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Some Generalizations of Q-Principally Injective Modules

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Abstract The purpose of this work is to investigate some more property of Q- finitely injective modules and generalize this idea to Q-small finitely injective modules. A quasi-f-injective module Q is non co-Hopfian if and only if there is a decomposition $Q = N_r \oplus (\oplus_{i=1}^{r} M_i)$ for any positive integer r, where $N_r \cong Q$ and $M_i \neq 0$ for $1 \leq i \leq r$. Also, we prove that a semi-regular module Q, an R-module P is Q-sf-injective if and only if P is Q-f-injective.

1 Introduction

It has been the interest of many researchers to study the finite injective and principally injective modules for many years. In 1969, R. N. Gupta [5] introduced the idea of f-injective modules and proved, a ring R is Noetherian if and only if any f-injective module over R is injective. Ramamurthy and Rangaswamy [12] proved that a finitely generated submodule N which is isomorphic to a direct summand of Q is a direct summand of Q and vice versa, also shown that over a right Noetherian ring each quasi injective module is equivalent to finitely quasi injective.
In 1991, R. Wisbauer [15] introduced the concept of quasi-principally injective modules (in short, qp-injective) under the terminology of semi-injective modules as a generalization of Q-p-injective modules. An R-module N is called M-generated, if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I, if I is finite then N is called finitely M-generated. In particular, a submodule K of M is called an M-cyclic submodule of M, if it is isomorphic to M/L for some submodule L of M equivalently to say that there exist an epimorphism from M to K. Sanh et al. [14] introduced the idea of Q-p-injective module which is a generalization of p-injective modules and they called a module Q is M-p-injective, if for any $\phi : U \to Q$ (where U is an M-cyclic submodule of M) there exists $\psi : M \to Q$ such that $\phi = \psi i$, where $i : U \to M$ is an inclusion. Q is known as quasi-principally injective (in short qp-injective or semi injective), if it is Q-p-injective. It was shown by Sanh et al. [14] that qp-injective modules satisfy (C_2) and (C_3) conditions. These work extends the results of Nicholson and Yousif [10]. Also, they proved that the finite direct sum of Q-p-injective modules is Q-p-injective. In 2012, Kumar et al. [9] generalized the idea of p-injective modules given in [14] to Q-small-principally injective (in short, Q-sp-injective) modules and quasi-sp-injective modules. It was shown that the notion of quasi-sp-injective and qp-injective modules are equivalent for a hollow modules. Now, we define some of the terminologies Let R be a ring and M be a right R-module. We say that M has the exchange property if whenever we have right R-module decompositions, $A = M \oplus N = \oplus_{i\in I} A_i$, for some indexing set I then there are submodules $A'_i \subset A_i$with $A = M \oplus_{i\in I} A_i$. We say that M has finite exchange property, if the index set I is finite. A module M is said to be co-Hopfian, if every injective endomorphism $f : M \to M$ is an automorphism and module M is said to be directly-finite, if it is not isomorphic to a proper direct summand of itself. A module in which its submodules are linearly ordered by inclusion is called uniserial. A module M is said to have the cancellation property if for modules H and K $M \oplus H \cong M \oplus K \implies H \cong K$. Equivalently, if $A \oplus H = B \oplus K$ with $A \cong B \implies H \cong K$. A module M is called weakly co-Hopfian, if every injective endomorphism is essential. We refer [15], for undefined notions and terminologies.

2 Q-Finitely Injective Modules

The purpose of this section is to examine some of the properties associated to Q-finitely injective modules. We prove that quasi-f-injective extending module is co-Hopfian if and only if it satisfies cancellation property.

Definition 2.1. [8] A right R-module B is known as A-finitely injective (A-f-injective) if every homomorphism from a finitely A-generated submodule of A to B can be extended to a homomorphism from A to B. Equivalently, for each $s_1, s_2, ..., s_n \in T = End(A)$, every homomorphism from $\alpha : s_1(A) + s_2(A) + \cdots + s_n(A) \to B$ there exists a homomorphism $\beta : A \to B$ such that $\alpha = \beta i$. The module B is quasi-f-injective, if it is B-f-injective.

