let $t \in (0:_{\mathcal{P}} e)$. Applying to mark the natural homomorphism from $\mathcal{R}$ to $\mathcal{R}/e$, there is a homomorphism $f : \mathcal{P} \to \mathcal{R}/q$ for that $f(\overline{c}) = ct$ for all $\overline{c} \in \mathcal{R}/t$.
As the diagram

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\ \mathcal{R}/e \arrow[r, two heads, "g"] & \mathcal{R}/e \arrow[r] & 0 \end{tikzcd}$$

has require row, which is completed with homomorphism $h : \mathcal{P} \to \mathcal{R}/e$ and construct the enlarged diagram. Thus $t = f(\overline{1}) = h(r\overline{1}) = rh(\overline{1})$. Hence (since $h(\overline{1}) \in (0:_{\mathcal{P}} e)$) we have$(0:_{\mathcal{P}} e) = r(0:_{\mathcal{P}} e)$, and the result follows. □

Lemma 2.2. *Assume $r_1, r_2 \ldots r_n$ are graded ideals of $\mathcal{R}$ and $\mathcal{P}$ a projective $\mathcal{R}$-module. Then.*

$$\sum_{i=1}^{n}(0:_{\mathcal{P}} r_i) = (0:_{\mathcal{P}} \cap_{i=1}^{n} r_i)$$

Proof. Let $t \in (0:_{\mathcal{P}} \cap_{i=1}^{n} r_i)$. Let $\pi : \mathcal{R} \to \mathcal{R}/\cap_{i=1}^{n} r_i$ and, for each $i = 1, ..., n$, $\pi_i : \mathcal{R} \to R/r_i$, be the natural homomorphisms. Then the monomorphism is

$$\theta : \mathcal{R}/\cap_{i=1}^{n} r_i \to \oplus_{i=1}^{n}(\mathcal{R}/r_i)$$

for that $\theta(\pi(r)) = \pi_1(r), \pi_2(r) \ldots, \pi_n(r)$ for all $r \in \mathcal{R}$. Also , the homomorphism $\psi :$ $mathcalP \to \mathcal{R}/\cap_{i=1}^{n} r_i$ for which $\phi(\pi(r)) = rt$ for all $r \in \mathcal{R}$. As $\mathcal{P}$ is projective, we may extend the diagram

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, dotted, "\psi"'] \arrow[d, "\phi"] & \\ \mathcal{R}/\cap_{i=1}^{n} r_i \arrow[r, two heads, "\theta"] & \oplus_{i=1}^{n}(\mathcal{R}/r_i) \arrow[r] & 0 \end{tikzcd}$$

which has required row by homomorphism $\phi : \mathcal{P} \to \bigoplus_{i=1}^{n}(\mathcal{R}/r_i)$ and construct the enlarged diagram commute. Now $t = \psi(\pi(1)) \in Im(\phi)$, and it is clear that $Im(\phi) \subseteq \Sigma_{i=1}^{n}(0:_{\mathcal{P}} r_i)$. It follows that $(0:_{\mathcal{P}} \cap_{i=1}^{n} r_i) \subseteq \Sigma(0:_{\mathcal{P}} r_i)$. Since the obverse insertion is clear, and is followed the result.

□

Theorem 2.3. *The set of a prime graded ideal of $\mathcal{R}$ is denoted by $Ass(\mathcal{R})$and belong to primary decomposition of zero graded ideal. Let a projective R-module P contain secondary representation along $Att\mathcal{P} \subseteq Ass\mathcal{R}$*
Specifically, consider $0 = e_1 \cap e_2 \cap \cdots \cap e_n$ be a normal primary decomposition for the zero graded ideal of $\mathcal{R}$, for $i = 1, \ldots, n$; e_i a p_i primary graded ideal. Then

$$P = (0:_{\mathcal{P}} e_1) + (0:_{\mathcal{P}} e_2) + \cdots + (0:_{\mathcal{P}} e_n). \tag{2.1}$$

Therefore, $(i = 1, \ldots, n)e_i(0:_{\mathcal{P}} e_i)$ is whether zero or p_i-secondary. Furthermore, if k is an integer s.t. $1 \le k \le n$, along

$$K = \{1, \ldots, k-1, k+1, \ldots n\},$$

Hence $\mathcal{P} = \Sigma_{i\in K}(0:_{\mathcal{P}} e_i)$ iff $\cap_{i\in K} q_i$ annihilates P; consequently, if $\mathcal{P}$ is a projective co generator of $\mathcal{R}$, then equation (1) is a minimal secondary representation for $\mathcal{P}$, and $Att(\mathcal{P}) = Ass(\mathcal{R})$.

