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Reduction of ideals for modules

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Abstract For a local Noetherian ring (B, n) with infinite residue field B/n and H is a finitely generated B-module, the ideals $L \subseteq K$ with the property that $K^{m+1}H = LK^mH$ is called the reduction of K for H. Some of results on reduction of ideals are extended reduction of ideals for H. Moreover, we give analogous result of P-A for reduction of K for H.

1 Introduction

In the Noetherian commutative ring B with identity, K is the ideal of B, the ideals play a significant role in the theory of reduction. It links the isolated complete intersection singularities theory, the Rees algebra and to study linear systems of divisors on non singular surfaces. The Rees algebra of K is defined by $\mathcal{B}(K) = \oplus_{m\geq 0} K^m$. There are several interesting relationship that holds between invariants of reduction of ideals. Many such relations are proved in the paper [8] of Rees. Many results have been done to generalize some of the work on reduction of ideals to local ring of higher dimension [9], [10],[11][15].

The modules in the reduction theory are not just a generalization of ideals. In some cases such as formation of tensor products, products, quotients are closed for modules but not for ideals. These obstacles have been forcing the module theoretic setup to study of reduction of ideals for modules. The construction of the reduction and the Rees algebras of ideals for modules was initiated by Sharp ([1], [2]). He studied general properties of this algebra, its behavior under annihilator of a module, extension of a ring of scalars and determinantal trick. Since then, the theory of reduction of ideals for modules has received a good deal of attention from geometers and algebraists and has been further refined and generalized. This theory has an interesting interwining of its structural and numerical aspects. The P-A theorem [3] has been several proofs beginning with one by Carroll([4], [5]), Caviglia [6], Trung [12], Parmeshvaran Srininasan[7], Singh, Kumar [13]. A recent paper of Goel, Roy, Verma [14] has a proof of Eakin- Sathaye theorem, joint reduction and good filtration of ideals.

The objective of this paper is to investigate the properties of reduction of ideals for modules and to extend some of the results on reduction of ideals to reduction of an ideal for modules. Using the theory of reduction we will give result of P-A [3] in the reduction of an ideal for a module. This theorem will show that if for any reduction L of K for module H is generated by s elements and K^mH is generated by less than $\binom{m+s}{s}$, then reduction number of K for H is at most m.

2 Preliminaries

We give some general facts and definitions about reduction of an ideal for a module, Rees algebra and fiber cone which will be used in Section 3.

Definition 2.1. Let B be a commutative ring with identity, K be an ideal of B and H be B-module. The Rees algebra of K for H is defined as

$$\mathcal{B}(H) = \bigoplus_{m\geq 0} K^mH.$$

Let (B, n) be a Noetherian local ring. Then fiber cone of K for H is defined by

$$F(H) = \frac{\mathcal{B}(H)}{n\mathcal{B}(H)} \simeq \bigoplus_{m \geq 0} \frac{K^m H}{nK^m H}.$$

The dimension of the fiber cone of K for H is called analytic spread of K for H. It is denoted by $\lambda(K, H)$.

Definition 2.2. [15] " Let H be B-module and the ideals $L \subseteq K$ of a ring B. Then L is said to a reduction of K for H if $L\ K^m H = K^{m+1} H$ for some $m \geq 0$. The reduction number of K for H is defined as"

$$r_K(H) = Min \{m \geq 0 \mid L\ K^m H = K^{m+1} H\}$$

Definition 2.3. [15] A reduction L of K is called a minimal reduction of K for H if no ideal is strictly contained in L a reduction of K for H.

For every ideal $L \subseteq K$ is a reduction of K for H such that $\lambda(K, H) \leq \mu(L)$, where $\mu(.)$ is the minimal generating function. If residue field is infinite, then $\lambda(K, H) = \mu(L)$.

Proposition 2.4. *For a finitely generated H, B-module and $L \subseteq K$ are ideals of a Noetherian ring B. Then L is a reduction of K in $B/ann(H)$ if and only if L is a reduction of K for H.*

Proof. Let L be a reduction of K for H. Then $LK^m H = K^{m+1} M$ for some $m \geq 0$. Let $H =< x_1, \ldots, x_t >$, $K^m =< y_1, \ldots, y_t >$ and $a \in K$ such that $ay_i \in K^{m+1}$ for $i = 1, \ldots, t$. Therefore,

$$ay_i\ x_i = \sum_{j=1}^{t} b_{ij}\ y_j\ x_j, \text{where } b_{ij} \in L\ for\ i = 1, \ldots, t.$$

$$\sum_{j=1}^{t} (a\delta_{ij} - b_{ij}) y_j x_j = 0,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. By multiplying adjoint of the matrix $(a\delta_{ij} - b_{ij})$, we have $\triangle x_j y_j = 0$, for all $j = 1, \ldots, t$, where $\triangle$ is the determinant of the matrix $(a\delta_{ij} - b_{ij})$. This implies that $\triangle K^m H = 0$ and $\triangle K^m \subseteq ann(H)$. Now expansion of $\triangle$ shows that L is a reduction of K in $B/ann(H)$.