Lemma 2.2. *[8] 1. If $\{X_i : i \in I\}$ be Q-f-injective modules, then $\Pi_{i\in I} X_i$ is Q-f-injective.*
2. *Let $\{M_i : i \in I\}$ be any family of Q-f-injective modules. If Q is finitely generated, then $\oplus_{i\in I} M_i$ is Q-f-injective.*
3. *Direct summand of Q-f-injective module is Q-f-injective.*
4. *Let K be a finitely N-generated submodule and N be a finitely Q-generated submodule of Q, then K is finitely Q-generated submodule of Q.*
5. *Let X be a finitely generated right ideal of R and $N_1 \subset^{\oplus} N$. If N is f-injective, then N_1 is X-f-injective.*
6. *Let L be a finitely Q-generated submodule of Q. If N is Q-f-injective, then N is both L-f-injective and Q/L-f-injective.*

Lemma 2.3 (Proposition 2.6, [8]). *Let P be a finitely M-generated submodule of M. If Q is M-f-injective, then it is P-f-injective, also any submodule of Q is P-f-injective. Moreover, if M is quasi-projective, then Q is M/P-f-injective.*

Proposition 2.4. *Consider $A = \oplus_{i=1}^{n} A_i$, where each A_i is A-f-injective module. Then Q is A-f-injective if and only if Q is A_i-f-injective for $1 \leq i \leq n$.*

Proof: If part is clear. Conversely, we consider Q is A_i-f-injective $1 \leq i \leq n$. Consider the inclusion $i : P \to A$, where P is finitely A-generated submodule and $\phi : P \to Q$ be any homomorphism. Now, construct a set $S = \{(K_i, \alpha_i) : K_i$ is finitely A-generated submodule containing P and $\alpha_i : K_i \to Q$ that extends $\phi : P \to Q\}$. Then by Zorn's lemma, we get a maximal member (L, g) of S such that $P \subset L \subset A$ and $g : L \to Q$ extends ϕ. Claim that $L = A$ and $A_i \subset L$, for every i. Since Q is A_i-f-injective for each i, then there exists $g_i : A_i \to Q$ such that $g_i = g$ on $L \cap A_i, 1 \leq i \leq n$. Now, define $h_i : L + A_i \to N$ by $h_i(l + a_i) = g(l) + g_i(a_i), \forall l \in L, a_i \in A_i$. Since $g_i = g$ on $L \cap A_i$, then h_i are well defined map. Also, since $K \subset L$ and g extends ϕ, then h_i extends ϕ. Hence, by maximality of $(L, g) \implies L + A_i = L \implies A_i \subset L = A$. Hence, Q is A-f-injective. □

Corollary 2.5. *Consider R is a finitely generated ring such that $R = \oplus_{i=1}^{n} X_i$. Then any R-module H is f-injective if and only if X_i-f-injective for each $1 \leq i \leq n$.*

Here, we define **(C_2):** A submodule of M is isomorphic to a direct summand of M, then it is a direct summand of M itself.
(C_3): If A and B are direct summand of M with $A \cap B = 0$. Then $A \oplus B$ is also a direct summand of M.
(C_4): A module M is said to be a C_4-module if and only if $A, B \subset^{\oplus} M$ with $A \cap B = 0$ and $A \cong B$, then $A \oplus B \subset^{\oplus} M$, equivalently, if A and B are submodules of M with $A \cap B = 0$ and $B \cong A \subset^{\oplus} M$, then $B \subset^{\oplus} M$.

