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PRINCIPALLY QUASI DUAL-BAER MODULES

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Abstract In this paper, we generalize quasi dual-Baer module to principally quasi dual-Baer (PQ dual-Baer) module. A module M is said to be PQ dual-Baer if for each cyclic submodule X of M, $D_{\mathbf{E}}(X) = \{f \in \mathbf{E} : Im(f) \subseteq X\}$ is a direct summand of $\mathbf{E} = End(M)$. We study some properties of PQ dual-Baer modules. We find some conditions for which the direct sum of arbitrary copies of PQ dual-Baer modules is PQ dual-Baer. We also study the ring of endomorphisms of PQ dual-Baer modules.

1 Introduction

All over the article we consider the ring R to be associative ring with identity element and module M to be unital. A ring R, in which annihilator of each right ideal (ideal) in R is a direct summand of R is known as Baer (quasi-Baer) ring ([4], [5], [7]). Baer ring is an attractive topic for researchers because it has a connection to functional analysis ([2], [4], [7]). A principally quasi-Baer (in short, PQ Baer) ring was defined by Birkenmeier et al. [3], which was actually a generalization of quasi-Baer ring. In theory, the ring R is described as PQ Baer if right annihilator of every principal ideal of R in R is a direct summand of R. Rizvi and Roman in [10], defined Baer like properties for an R-module M and called a module M Baer (quasi-Baer) if the left annihilator of every submodule (fully invariant) of M in $\mathbf{E} = End(M)$ is a direct summand of $\mathbf{E}$ ([4], [10]). Motivated by this nice structure of Baer module much more work have been done by many authors in literature (see, [1], [4], [6], [8], [10], [12], [13]). In [13], Ungor et al. introduced PQ-Baer modules and Dana et al. [6] and G. Lee [13] also studied PQ-Baer modules in different aspects. According to them the left annihilator of $\mathbf{E}m$ (or cyclic submodule of M) in $\mathbf{E} = End_R(M)$ for every $m \in M$ must be a direct summand of $\mathbf{E}$ for a module M to be PQ-Baer. The dual concept of Baer modules is being considered for extending the theory of Baer modules. In [12], Tutuncu et al. presented the idea of dual notion of Baer modules and termed a module M to be dual-Baer if for every submodule X of M, $D_{\mathbf{E}}(X) = \{\alpha \in \mathbf{E} : Im(\alpha) \subseteq X\} = \mathbf{E}e$ for some $e^2 = e \in \mathbf{E} = End_R(M)$. The dual-Baer module have some nice connections with semisimple ring, Harada ring and lifting module (see [12]). Dual concept of Baer modules also have an attraction for further study. So in [11], Tribek et al. introduced quasi dual-Baer module and they defined a module M as quasi dual-Baer if for every ideal $\mathbf{T}$ of $\mathbf{E} = End_R(M)$, $E_M(\mathbf{T}) = \Sigma_{f\in\mathbf{T}} Im(f)$ is a direct summand of M.

Motivated by above generalizations of Baer modules, we introduce the class of principally quasi (in short, PQ) dual-Baer modules which properly contain the class of quasi dual-Baer modules. We define the module M to be PQ dual-Baer if for all $m \in M$, $D_{\mathbf{E}}(\mathbf{E}m) = \mathbf{E}f$ for some $f^2 = f \in \mathbf{E} = End_R(M)$.

In section 2, we define and study PQ dual-Baer modules. By providing an example we show that a PQ dual-Baer module need not be a quasi dual-Baer module (see Example 2.9). While a PQ dual-Baer module whose ring of endomorphisms has FI-GSSP is a quasi dual-Baer module (see Proposition 2.10). It is proved in (Proposition 2.12) that inheritance of PQ dual-Baer properties occur through direct summand of PQ dual-Baer modules. We characterize regular (von Neumann) and semisimple Artinian ring in terms of the PQ dual-Baer module (see Proposition 2.13 and Proposition 2.14). We find conditions over which the direct sum of PQ dual-Baer modules is PQ dual-Baer (see Proposition 2.18 and Theorem 2.19). In the last section, we study the endomorphism ring of PQ dual-Baer modules. It is shown that the ring of endomorphisms of a PQ dual-Baer module generally is a PQ-Baer ring (see Proposition 3.1) while it is not in