Conversely, L is a reduction of K in $B/ann(H)$. Then there exist $m > 0$ such that $K^{m+1} \subseteq LK^m + ann(H)$. This implies that $K^{m+1} H \subseteq LK^m H$ and L is a reduction of K for H. □

Definition 2.5. An ideal K is said to be nilpotent ideal for H, if $K^m \subseteq ann(H)$ for some $m > 0$.

Remark 2.6. If an ideal K is a nilpotent ideal for H, then any ideal contained in K is a reduction of K for H.

Proof. Let $L \subseteq K$ be ideals of a ring B. Since K is a nilpotent ideal for H, $K^m \subseteq ann(H)$ and $K^m H = 0$ for some $m > 0$. This implies that $LK^m H = L.0H = K^{m+1} H = 0$ and L is a reduction of K for H. □

Proposition 2.7. *Let (B, n) be a local Noetherian ring and K be n-primary ideal of B. Suppose H is a finitely generated B-module. Then $dim(H) = 0$ if and only if K is a nilpotent for H.*

Proof. Let $dim(\frac{B}{ann(H)}) = dim(H) = 0$. Note that $(\frac{B}{ann(H)}, \frac{n}{ann(H)})$ is a local Noetherian ring with $dim(\frac{B}{ann(H)}) = 0$. Therefore $\frac{B}{ann(H)}$ is an Artinian ring. Consider the following chain of ideals of $\frac{B}{ann(H)}$

$$\overline{K} \supseteq \overline{K^2} \supseteq \cdots \supseteq \overline{K^m} \supseteq \overline{K^{m+1}},$$

where $\overline{K^m} = K^m + ann(H)/ann(H)$. Since $\dfrac{B}{ann(H)}$ is an Artinian ring, $\overline{K^m} = \overline{K^{m+1}}$ for $m >> 0$. By Nakayama lemma, $\overline{K^m} = 0$ and $K^m \subseteq ann(H)$.

Conversely, suppose K is a nilpotent ideal for H. Then there exist $m > 0$ such that $K^m \subseteq ann(H)$. So that

$$rad(K^m) \subseteq rad(ann(H))$$

$$rad(K) \cap \cdots \cap rad(K) = rad(K^m) \subseteq rad(ann(H)) = \bigcap_{ann(H) \subseteq X} X,$$

where X are minimal prime ideals over $ann(H)$. Since K is n-primary ideal, $rad(K) = n$ and $n \subseteq X$. Therefore $n = X$ and $\dfrac{n}{ann(H)}$ is the only prime ideal of $\dfrac{B}{ann(H)}$. Then $dim(H) = dim(\dfrac{B}{ann(H)}) = 0$.

□

3 Reductions for modules

Lemma 3.1. *Let (B, n) be a local Noetherian ring. Suppose $L \subset K$ are ideals of B and H is finitely generated B-module. Then $L + nK$ is a reduction of K for H if and only if L is a reduction of K for H.*

Proof. Let $L+nK$ be a reduction of K for H. Then there exists $m \geq 0$ such that $K^{m+1}H = (L+nK)K^mH \subseteq L\ K^mH + nK^{m+1}H \subseteq K^{m+1}H$. So equality holds throughout and $K^{m+1}H = L\ K^mH + nK^{m+1}H$. We go through modulo $L\ K^mH$, $\dfrac{K^{m+1}H}{L\ K^mH} = \dfrac{L\ K^mH + nK^{m+1}H}{L\ K^mM} = n\left(\dfrac{K^{m+1}H}{L\ K^mH}\right)$. By Nakayama lemma, $\dfrac{K^{m+1}H}{L\ K^mH} = 0$. So that $K^{m+1}H = K\ L^mH$. Conversely, if L is a reduction of K for H, then $K^{m+1}H = L\ K^mH$ for some $m \geq 0$ and $K^{m+1}H = L\ K^mH \subseteq (L + nK)K^mH \subseteq LK^mH + nK^{m+1}H \subseteq K^{m+1}H$. So equality holds throughout and $L + nK$ is a reduction of K for H. □

Proposition 3.2. *Let (B, n) be a local Noetherian ring, K be an ideal of B and H be finitely generated B-module. Suppose $u_i - v_i \in nK$ for $i = 1, \ldots, r$. Then $(u_1, \ldots, u_r)$ is a reduction of K for H if and only if $(v_1, \ldots, v_r)$ is a reduction of K for H.*

Proof. Let $L = (u_1, \ldots, u_r)$ be a reduction of K for H. Then $L + n\ K$ is a reduction of K for H (Lemma 3.1). Note That $u_i + nK = v_i + nK$ for $i = 1, \ldots, r$. Therefore, $(u_1, \ldots, u_r) + nK = (v_1, \ldots, v_r) + nK$ and $(v_1, \ldots, v_r) + nK$ is a reduction of K for H. By Lemma 3.1, $(u_1, \ldots, u_r)$ is a reduction of K for H. □

Proposition 3.3. *Let H be a finitely generated B-module and (B, n) be a local Noetherian ring. Suppose $L \subseteq K$ are ideals of B. Then $L = (x_1 \ldots, x_r)$ is a reduction of K for H if and only if $dim\left(\dfrac{F(H)}{(x_1, \ldots, x_r)H}\right) = 0$.*

Proof. Let L be a reduction of K for H. Then there exists $m > 0$ such that $LK^mH = K^{m+1}H$. By Lemma 3.1, $L + nK$ is a reduction of K for H. This implies that $K^{m+1}H = L\ K^mH + nK^{m+1}H$ and the length of module of $\dfrac{K^mH}{(x_1 \ldots, x_t)K^{m-1}H + nK^mH}$ is equal to zero for $m >> 0$. Therefore $dim\left(\dfrac{F(H)}{(x_1, \ldots, x_r)H}\right) = 0$.