Proposition 2.6. 1. *Suppose that Q, N_1 and N_2 are R-modules. If $N_1 \cong N_2$ and N_1 is Q-f-injective then N_2 is Q-f-injective.*
2. *Any quasi-f-injective module satisfies the conditions (C_2) and (C_3).*

Proof: Straight forward. □

Proposition 2.7. *Any quasi-f-injective module satisfies (C_4) condition.*

Proof: Since a quasi-f-injective module satisfies (C_2) and (C_3) conditions so by an implication we have $(C_2) \implies (C_3) \implies (C_4)$, the result holds. □

Proposition 2.8. *Let P be a finitely Q-generated submodule of Q. If A is Q-f-injective then any $A_1 \subset^{\oplus} A$ is P-f-injective.*

Proof: Straight forward. □

Theorem 2.9. *Consider a module Q is a quasi-f-injective. Then Q is a non co-Hopfian if and only if there is a decomposition $Q = N_r \oplus (\oplus_{i=1}^{r} M_i)$ for any positive integer r, where $N_r \cong Q$ and $M_i \neq 0$ for $1 \leq i \leq r$.*

Proof: We consider Q is a non co-Hopfian module, then any one-one endomorphism $\phi : Q \to Q$ which is not an automorphsim. Let $\phi(Q) = N_1, N_1 \neq Q$ and $g : N_1 \to Q$ be an isomorphism. As Q is quasi-f-injective, so $\psi : Q \to Q$ exists such that $\psi|_{N_1} = g$. Therefore $Q = N_1 \oplus ker\psi = N_1 \oplus M_1$, where $M_1 = ker\psi \neq 0$. Again, since N_1 is non co-Hopfian then by similar argument we get $N_1 = N_2 \oplus M_2$ with $N_2 \cong N_1$ and $M_2 \neq 0$, thus $Q = N_2 \oplus (M_1 \oplus M_2)$. Now, continuing this process in the similar manner we get the desired result, i.e. $Q = N_r \oplus (\oplus_{i=1}^{r} M_i)$ for any positive integer r, where $N_r \cong Q$ and $M_i \neq 0$ for $1 \leq i \leq r$.
Conversely, we assume that $Q = N_r \oplus (\oplus_{i=1}^{r} M_i)$ for any positive integer r, where $N_r \cong Q$ and $M_i \neq 0$ for $1 \leq i \leq r$. Then Q is non co-hopfian as it is not directly finite. □

Proposition 2.10. *Consider Q is a quasi-f-injective, uniserial module, then each one-one endomorphism of Q is onto, i.e. Q is co-Hopfian.*

Proof: Consider a one-one map $\sigma \in End(Q)$ and Q is a quasi-f-injective then $\sigma(Q) \subset^{\oplus} Q$. The uniserial module Q is indecomposable, then it follows that $\sigma(Q) = Q$. Hence, σ is an automorphism i.e. Q is co-Hopfian. □

Corollary 2.11. *Every quasi-f-injective uniserial module is a weakly co-Hopfian module.*

Remark 2.12. Clearly, any quasi-f-injective module does not satisfies (C_1) condition. We observe that, an extending quasi-f-injective module is a continuous and quasi-continuous.

Corollary 2.13. *Every directly finite quasi-f-injective module with (C_1) condition has the cancellation property.*

Theorem 2.14. *Consider Q is a quasi-f-injective module with (C_1) condition. Then Q is co-Hopfain if and only if it satisfies the cancellation property.*

Proof: Let Q be co-Hopfian then it is directly finite and so from the Corollary 2.13 that Q satisfies cancellation property.
Conversely, we consider Q is non co-Hopfian and has cancellation property. Then there is a decomposition of $Q = N_1 \oplus M_1$, where $N_1 \cong Q$ and $M_1 \neq 0$. But, Q has cancellation property then we have $M_1 = 0$ which is not possible, hence our supposition is wrong. Thus, Q is co-Hopfian. □

Proposition 2.15. *For a quasi-f-injective module Q with (C_1) condition the following assertions are equivalent:*