Proof. Shows that$(0 :_{\mathcal{P}} e_i)$ is either zero or p_i- secondary, by Lemma 2.2 we have $\mathcal{P} = (0 :_{\mathcal{P}} \cap_{i=1}^{n} e_i) = \Sigma(0 :_{\mathcal{P}} e_i)$. Through Lemma 2.2 getting more information about that, if the integer k satisfies $1 \leq k \leq n$, then

$$\Sigma_{i\in k}(0 :_{\mathcal{P}} e_i) = (0 :_{\mathcal{P}} \cap_{i\in k} e_i);$$

the final module is clearly equal to $\mathcal{P}$ if and only if $\cap_{i\in k} e_i$ annihilates $\mathcal{P}$.
Now suppose $\mathcal{P}$ is a projective co generator of $\mathcal{R}$. A certain condition to prove final contention of theorem, for each $k = 1, \ldots, n$. The graded ideal $\cap_{i\in k} e_i$ does not annihilate $\mathcal{P}$. It is appropriately acceptable to show that, If s is an erratic non zero ideal of $\mathcal{R}$, then $\mathcal{P}$ is not annihilated by s. For the end of it, assume z be a non zero element of s. Since $\mathcal{P}$ is a projective co generator of $\mathcal{R}$, there exists a homomorphism $\eta : \mathcal{R} \to \mathcal{P}$. such that $\eta(z) \neq 0$. Then $z\eta(1) = \eta(z) \neq 0$, so $\eta(1)$ is an element of $\mathcal{P}$ where z does not annihilates $\mathcal{P}$ and it is not annihilated by s. □

Theorem 2.4. *A projective module $\mathcal{P}$ over a ring $\mathcal{R}$ is an indecomposable iff $\mathcal{P} \equiv \mathcal{P}(\mathcal{R}/I)$, where I is an irreducible graded left ideal of $\mathcal{R}$. In this case, for each $t \neq 0 \in \mathcal{P}$, $O(t)$ is an irreducible graded left ideal and $\mathcal{P} \equiv \mathcal{P}(R/O(t))$.*

Proof. Suppose that i be an irreducible graded left ideal of R, and E, F graded left ideal of $\mathcal{R}$ equivalently $E(I \cap F)I = 0$. Thus
$E \cap F = I$, and then either $E = F$ or $F = I$. consequently $\mathcal{P}(\mathcal{R}/I)$ is indecomposable by ([3], Proposition-2.2)
Contrariwise, imagine that $\mathcal{P}$ is an irreducible, projetcive module, $t \neq 0 \in \mathcal{P}$, and $I = O(t)$. By using the concept of ([3] Proposition-2.2) $\mathcal{P} = \mathcal{P}(\mathcal{R}t)$; and subsequently $\mathcal{R}t \equiv \mathcal{R}/I$, we have $\mathcal{P} = \mathcal{P}(\mathcal{R}/I)$. Let $I = E \cap F$ be a singular decomposition of I by the left graded ideals E, F. We secure $\mathcal{R}/I$ in $\mathcal{B} = \mathcal{P}(\mathcal{R}/E) \oplus \mathcal{P}(\mathcal{R}/F)$, and let C be a projective enclosure of $\mathcal{R}/I$ in $\mathcal{B}$. As a consequence of singularity of I, $\mathcal{R}/I \cap \mathcal{R}/K \neq 0$. Accordingly, by ([3], Lemma 1.1) along with the indecomposability of C, C assign monomorphically into $P(R/E)$. The reflection of C is a projective module including $\mathcal{R}/E = \mathcal{P}(\mathcal{R}/E)$. Therefore $\mathcal{P}(\mathcal{R}/E)$ is indecomposable equally, $\mathcal{P}(\mathcal{R}/F)$ is indecomposable. By using ([3], Theorem 2.3) $\mathcal{P}(\mathcal{R}/I) \equiv \mathcal{P}(\mathcal{R}/E) \oplus \mathcal{P}(\mathcal{R}/F)$. But this controverts the indecompositility of $\mathcal{P}(\mathcal{R}/I)$, and thus I is irreducible. □