general, PQ dual-Baer ring. By taking the class of finitely generated PQ dual-Baer modules, we prove that the ring of endomorphisms $\mathbf{E} = End_R(M)$ of M is PQ dual-Baer if ${}_{\mathbf{E}}\mathbf{E}$ has SSP (see Proposition 3.2).
The notations $\subseteq$, $\leq$, $\leq^{\oplus}$, $\leq^{e}$ and $\trianglelefteq$ will be fixed to denote a subset, a submodule, a direct summand, an essential submodule and a submodule invariant by endomorphism (or an ideal) respectively. For right R-module X, $r_X(\mathbf{T}) = \{x \in X : \mathbf{T}(x) = 0\}$ and $l_{\mathbf{E}}(Y) = \{\alpha \in \mathbf{E} : \alpha(Y) = 0\}$ where $\mathbf{T} \leq {}_{\mathbf{E}}\mathbf{E}$ and $Y \leq X$, will denote right annihilator in X of $\mathbf{T}$ and left annihilator in $\mathbf{E}$ of Y respectively. We also denote $D_{\mathbf{E}}(Y) = \{\alpha \in \mathbf{E} \ : \ Im(\alpha) \subseteq Y\}$ for $Y \subseteq X$ and $E_X(\mathbf{T}) = \sum_{\alpha \in \mathbf{T}} Im(\alpha)$ for $\mathbf{T} \subseteq \mathbf{E}$ and $\mathbf{E} = End_R(M)$ (ring of endomorphisms of an R-module M).

2 Principally quasi dual-Baer module

Definition 2.1. We define a module M principally quasi (in short, PQ) dual-Baer if for each cyclic submodule X of M, $D_{\mathbf{E}}(X)$ is a direct summand of $\mathbf{E}$.
In other words the module M is PQ dual-Baer if for every $m \in M$ there is a $f^2 = f \in \mathbf{E}$ such that $D_{\mathbf{E}}(\mathbf{E}m) = \mathbf{E}f$. Generally we say a ring R right PQ dual-Baer if R_R is a PQ dual-Baer right R-module.

Example 2.2. (i) The $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ are PQ dual-Baer.

(ii) An injective indecomposable module is PQ dual-Baer module.

(iii) Every dual-Baer is a PQ dual-Baer module.

(iv) R_R is a PQ dual-Baer right R-module if R is a right regular ring.

(v) Every PQ dual-Baer module is dual-Rickart.

Let M be an R-module and $\mathbf{E} = End_R(M)$. An idempotent $f^2 = f \in \mathbf{E}$ is right (left) semicentral if $fg = fgf$ ($gf = fgf$) for each $g \in \mathbf{E}$. We fix the set $\mathbb{S}_r(\mathbf{E})$ to denote idempotent elements of $\mathbf{E}$ which are right semicentral also.

Lemma 2.3. *If M is a PQ dual-Baer module with $\mathbf{E} = End_R(M)$ then for $m \in M$, there is a $f \in \mathbb{S}_r(\mathbf{E})$ such that $D_{\mathbf{E}}(\mathbf{E}m) = \mathbf{E}f$.*

Proof. Let M be a PQ dual-Baer module and $m \in M$. Then there exists $f^2 = f \in \mathbf{E}$ such that $D_{\mathbf{E}}(\mathbf{E}m) = \mathbf{E}f$. Since $\mathbf{E}f\phi(\mathbf{E}m) \subseteq \mathbf{E}f(\mathbf{E}m) \subseteq \mathbf{E}m$, for every $\phi \in \mathbf{E}$. Therefore $\mathbf{E}f\phi \subseteq D_{\mathbf{E}}(\mathbf{E}m)$, which implies that $f\phi = f\phi f$. Hence $f \in \mathbb{S}_r(\mathbf{E})$. □

Remark 2.4. From Lemma 2.3 it is clear, if M is PQ dual-Baer module then the idempotent $f \in \mathbf{E} = End_R(M)$ such that $D_{\mathbf{E}}(\mathbf{E}m) = \mathbf{E}f$ is right semicentral.

A module X has summand sum property (SSP) (generalised summand sum property (GSSP)), if sum of finitely (resp. infinitely) many summands of X is also a summand of X. Furthermore a ring $\mathbf{E}$ of endomorphisms has SSP (GSSP) if $\mathbf{E}$-module $\mathbf{E}_{\mathbf{E}}$ has SSP (GSSP). While X has FI-SSP (FI-GSSP) if sum of summands which are fully invariant as well in X, is also a summand of X.
The following proposition shows when finitely generated modules are PQ dual-Baer.