1. Q is a clean module;

2. Q has finite exchange property;

3. Q has full exchange property.

Proof: Proof follows from [3], Thoerem 4.3 and Remark 2.12. □

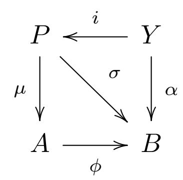
Lemma 2.16. *Consider P is a fully invariant finitely Q-generated submodule and $Q = \oplus_{i \in I} N_i$, where each N_i's are finitely Q-generated direct summands of Q. Then $P = \oplus_{i \in I}(P \cap N_i)$.*

Proposition 2.17. *Every quasi-f-injective duo module Q has SIP and SSP.*

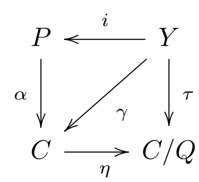
Proof: Consider $Q_1, Q_2 \subset^{\oplus} Q$ and Q is quasi-f-injective duo module. We claim that $Q_1 \cap Q_2$ and $Q_1 + Q_2$ both are direct summands of Q. For this, we assume $Q = Q_1 \oplus Q_1^{'} = Q_2 \oplus Q_2^{'}$. We observe that every direct summand of Q is finitely Q-generated and fully invariant submodule of Q. Now, Q_2 can be expressed as $Q_2 = Q_2 \cap (Q_1 \oplus Q_1^{'}) = (Q_2 \cap Q_1) \oplus (Q_2 \cap Q_1^{'})$. Thus, $Q = Q_2 \oplus Q_2^{'} = (Q_2 \cap Q_1) \oplus (Q_2 \cap Q_1^{'}) \oplus Q_2^{'}$. So, $Q_1 \cap Q_2 \subset^{\oplus} Q$ and hence Q has SIP. Next, $Q_1 + Q_2 = Q_1 + (Q_2 \cap Q_1) \oplus (Q_2 \cap Q_1^{'}) = Q_1 \oplus (Q_2 \cap Q_1^{'})$. It is clear for quasi-f-injective module, direct sum of two disjoint direct summand is again a direct summand. Hence $Q = Q_1 + Q_2 = Q_1 \oplus (Q_2 \cap Q_1^{'})$, thus has SSP. □

Theorem 2.18. *For a projective R-module P the following assertions are equivalent:*
1. Every factor of P-f-injective module is P-f-injective;
2. Every factor of P-injective module is P-f-injective;
3. Every factor of an injective module is P-f-injective;
4. Every finitely P-generated submodule of P is projective.

Proof: $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are obvious.
$(3) \Rightarrow (4)$ For two R-module A and B we consider $\phi : A \to B$ is an onto homomorphism and A is an injective module. Suppose that Y is finitely P-generated submodule of P and $\alpha : Y \to B$ is a homomorphism. By (3), B is P-f-injective, then there is a $\sigma : P \to B$ such that $\sigma i = \alpha$. Since P is projective, σ can be lifted to $\mu : P \to A$ such that $\phi\mu = \sigma$. Clearly, $\phi\mu i = \alpha$ this implies that α lifts, where $\mu i : Y \to A$. Hence, Y is projective.



$(4) \Rightarrow (1)$ Consider Y is a finitely P-generated submodule, and C is P-f-injective module. Consider $Q \subset C$, and $\eta : C \to C/Q$ is natural epimorphism. Since Y is P-projective, $\tau : Y \to C/Q$ can be lifted to $\gamma : Y \to C$. Since C is P-f-injective, γ can be extended to $\alpha : P \to C$. Hence, $\eta\alpha : P \to C/Q$ extends τ. □



3 Q-Small-Finitely Injective Modules

Here, we give the idea of Q-small-finitely injective module and discuss its properties and quasi-small-finitely injective modules, which generalizes the notions of Q-f-injective modules and quasi-f-injective modules. Also, discuss the several equivalent conditions.

Definition 3.1. A module P is said to be Q-small finitely injective (in short, Q-sf- injective) if every homomorphism from a small finitely Q-generated submodule of Q to P can be extended to a homomorphism from Q to P. P is called quasi-sf-injective if it is P-sf-injective.