Theorem 2.5. *Assume R is graded Noetherian ring.*

(a) There is a bijective correspondence between the graded prime ideals of $\mathcal{R}$ and isomorphism types of (non-zero) indecomposable projective R modules given by $P(\mathcal{R}/p) \leftrightarrow p$.

(b) Every projective $\mathcal{R}$-module can be expressed essentially uniquely as a direct sum of indecomposable projective $\mathcal{R}$-modules.

Proof. (a) This is distinguishable that a graded prime ideal is irreducible; and then $\mathcal{P}(\mathcal{R}/Q)$ is an indecomposable, projective module by Theorem 2.4. Assume that Q_1, Q_2 are two graded prime ideals of $\mathcal{R}$ equivalently $\mathcal{P}(\mathcal{R}/Q_1) \cong \mathcal{P}(\mathcal{R}/Q_2)$. We take up that $\mathcal{P}(\mathcal{R}/Q_1)$ secure $\mathcal{R}/Q_1$ and $\mathcal{R}/Q_2$. Consequently by using ([3], Proposition-2.2) $\mathcal{R}/Q_1 \bigcap \mathcal{R}/Q_2 \neq 0$. Nonetheless, each non zero elements of $\mathcal{R}/Q_1$ (resp. $\mathcal{R}/Q_2$) has align ideal Q_1 (resp. Q_2). Then $Q_1 = Q_2$ and the mapping $Q \to \mathcal{P}(\mathcal{R}/Q)$ is one one onto.
(b) Assume that $\mathcal{P}$ be any indecomposable, projective r-module. Thus By Theorem 2.4 there is an irreducible ideal S of $\mathcal{R}$ equivalently
$\mathcal{P} \equiv \mathcal{P}(\mathcal{R}/S)$. A unique prime ideal Q associates S which is primary ideal([13], Lemma 1.8.3). If $S = Q$, then we are accomplished. Thus, let $S \neq Q$. In such way, there is minimal integer $m > 1$ such that $Q^m \subset S$. Now hold $x \in Q^{m-1}$ equivalently, $x \notin S$ and denote the image of x in $\mathcal{R}/S$ by $\overline{x}$. Apparently, $O(\overline{x}) \supset Q)$; in other words if $y \in O(\overline{x}$, then $xy \in S$, and so $y \in Q$, presenting that $O(\overline{x}) = Q$. Accordingly, there is an element of $\mathcal{P}(\mathcal{R}/S)$ with align ideal Q, and then $\mathcal{P}(\mathcal{R}/S) \equiv \mathcal{P}(\mathcal{R}/Q)$ by Theorem 2.4. □

Theorem 2.6. *Let $\mathcal{R}$ be graded Noetherian ring, and p a prime ideal of $\mathcal{R}$. Then*

(a) Some power of p annihilates each element of $\mathcal{P}(\mathcal{R}/p)$.

(b) An automorphism is produced by the multiplication of an element $r \in \mathcal{R} - p$

(c) $\cap_{j=1}^{\infty} p^j$ is annihilator of $\mathcal{P}(\mathcal{R}/p)$ where p^j indicates the j^{th} prime power of p.

Proof. (a) Conspicously, $\mathcal{A}_j$ is graded submodule of $\mathcal{P}$, along with $\mathcal{A}_j \subset \mathcal{A}_{j+1}$. Suppose $t \notin 0 \in \mathcal{P}$; thus by using ([3] Lemma 3.2(1)) $O(t)$ is a Q-primary ideal. Thus there stands positive integer j equivalently $Q^j \subset O(t)$, and so $t \in \mathcal{A}_j$. Resultantly $\mathcal{P} = \cup \mathcal{A}_j$.