Proposition 2.5. *If a module M is PQ dual-Baer with endomorphism ring $\mathbf{E} = End_R(M)$ and ${}_{\mathbf{E}}\mathbf{E}$ has SSP then $D_{\mathbf{E}}(X)$ is a direct summand of $\mathbf{E}$, for every submodule $X =< x_1, x_2, ..., x_n >$ of M.*

Proof. Let $X = \sum_{i=1}^{n} \mathbf{E}x_i$ be a submodule generated by $x_i \in M$ where $(1 \leqslant i \leqslant n)$ and $(n \in \mathbb{N})$. It is routine to check that $D_{\mathbf{E}}(X) = D_{\mathbf{E}}(\Sigma_{i=1}^{n}\mathbf{E}x_i) = \Sigma_{i=1}^{n} D_{\mathbf{E}}(\mathbf{E}x_i)$. Since M is PQ dual-Baer therefore from Lemma 2.3, there exists $e_i^2 = e_i \in \mathbb{S}_r(\mathbf{E})$ such that $D_{\mathbf{E}}(\mathbf{E}x_i) = \mathbf{E}e_i$ for every $1 \leqslant i \leqslant n$. Thus $D_{\mathbf{E}}(X) = \Sigma_{i=1}^{n}\mathbf{E}e_i$. Since ${}_{\mathbf{E}}\mathbf{E}$ has SSP, $\Sigma_{i=1}^{n}\mathbf{E}e_i$ is also a direct summand of $\mathbf{E}$. □

Proposition 2.6. *For a module M the following conditions are equivalent* *(a)* M *is a PQ dual-Baer module;*

(b) For any cyclic submodule $P \leq M$*, there is a decomposition* $M = P_1 \oplus P_2$ *with* $P_1 \leq^{\oplus} P$ *and* $Hom(M, P \cap P_2) = 0$*.*

Proof. $(a) \Rightarrow (b)$. Let P be a cyclic submodule of M and $\mathbf{E} = End_R(M)$. Then by (a), there must be an element $f^2 = f \in \mathbf{E}$, for which $D_{\mathbf{E}}(P) = \mathbf{E}f$. Suppose that $P_1 = fM$ and $P_2 = (1-f)M$ which implies $M = P_1 \oplus P_2$. Also $E_M(D_{\mathbf{E}}(P)) = E_M(\mathbf{E}f) = fM = P_1 \leq^{\oplus} P$, therefore $P = P_1 \oplus (P \cap P_2)$. Now take, $g \in \mathbf{E}$ be such that $g(M) \subseteq P \cap P_2$ which implies that $g \in D_{\mathbf{E}}(P)$. So there exists $h \in \mathbf{E}$ such that $g = hf$. Thus $g(M) \subseteq P_1$. Since $g(M) \subseteq P_2$ which yields that $g = 0$. Hence $Hom_R(M, P \cap P_2) = 0$.
$(b) \Rightarrow (a.)$ Let $\mathbf{E} = End(M)$ and $P = \mathbf{E}m$ where $m \in M$. Clearly P is cyclic submodule of M so by condition (b), there is a decoposition of M such that $P_1 \oplus P_2 = M$, $P_1 \subseteq P$ and $Hom(M, P \cap P_2) = 0$. Let $P_1 = fM$ for some idempotent $f^2 = f \in \mathbf{E}$. Then it is clear that $\mathbf{E}f \subseteq D_{\mathbf{E}}(P)$. Let $g \in D_{\mathbf{E}}(P)$ and π be a projection map from P to $P \cap P_2$. Then $\pi\phi = 0$ which implies that $g(M) \subseteq f(M)$. Thus $g(1-f) = 0 \Rightarrow g = gf \in \mathbf{E}f$ which gives $D_{\mathbf{E}}(P) \subseteq \mathbf{E}f$. Therefore $D_{\mathbf{E}}(P) = \mathbf{E}f$. Hence M is PQ dual-Baer module. □

Corollary 2.7. *If all the cyclic submodules of* M *are direct summands of* M *then* M *is a PQ dual-Baer module.*

Corollary 2.8. *In the case of an indecomposable module* X*, the following are equivalent*

(1) X *is PQ dual-Baer module;*

(2) $Hom(X, Y) = 0$*, for every cyclic submodule* Y *of* X*.*

Proof. $(1) \Rightarrow (2)$ easily seen from Proposition 2.6.
$(2) \Rightarrow (1)$ Let P be a cyclic submodule of M and $\mathbf{E} = End_R(M)$. Since M is indecomposable and $Hom(M\,P) = 0$, $D_{\mathbf{E}}(M) = \mathbf{E}$ and $D_{\mathbf{E}}(P) = 0$. Hence $D_{\mathbf{E}}(P) \leq^{\oplus} \mathbf{E}$ which proves that M is PQ dual-Baer module. □

It is clear from the definition that the following hierarchy is true in general.
Dual-Baer $\Rightarrow$ Quasi dual-Baer $\Rightarrow$ PQ dual-Baer.
We provide some examples that show the converse of the above implications need not be true.