Lemma 3.2.
1. Direct summands of Q-sf-injective module is Q-sf-injective.
2. Let $\{X_i : i \in I\}$ be Q-sf-injective modules. Then $\Pi_{i \in I} X_i$ is Q-sf-injective.

Proposition 3.3. *For a quasi-sf-injective module Q, we have:*
1. Any fully invariant small finitely Q-generated submodule of Q is an Q-sf-injective.
2. A quasi-sf-injective module Q satisfy (C_2) and (C_3) conditions.

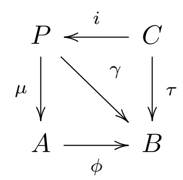
Proof: Straight forward. □

Proposition 3.4. *Consider P, Q and T are R-modules with $P \cong Q$. If P is T-sf-injective, then Q is T-sf-injective module.*

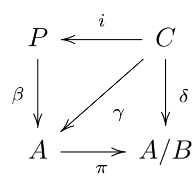
Proof: Straight forward.□

Theorem 3.5. *A projective R-module P the following assertions are equivalent:*
1. Every quotient of P-sf-injective module is P-sf-injective;
2. Every quotient of P-f-injective module is P-sf-injective;
3. Every quotient of P-injective is module P-sf-injective;
4. Every quotient of an injective module is P-sf-injective;
5. Every small finitely P-generated submodule of P is projective.

Proof: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.
$(4) \Rightarrow (5)$ Consider $\phi : A \to B$ is an epimorphism in which A is an injective module. Let C be small finitely P-generated submodule of P and $\tau : C \to B$ is a homomorphism. By (3), B is P-sf-injective, τ can be extended to $\gamma : P \to B$ such that $\gamma i = \tau$, where $i : C \to P$ is inclusion map. Since P is projective, γ can be lifted to $\mu : P \to A$ such that $\phi\mu = \gamma$. Then clearly, $\phi\mu i = \tau$ which implies that τ lifts, where a homomorphism $\mu i : C \to A$. Hence, C is projective.



$(5) \Rightarrow (1)$ Consider C is a small finitely P-generated submodule of P, and N is an P-sf-injective R-module. Let B be a submodule of A, and $\pi : A \to A/B$ be canonical epimorphism. From (5), C is projective, any $\delta : C \to A/B$ can be lifted to $\gamma : C \to A$. Since A is P-sf-injective, γ can be extended to $\beta : P \to A$. Thus, $\pi\beta : P \to A/B$ extends δ.



□

Theorem 3.6. *Consider Q is a hollow R-module. Then P is Q-f-injective module if and only if P is Q-sf-injective module.*

Proof: First part of theorem is clear. Now, we prove the other part. For this we assume that P is Q-sf-injective module. Let L be a finitely Q-generated submodule of Q. Clearly, L is small finitely Q-generated submodule of Q because Q is hollow. Therefore, $\alpha : L \to P$ can be extended to $\beta : Q \to P$. Hence, P is Q-f-injective module. □

Theorem 3.7. *If Q is a semi-regular module. Then P is a Q-sf-injective if and only if P is a Q-f-injective.*

Proof: $(\Rightarrow)$ Let $\gamma : H \to P$ be a homomorphism, where H is a finitely Q-generated submodule. Since Q is semi regular, then there exists a decomposition $Q = Q_1 \oplus Q_2$, where $Q_1 \subseteq H$ and $H \cap Q_2$ is small in H. Hence, $Q = H + Q_2$, $H = Q_1 \oplus (H \cap Q_2)$ and so $H \cap Q_2$ is a finitely Q-generated submodule of H. Therefore, there is $\tau : Q \to P$ such that $\tau(x) = \gamma(x)$ for all $x \in H \cap Q_2$. Now, we take a homomorphism $\psi : Q \to P$ defined by $\psi(m) = \gamma(a) + \tau(q)$ for any $m = a + q$, $a \in H, q \in Q_2$. Now, we show that γ is well defined. Take, $a_1 + q_2 = a_2 + q_2^{'}$ where $a_1, a_2 \in H, q_2, q_2^{'} \in Q_2$, then $a_1 - a_2 = q_2^{'} - q_2 \in H \cap Q_2$. Hence, $\gamma(a_1 - a_2) = \tau(q_2^{'} - q_2) \implies \psi(a_1 + q_2) = \psi(a_2 + q_2^{'})$. Thus ψ is a homomorphism and $\psi|_H = \gamma$.
$(\Leftarrow)$ Obvious. □