(b) By using ([3], Lemma 3.2(1)) we get $\cap_{x \in \mathcal{A}_j} O(t)$ is the intersection of all irreducible Q-primary ideals encompassing Q^j. It is simply found that this intersection is equal to Q^j.

(c) By using part (a) and (b) we get the result of this part.

□

In the next theorem, we will use the notation $occ(\mathcal{P})$ which is explained in term of projective $\mathcal{R}$-module $\mathcal{P}$. We consider the set $(\mathcal{P}_\alpha)_{\alpha \in \mu}$ of prime ideals. In the next theorem, $occ(\mathcal{P})$ represent the projective cover of $\mathcal{R}$ module.

Theorem 2.7. *Let $\mathcal{R}$ be Noetherian graded ring and $\mathcal{P}$ a projective $\mathcal{R}$-module. Then*

$$Att(\mathcal{P}) = \{p' \in Ass(\mathcal{R}) : p' \subseteq p \;\; \textit{for some} \;\; p \in occ(\mathcal{P})\}$$

Proof. It is known that $0 = e_1 \cap e_2 \cap \cdots \cap e_n$ is a normal primary decomposition for the zero graded ideal of $\mathcal{R}$, for $i = 1, \ldots, n$; e_i a p_i- secondary.

$$\mathcal{P} = (0 :_{\mathcal{P}} e_1) + (0 :_{\mathcal{P}} e_2) + \cdots + (0 :_{\mathcal{P}} e_n) \tag{2.2}$$

and (for $i = 1, \ldots, n$), $(0 :_{\mathcal{P}} e_i)$ is either zero or p_i secondary. To carry away any zero term from the right hand side and remove any oratorical terms then we can get a minimal secondary representation for $\mathcal{P}$ from equation (2). We shall do this theorem in two parts (a) and (b). In part (a), it will show $(0 : \mathcal{P}e_i) = 0$ for any i for which p does not involve p_i in $occ(\mathcal{P})$ and in part(b)p associating with $occ(\mathcal{P})$ involves p_k where k is an integer $(1 \leqq k \leqq n)$, then $\Sigma_{i \in k}(0 :_{\mathcal{P}} e_i) \neq \mathcal{P}$, where $K = \{1, \ldots, k-1, k+1, \ldots, n\}$ so that $(0 : \mathcal{P}e_i)$ can not be discarded from equation (2) at any point of the subtraction measure.
(a). Assume that $p_i \nsubseteq p$ for all $p \in occ(\mathcal{P})$, for any integer i where $(1 \leqq i \leqq n)$. If the family of graded prime ideals of $\mathcal{R}$ is $(p_\delta)_{\delta \in \Upsilon}$ for which $\mathcal{P} \cong \oplus_{\delta \in \Upsilon} \mathcal{P}(\mathcal{R}/p_\delta)$ then $(0 :_{\mathcal{P}} e_i) \cong \bigoplus_{\delta \in \Upsilon} (0 :_{\mathcal{P}} (\mathcal{R}/p_\delta)e_i)$. Thus, the purpose $(0 :_{\mathcal{P}} e_i) = 0$ is enough to reveal that $(0 :_{\mathcal{P}} (\mathcal{R}/p_\delta)e_i) = 0$ for all $p \in occ(\mathcal{P})$. Now we know for p , $e_i \nsubseteq p$(for $r(e_i) = p_i) = 0$, so assume $r \in (e_i - p)$. Using Theorem 2.5(b) an automorphism of $\mathcal{P}(\mathcal{R}/p)$ is arranged by multiplication of r in $\mathcal{P}(\mathcal{R}/p)$ and therefore, $(0 :_{\mathcal{P}} (\mathcal{R}/p)e_i) = 0$, as needed.
(b). Assume that k is an integer (with $1 \leqq k \leqq n$) for which $p_k \subseteq p$ for few $p \in occ(\mathcal{P})$. $\Sigma_{i \in k}(0 :_{\mathcal{P}} e_i) \neq \mathcal{P}$ will be shown. Let it not be the case; then by using Theorem 2.3, $a_k = \bigcap_{i \in k} e_i$ annihilates $\mathcal{P}$ and so a_k annihilates $\mathcal{P}(\mathcal{R}/p)$. Hence by Theorem 2.5$(c)$, $a_k \subseteq \bigcap_{j=1}^{\infty} p^j$. At this time $a_k \nsubseteq e_k$; suppose $r \in (a_k - e_k)$. Then $r \in \bigcap_{j=1}^{\infty} p^j$ is the kernel of natural homomorphism from $\mathcal{R}$ to $\mathcal{R}_p$. Hence there exist $b \in \mathcal{R} - p$ such that $(p_k \subseteq p)b \in \mathcal{R}_{p_k}$, this is contradiction to the fact that e_k is p_k-primary.