Example 2.9. (i) Consider the ring $R = \begin{pmatrix} J & K/J \\ 0 & J \end{pmatrix}$, where J is a simple domain that is not a division ring and K is the ring of quotients of J. Then the R-module $M = \begin{pmatrix} J & K/J \\ 0 & 0 \end{pmatrix}$ is a quasi dual-Baer module from [11, Example 2.9(iii)], but it is not dual-Baer.

(ii) Let $A = \mathbb{Z}$ and $M = \prod_p \mathbb{Z}_p$ (p be a prime) be an A-module. The evidence is clear that M a PQ dual-Baer A-module, while from [11, Example 3.4], M is not a quasi dual-Baer module.

Proposition 2.10. *A module* M *is a quasi dual-Baer if and only if* M *is a PQ dual-Baer and* ${}_{\mathbf{E}}\mathbf{E}$ *has FI-GSSP, where* $\mathbf{E} = End_R(M)$*.*

Proof. The module M is PQ dual-Baer because it is a quasi dual-Baer module so for sufficient condition, it only remains to prove that left $\mathbf{E}$-module ${}_{\mathbf{E}}\mathbf{E}$ has FI-GSSP. For it, let $T = \Sigma_{i\in\Lambda}\mathbf{E}e_i$ and each $e_i \in \mathbb{S}_r(\mathbf{E})$. Then $\Sigma_{i\in\Lambda}\mathbf{E}e_i = \Sigma_{i\in\Lambda}D_{\mathbf{E}}(e_iM) = D_{\mathbf{E}}(\Sigma_{i\in\Lambda}e_iM) = \mathbf{E}e$ for some $e \in \mathbb{S}_r(\mathbf{E})$. Therefore $\mathbf{E}$-module ${}_{\mathbf{E}}\mathbf{E}$ has FI-GSSP.
Conversely, let $N \trianglelefteq M$. Since $D_{\mathbf{E}}(N) = \Sigma_{n\in N}D_{\mathbf{E}}(\mathbf{E}n)$ and M is PQ dual-Baer module, there is an $e_i \in \mathbb{S}_r(\mathbf{E})$ for each i such that $D_{\mathbf{E}}(\mathbf{E}n) = \mathbf{E}e_i$. By hypothesis, ${}_{\mathbf{E}}\mathbf{E}$ has FI-GSSP, therefore $D_{\mathbf{E}}(N) = \Sigma_{i\in\Lambda}\mathbf{E}e_i \leq^{\oplus} \mathbf{E}$. Hence M is a quasi dual-Baer module. □

Proposition 2.11. *If the module* M *is finitely generated and* $\mathbf{E}$ *is a principal ideal domain with SSP, then the following statements are equivalent:*

(1) M *is a dual-Baer;*

(2) M is a quasi dual-Baer;

(3) M is a PQ dual-Baer.

Proof. It follows from Proposition 2.5. □

In the following proposition we show that direct summand is inherited for PQ dual-Baer modules.

Proposition 2.12. *Let X be a direct summand of a module M. Then X is PQ dual-Baer if M is PQ dual-Baer.*

Proof. Let $\mathbf{E} = End_R(M)$, $X \leq^{\oplus} M$ and $x \in X$. So we have $f^2 = f \in \mathbf{E}$ such that $X = fM$ and $T = End(X) \cong f\mathbf{E}f$. Since M is a PQ dual-Baer module, there exists a $g \in \mathbb{S}_r(\mathbf{E})$ such that $I = D_{\mathbf{E}}(\mathbf{E}x) = \mathbf{E}g$. Since by Lemma 1.3 of [1], $I \trianglelefteq \mathbf{E}$, therefore $fIf = f\mathbf{E}f \cap I$. Since $g \in \mathbb{S}_r(\mathbf{E})$, $gf = gfg$. Therefore $fIf = f\mathbf{E}gf = f\mathbf{E}gfg = (f\mathbf{E}gf)(fg)$ which implies $fIf \leq^{\oplus} f\mathbf{E}f$. Now we claim that $D_T(Tx) = fIf$. For it, let $h \in I$, $fhf(M) = fh(fM) = fh(X) \subseteq f(\mathbf{E}x) \subseteq (f\mathbf{E}f)x = Tx$ which yields $fhf(M) \in D_T(Tx)$. Thus $fIf \subseteq D_T(Tx)$. Now assume that $0 \neq f\phi f \in f\mathbf{E}f$ such that $f\phi f(X) \subseteq Tx$ where $\phi \in \mathbf{E}$. Since $X = fM$, $fhf(M) = f\phi f(X) \subseteq Tx \subseteq \mathbf{E}x$, so $f\phi f \in D_{\mathbf{E}}(\mathbf{E}x) = I$. But $f\phi f = ff\phi ff = f(f\phi f)f \in fIf$. Therefore $D_T(Tx) = fIf$ for all $x \in X$. Hence X is a PQ dual-Baer module. □

Proposition 2.13. *The following assumptions are equivalent for a module M:*

(1) Every S-module is PQ dual-Baer;

(2) Every projective S-module is PQ dual-Baer;

(3) The free module $S^{(S)}$ is PQ dual-Baer;

(4) The S is semisimple and Artinian ring.