Remark 3.8. The following implications shows the two way generalizations of quasi-sf-injective module.

$$\begin{array}{ccc} & & \text{Quasi injective module} \\ & & \Downarrow \\ \text{Quasi-f-injective module} & \Leftarrow & \text{Quasi-p-injective module} \\ \Downarrow & & \Downarrow \\ \text{Quasi-sf-injective module} & \Leftarrow & \text{Quasi-sp-injective module} \end{array}$$

Now, we give a counter example that reverse implication does not hold.

Example 3.9. Consider $\mathbb{Z}/p\mathbb{Z}$ (where p is prime) as $\mathbb{Z}$-module. Then $\mathbb{Z}/p\mathbb{Z}$ is $\mathbb{Z}$-p-injective but not $\mathbb{Z}$ injective module.

Proposition 3.10. *An epiretractable hollow module Q is quasi-injective if and only if it is quasi-sf-injective.*

Proof: Straight forward. □

Proposition 3.11. *For an epiretractable module Q, the following assertions are equivalent:*
1. Q is quasi-injective;
2. Q is quasi-p-injective;
3. Q is quasi-f-injective.

Proof: Straight forward. □

Proposition 3.12. *For a hollow module Q, these assertions are equivalent:*
1. Q is quasi-f-injective;
2. Q is quasi-sp-injective;
3. Q is quasi-sf-injective.

Proof: Straight forward. □

In [9] the idea of small module homomorphism has been given as a homomorphism $\gamma : X \to Y$ such that image of γ is a small submodule of Y.

Theorem 3.13. *Consider a module Q is a quasi-f-injective and each small endomorphisms $\alpha_i \in S = End(Q), 1 \leq i \leq n$ such that $\sum_{i=1}^{n} S\alpha_i$ is direct. Then $\phi : \sum_{i=1}^{n} \alpha_i(Q) \to Q$ can be extended to a $\psi : Q \to Q$.*

Proof: Since α_i, for $i = 1, 2, \cdots n$ is small module endomorphism and Q is quasi-f-injective, there is $\psi_i : Q \to Q$ such that, $\psi_i\alpha_i = \phi\alpha_i$ and consequently $\sum_{i=1}^{n} \psi_i\alpha_i = \sum_{i=1}^{n} \phi\alpha_i$. Since $(\sum_{i=1}^{n} \alpha_i)(Q) \subseteq \sum_{i=1}^{n} \alpha_i(Q)$, ϕ can be extended to $\psi : Q \to Q$ such that, $\psi(\sum_{i=1}^{n} \alpha_i)(m) = \phi(\sum_{i=1}^{n} \alpha_i)(m)$ for any $m \in Q$. That is $\sum_{i=1}^{n} \psi\alpha_i = \sum_{i=1}^{n} \phi\alpha_i$. It follows $\sum_{i=1}^{n} \psi\alpha_i = \sum_{i=1}^{n} \psi_i\alpha_i$. The direct sum $\oplus_{i=1}^{n} S\alpha_i$ implies $\psi\alpha_i = \psi_i\alpha_i$ for all $i = 1, ..., n$. Therefore for any $x \in \sum_{i=1}^{n} \alpha_i(Q) \implies \phi(x) = \psi(x)$. Hence, the theorem. □

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