3 Conclusion

In this paper we distressed with conversion of primary decomposition into secondary representation. And shown the secondary representation of projective module over graded Noetherian ring. □

References

[1] Bell P Jason, Commutative Algebra(Lecture notes), *Springer*, 29(2010).

[2] Cartan H. and Eilenberg S., Projective Modules. $\oint$1.2 in Homological Algebra. *Princeton, NJ: Princeton University Press*,(1956), 6-8.

[3] Eben Matlis, Injective Module Over Noetherian Ring, *Pacific Journal Of Mathematics*, 8(1958), 511-528.

[4] Rodney Y. Sharp, Secondary Representation Of Injective Modules Over a Commutative Noetherian Ring, *Proceedings of the Edinburgh Mathematical Society*, 20(1976), 143-151.

[5] Hilton, P. J. and Stammbach U., Free and Projective Modules and Projective Modules over a Principal Ideal Domain, $\oint 4$ and $\oint$ 5 in A Course in Homological Algebra, 2nd ed. *New York: Springer-Verlag*, (1997), 22-28.

[6] Jacobson N., Projective Modules. $\oint 3.10$ in Basic Algebra II. *San Francisco, CA: W. H. Freeman and Company*, (1980), 148-155.

[7] Kunz E., Projective Modules. $\oint 3$ in Introduction to Commutative Algebra and Algebraic Geometry. *Boston, MA: Birkhäuser*, (1985) 110-112.

[8] Lam, T. Y., Projective Modules. $\oint 2$ in Lectures on Modules and Rings. *New York: Springer-Verlag*, (1999), 21-59.

[9] Leinster Tom, The Bijection Between Projective Indecomposable and Simple Module, *Bull. Belg. Math. Soc. Simon Stevin*, 22(2015), 725-735.

[10] Mac Lane S., Free and Projective Modules in Homology. *Berlin: Springer-Verlag*, (1967), 19-21.

[11] Mishra Kumar Ratnesh, Kumar Datt Shiv, and Behra Srinivas, On Projective Modules and Computation of Dimension of a Module Over Laurent Polynomial *International Scholarly Research Network ISRN Algebra*, (2011), 12.

[12] Mishra Kumar Ratnesh, Kumar Datt Shiv and Sridharan Raja Completion of unimodular row to an invertible matrix, *Mitteilungen Klosterneuburg Journal*, 64(2014), 12.

[13] Northcott D., Ideal theory, *Cambridge University Press*, (1953).

[14] Northcott D. G., Projective Modules in An Introduction to Homological Algebra, *Cambridge, England: Cambridge University Press*, (1996), 63-67.

[15] Passman, D. S., A Course in Ring Theory. *Pacific Grove, CA: Wadsworth and Brooks/Cole*, (1991), 18-20.

[16] Rowen, L. H., Projective Modules (An Introduction). $\oint 2.8$ in Ring Theory, *San Diego, CA: Academic Press*, 1(1988), 225-237.

Author information

Ratnesh Kumar Mishra, Pratibha and Rakesh Mohan, Pratibha Department of Applied Sciences, Faculty of Engineering and Technology, Parul University, Vadodara 391760, Gujarat, India.
E-mail: pratibha.tyagi17542@paruluniversity.ac.in

WARNING: Title too long for running head.
PLEASE supply a shorter form with \headlinetitle