Proof. The implications (1) ⇒ (2) ⇒ (3) are easy to verify.
(3) ⇒ (4) For every right ideal J of S, there must be a free module K_S and an epimorphism π for which $\pi(K) = J$. Since $K \leq^{\oplus} S^{(S)}$, K_S is PQ dual-Baer.
Thus $\pi K_S = J \leq^{\oplus} K_S$ which implies that $J \leq^{\oplus} S_S$. Hence ring S is semisimple and Artinian.
(4) ⇒ (1) It follows easily. □

Now we characterize PQ dual-Baer module over regular ring.

Proposition 2.14. *The following conditions are equivalent:*

(a) Each finitely generated free S-module is PQ dual-Baer;

(b) The free S-module $S^{(n)}$ is PQ dual-Baer where $n \in \mathbb{N}$;

(c) S is regular ring.

Proof. $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are trivial.
$(c) \Rightarrow (a)$. It is obvious to have that $End(S^{(n)}) \cong Mat_n(R)$ for every $n \in \mathbb{N}$. Since ring S is regular, so $Mat_n(S)$ is also regular ring. Hence $S^{(n)}$ is PQ dual-Baer R-module. □

Proposition 2.15. *If a ring S is regular which is neither semisimple nor Artinian. Then each free S-module is PQ dual-Baer while it is not dual-Baer.*

Proof. From proposition 2.14, every free module M which is finitely generated over the ring S is a PQ dual-Baer. Since S is not semisimple, by [12, Corollary 2.10], module M is not PQ dual-Baer. □

Example 2.16. The ring $J = \Pi_{i=1}^{\infty}\mathbb{Z}_p$ (where p is a prime) is regular which is clearly neither semisimple nor Artinian. Hence from Proposition 2.15, every finitely generated free J-module M is a PQ dual-Baer module but M is not dual-Baer.

Now we provide an example which shows that direct sum of PQ dual-Baer modules, generally need not be a PQ dual-Baer module.

Example 2.17. If $A = \mathbb{Z}$, $P = \mathbb{Z}_{p^\infty}$ and $Q = \mathbb{Z}_p$ (p is any prime). Then P and Q are PQ dual-Baer A-modules. While by [9, Example 2.10], $P \oplus Q$ is not an A dual-Rickart module. Hence $P \oplus Q$ can not be a PQ dual-Baer A-module.

We discuss in the following proposition, when direct sum of two PQ dual-Baer modules is PQ dual-Baer.

Proposition 2.18. *If M_1 and M_2 are PQ dual-Baer modules such that $Hom(M_\alpha\, , M_\beta) = 0$ for every $\alpha \neq \beta$, $\alpha, \beta = 1, 2$, then $M_1 \oplus M_2$ is a PQ dual-Baer module.*

Proof. Let $M = M_1 \oplus M_2$ with $\mathbf{E}_1 = End(M_1)$ and $\mathbf{E}_2 = End(M_2)$. Since $Hom(M_\alpha\, M_\beta) = 0$ for every $\alpha \neq \beta$, $\mathbf{E} = End(M) = \mathbf{E}_1 \oplus \mathbf{E}_2$. Therefore, for every $m = (m_1,\ m_2) \in M$, $D_{\mathbf{E}}(\mathbf{E}m) = D_{\mathbf{E}_1}(\mathbf{E}_1 m_1) \oplus D_{\mathbf{E}_2}(\mathbf{E}_2 m_2)$. From hypothesis M_i is PQ dual-Baer module, so there exists $e_i^2 = e_i \in \mathbf{E}_i$ such that $D_{\mathbf{E}_i}(\mathbf{E}_i m_i) = \mathbf{E}e_i$ for each $i = 1, 2$. Thus $D_{\mathbf{E}}(\mathbf{E}m) = \mathbf{E}_1 e_1 \oplus \mathbf{E}_2 e_2 \leq^{\oplus} \mathbf{E}$. Hence M is a PQ dual-Baer module. □

In the following result we study when direct sum of arbitrary copies of PQ dual-Baer modules is PQ dual-Baer.

Theorem 2.19. *Let M be a PQ dual-Baer module with ring of endomorphisms $\mathbf{E}$ of M. Then $\bigoplus_{i\in\mathbf{I}} M_i$ where $M_i = M$ for each $i \in \mathbf{I}$, is PQ dual-Baer if ${}_{\mathbf{E}}\mathbf{E}$ has generalized summand sum property (GSSP).*

Proof. Let module M be PQ dual-Baer and $M^{(\mathbf{I})} = \bigoplus_{i\in\mathbf{I}} M_i$ where $M_i = M$ for each $i \in \mathbf{I}$ and $\mathbf{I}$ is an arbitrary index set. First we assume $\mathbf{I} = \mathbb{N}$. Let $m = (m_i)_{i\in\mathbf{I}} \in M^{(\mathbf{I})}$ and E_{ij} denote a $(\mathbf{I} \times \mathbf{I})$ matrix of $\mathbf{H} = End(M^{(\mathbf{I})})$ with $1_{\mathbf{E}}$ (identity element of $\mathbf{E}$) at $(i, j)th$ position and 0 on remaining places. Clearly, $E_{ij}(m) \in M^{(\mathbf{I})}$ such that m_j at i-th position and 0 elsewhere. So there is a $n \in \mathbb{N}$ such that for each $l > n$, $m_l = 0$, that means $E_{ll}(m) = 0$, which implies $m = \Sigma_{i=1}^{n} E_{ii}(m)$. Then from the claim of [8, Theorem 3.8], we get $\mathbf{H}(m) = \bigoplus_{j\in\mathbf{I}}(\Sigma_{i=1}^{n}\mathbf{E}_{ji}(m_i))$, where $\mathbf{E}_{ji} = Hom(M_i,\ M_j) = \mathbf{E}$. Consider $X_j = \Sigma_{i=1}^{n}\mathbf{E}_{ji}(m_i)$ for every $j \in \mathbf{I}$. It is routine to check that $D_{\mathbf{E}}(X_j) = D_{\mathbf{E}}(\Sigma_{i=1}^{n}\mathbf{E}_{ji}(m_j)) = \Sigma_{i=1}^{n} D_{\mathbf{E}}(\mathbf{E}_{ji}(m_j))$. Since M is PQ dual-Baer module and ${}_{\mathbf{E}}\mathbf{E}$ has GSSP, from Proposition 2.5, $D_{\mathbf{E}}(X_j) = \mathbf{E}e$ for some $e^2 = e \in \mathbf{E}$. Let $1_{\mathbf{H}}$ be the identity of $\mathbf{H}$ and take $e1_{\mathbf{H}} \in \mathbf{H}$ which is a diagonal matrix having e at diagonals. Then $e1_{\mathbf{H}}$ is an idempotent element of $\mathbf{H}$. Since $e1_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}}(X_j)) = \bigoplus_{j\in\mathbf{I}} e(X_j) \subseteq \bigoplus_{j\in\mathbf{I}}(X_j)$, $\mathbf{H}e1_{\mathbf{H}} \subseteq D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} X_j)$. Again let $\psi = [\psi_{kj}] \in D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} X_j)$, then $\psi(\bigoplus_{j\in\mathbf{I}} X_j) \subseteq \bigoplus_{j\in\mathbf{I}} X_j$ which implies that $\psi_{kj}(X_j) \subseteq X_j$ for all $j, k \in \mathbf{I}$. So $\psi_{kj} \in D_{\mathbf{E}}(X_j) = \mathbf{E}e$ for some idempotent $e \in \mathbf{E}$ because M is PQ dual-Baer module. Therefore $\psi_{kj} = \psi_{kj}e$ for all $j, k \in \mathbf{I}$. Hence $D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} X_j) \subseteq \mathbf{H}e1_{\mathbf{H}}$. So we get $D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} X_j) = \mathbf{H}e1_{\mathbf{H}}$. Thus $D_{\mathbf{H}}(\mathbf{H}m) = \mathbf{H}e1_{\mathbf{H}}$. By following the similar steps it is easy to prove the theorem when $\mathbf{I}$ is an arbitrary index set. □

3 Endomorphism ring of PQ dual-Baer modules

This section is devoted for study of ring of endomorphisms of a PQ dual-Baer module.
Following proposition suggests that ring of endomorphisms of a PQ dual-Baer module is PQ Baer.

Proposition 3.1. *Let M be a module and $\mathbf{E}$ be its endomorphism ring. Then $\mathbf{E}$ is PQ Baer ring if M is PQ dual-Baer module.*

Proof. Let M be a PQ dual-Baer module, $m \in M$ and T be a principal ideal of $\mathbf{E}$. Then there exists $f^2 = f \in \mathbf{E}$ such that $D_{\mathbf{E}}(Tm) = \mathbf{E}f$. For every $g \in T$, $Im(g) \subseteq \Sigma_{g\in D_{\mathbf{E}}(Tm)} Im(g) = \Sigma_{g\in\mathbf{E}f} Im(g) = E_M(\mathbf{E}f) = fM$. So for every $g \in T$, $(1 - f)\phi M = 0$ which implies that $(1 - f)g = 0$. Therefore $(1 - f) \in l_{\mathbf{E}}(T)$. For $\mathbf{E}$ to be a PQ Baer ring, it is enough to prove that $l_{\mathbf{E}}(T) = \mathbf{E}(1 - f)$. For it let, $h \in l_{\mathbf{E}}(T)$ then $h(D_{\mathbf{E}}(Tm)) = 0 \Rightarrow h(\mathbf{E}f) = 0 \Rightarrow hf = 0$. Therefore $h = h(1 - f) \in \mathbf{E}(1 - f)$. Thus $l_{\mathbf{E}}(T) \subseteq \mathbf{E}(1 - f)$. Now assume that $h \in \mathbf{E}(1 - f)$ then for every $m \in M$, $hT(m) = h(1 - f)T(m) \subseteq h(1 - f)(fM)$ because for every $h \in T$, $Im(h) \in fM$. So $hT(m) = 0$ for every $m \in M \Rightarrow hT = 0 \Rightarrow h \in l_{\mathbf{E}}(T)$. Hence $\mathbf{E}$ is PQ Baer ring. □

It is not necessary for the converse of the above proposition to be true. In fact, the $\mathbb{Z}$-module $\mathbb{Z}$ is not a PQ dual-Baer while $End_{\mathbb{Z}}\mathbb{Z} \simeq \mathbb{Z}$ is a PQ Baer ring.
In the next proposition we find the condition under which endomorphism ring of PQ dual-Baer module is a PQ dual-Baer.

Proposition 3.2. *Let M be a finitely generated PQ dual-Baer module with $\mathbf{E} = End_R(M)$ and ${}_{\mathbf{E}}\mathbf{E}$ has SSP, then endomorphism ring of M is PQ dual-Baer.*

Proof. Let M is PQ dual-Baer module with $f \in \mathbf{E}$. Assume that M is generated by $m_1, m_2, ..., m_n$ where each $m_i \in M$ and $n \in \mathbb{N}$. For every $\psi \in D_{\mathbf{E}}(\mathbf{E}\phi)$, $\psi(\mathbf{E}\phi) \subseteq \mathbf{E}\phi$ and $\psi((\mathbf{E}\phi)M) \subseteq (\mathbf{E}\phi)M$. Thus $\psi((\mathbf{E}\phi)(m_i)) \subseteq (\mathbf{E}\phi)(m_i)$ for all $1 \leqslant i \leqslant n$. Therefore $\psi \in D_{\mathbf{E}}(\mathbf{E}(\phi(m_i)))$ for each i. Since M is PQ dual-Baer module, there exist $e_i \in \mathbb{S}_r(\mathbf{E})$ such that $D_{\mathbf{E}}(\mathbf{E}(\phi(m_i))) = \mathbf{E}e_i$ for all $1 \leqslant i \leqslant n$. Hence $\psi \in \Sigma_{i=1}^n \mathbf{E}e_i$ and so $D_{\mathbf{E}}(\mathbf{E}\phi) \subseteq \Sigma_{i=1}^n \mathbf{E}e_i$. Now let $f \in \Sigma_{i=1}^n \mathbf{E}e_i$ and $m \in M$ be arbitrary. Then for $r_i \in R$ $(1 \leqslant i \leqslant n)$, $f(\mathbf{E}\phi(m)) = f(\Sigma_{i=1}^n \mathbf{E}\phi(m_i r_i)) = f(\Sigma_{i=1}^n (\mathbf{E}\phi(m_i)r_i))$. Clearly $\Sigma_{i=1}^n (\mathbf{E}\phi(m_i))r_i$ is finitely generated submodule of M. Since M is PQ dual-Baer module and ${}_{\mathbf{E}}\mathbf{E}$ has SSP so by proposition 2.5, $f(\Sigma_{i=1}^n (\mathbf{E}\phi(m_i)r_i)) \subseteq \Sigma_{i=1}^n (\mathbf{E}\phi(m_i))r_i$. Thus $f(\mathbf{E}\phi) \subseteq \mathbf{E}\phi$ that implies $f \in D_{\mathbf{E}}(\mathbf{E}\phi)$. Hence $\Sigma_{i=1}^n \mathbf{E}e_i = D_{\mathbf{E}}(\mathbf{E}\phi)$. Since ${}_{\mathbf{E}}\mathbf{E}$ has SSP, $D_{\mathbf{E}}(\mathbf{E}\phi) \leq^{\oplus} \mathbf{E}$. Hence $\mathbf{E}$ is a PQ dual-Baer ring. □

Proposition 3.3. *If $X = \bigoplus_{\lambda \in \mathbf{I}} M_\lambda$ where $M_\lambda = M$ for each $\lambda \in \mathbf{I}$. If the endomorphism ring of X is a PQ dual-Baer ring then $\mathbf{E} = End_R(M)$ is a quasi dual-Baer ring..*

Proof. Let M be a PQ dual-Baer module with $\mathbf{E} = End(M)$ and $T \trianglelefteq \mathbf{E}$. Consider $\mathbf{I} = |T|$ and $\mathbf{H} = End(X)$. Clearly $CFM_{\mathbf{E}} \subseteq \mathbf{H} \subseteq Mat_{\mathbf{I}}(\mathbf{E})$. Set $\psi = diag[\psi_1, \psi_2, ..., \psi_i, ...]_{i \in \mathbf{I}} \in \mathbf{H}$. We claim that $D_{\mathbf{H}}(\mathbf{H}\psi) = \mathbf{H} \cap Mat_{\mathbf{I}}(\Sigma_{\psi_i \in T} D_{\mathbf{E}}(\mathbf{E}\psi_i))$. For it let $\phi = [\phi_{ij}] \in D_{\mathbf{H}}(\mathbf{H}\psi)$ be arbitrary. Then $\phi(\mathbf{H}\psi) \subseteq \mathbf{H}\psi$. Denote by E_{ii} a matrix in $\mathbf{H}$ with $1_{\mathbf{E}}$ at (i, i)-th position and 0 on remaining places. Then $E_{ii}\phi E_{jj}(\mathbf{H}E_{kk}\psi E_{kk}) \subseteq \mathbf{H}E_{kk}\psi E_{kk} \Rightarrow \phi_{ij}(\mathbf{E}\psi_k) \subseteq \mathbf{E}\psi_k$ for each $i, j, k \in \mathbf{I}$. Thus $\phi_{ij} \in \Sigma_{\psi_k \in T} D_{\mathbf{E}}(\mathbf{E}\psi_k)$ for every $i, j \in \mathbf{I}$. Therefore $\phi \in \mathbf{H} \cap Mat_{\mathbf{I}}(\Sigma_{\psi_k \in T} D_{\mathbf{E}}(\mathbf{E}\psi_k))$. For the reverse inclusion, let $\theta = [\theta_{ij}] \in \mathbf{H} \cap Mat_{\mathbf{I}}(\Sigma_{\psi_k \in T} D_{\mathbf{E}}(\mathbf{E}\psi_k))$ be arbitrary. Then $\theta_{ij} \in \Sigma_{\psi_k \in T} D_{\mathbf{E}}(\mathbf{E}\psi_k)$ for every $i, j \in \mathbf{I}$. Thus $\theta_{ij}(\mathbf{E}\psi_k) \subseteq \mathbf{E}\psi_k$ for all $i, j, k \in \mathbf{I}$. Therefore $\theta(\mathbf{H}\psi) \subseteq \mathbf{H}\psi$. Hence, $\theta \in D_{\mathbf{H}}(\mathbf{H}\psi)$ which proves our claim. Now assume that $P = \Sigma_{\psi_k \in T} D_{\mathbf{E}}(\mathbf{E}\psi_k)$. So from our claim $\mathbf{H} \cap Mat_{\mathbf{I}}(P) = D_{\mathbf{H}}(\mathbf{H}\psi)$. Since from assumption $\mathbf{H}$ is PQ dual-Baer ring, there must exist $F^2 = F = [F_{ij}] \in \mathbf{H}$ for that $D_{\mathbf{H}}(\mathbf{H}\psi) = \mathbf{H}F$. It clearly follows that $E_{ii}FE_{ii} = F_{ii}E_{ii}$ is a right semicentral idempotent of $E_{ii}\mathbf{H}E_{ii}$. Thus $PE_{ii} = E_{ii}(\mathbf{H} \cap Mat_{\mathbf{I}}(P))E_{ii} = E_{ii}\mathbf{H}FE_{ii} = E_{ii}\mathbf{H}FE_{ii}FE_{ii}$. Thus $P = PF_{ii} \subseteq \mathbf{E}E_{ii}$ for all $i \in \mathbf{I}$. Since $\mathbf{H}F = \mathbf{H} + Mat_{\mathbf{I}}(P)$, $\mathbf{E}F_{ii} \subseteq P$. Hence $P = \mathbf{E}E_{ii}$ with $F_{ii} \in \mathbf{E}$. Therefore, $\mathbf{E}$ is quasi dual-Baer ring. □

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