Suppose $dim\left(\dfrac{F(H)}{(x_1, \ldots, x_r)H}\right) = 0$. Hence it has finite length and $\dfrac{F(H)}{(x_1, \ldots, x_r)H}$ has finitely many non zero graded components. This implies that $\dfrac{K^mH}{(x_1 \ldots, x_t)K^{m-1}H + nK^mH} = 0$ for $m >> 0$. Therefore $K^mH = (x_1, \ldots, x_r)K^{m-1}H + nK^mH$ and going modulo $(x_1, \ldots, x_r)K^{m-1}H$, we have $\dfrac{K^m H}{(x_1,\dots,x_r)K^{m-1}H} = \dfrac{(x_1,\dots,x_r)K^{m-1}H + nK^m H}{(x_1,\dots,x_r)K^{m-1}H} = n\Big(\dfrac{K^m H}{(x_1,\dots,x_r)K^{m-1}H}\Big)$. Then by Nakayama lemma, $\dfrac{K^m H}{(x_1,\dots,x_r)K^{m-1}H} = 0$. So $K^m H = (x_1,\dots,x_r)K^{m-1}H$. Hence $(x_1,\dots,x_r)$ is a reduction of of K for H.

□

Theorem 3.4. *Suppose that H is a finitely generated B-module and (B,n) is a local Noetherian ring with infinite residue field $B/n := k$. Let $L = (x_1,\dots,x_m) \subseteq K$ be ideals of R. Then there exists a non empty open subset of k^{sm}, where $s \le m$ the set*

$$W = \left\{ \begin{bmatrix} \overline{a_{11}} & \cdot & \cdot & \overline{a_{1m}} \\ \overline{a_{21}} & \cdot & \cdot & \overline{a_{2m}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \overline{a_{s1}} & \cdot & \cdot & \overline{a_{sm}} \end{bmatrix} \in B_{s\times m}(k) \mid \begin{bmatrix} a_{11} & \cdot & \cdot & a_{1m} \\ a_{21} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{s1} & \cdot & \cdot & a_{sm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_s \end{bmatrix} \right.$$

$\left. (y_1,\dots,y_s)K^t H = K^{t+1}H \right\}$ *for all s and m.*

Proof. Note that W is a well defined set. If

$$\begin{bmatrix} \overline{a_{11}} & \cdot & \cdot & \overline{a_{1m}} \\ \overline{a_{21}} & \cdot & \cdot & \overline{a_{2m}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \overline{a_{s1}} & \cdot & \cdot & \overline{a_{sm}} \end{bmatrix} = \begin{bmatrix} \overline{b_{11}} & \cdot & \cdot & \overline{b_{1m}} \\ \overline{b_{21}} & \cdot & \cdot & \overline{b_{2m}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \overline{b_{s1}} & \cdot & \cdot & \overline{b_{sm}} \end{bmatrix},$$

then $a_{ij} - b_{ij} \in n$ for all i, j and

$$\begin{bmatrix} a_{11} & \cdot & \cdot & a_{1m} \\ a_{21} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{s1} & \cdot & \cdot & a_{sm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_s \end{bmatrix} \ \& \ \begin{bmatrix} b_{11} & \cdot & \cdot & b_{1m} \\ b_{21} & \cdot & \cdot & b_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ b_{s1} & \cdot & \cdot & b_{sm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_s \end{bmatrix}.$$

Then $y_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m$, $z_i = b_{i1}x_1 + b_{i2}x_2 + \dots + b_{im}x_m$ and $y_i - z_i = (a_{i1} - b_{i1})x_1 + \dots + (a_{im} - b_{im})x_m$. Since $a_{ij} + n = b_{ij} + n$ and $y_i - z_i \in nK$, $(z_1,\dots,z_r)K^t H = K^{t+1}H$ (Proposition 3.2). So that W is a well defined set. Note that $K^t =< x_1^{s_1}.\dots.x_m^{s_m} \mid s_1+s_2+\dots+s_m = t >$ and $K^{t+1} =< x_1^{s_1}.\dots.x_m^{s_m} \mid s_1+s_2+\dots+s_m = t+1 >$.

Consider the minimal generating sets of K'^t and K'^{t+1}, where $K'^t = \dfrac{K^t + ann(H)}{ann(H)}$ over $B/ann(H)$. Then K'^t/nK'^t and K'^{t+1}/nK'^{t+1} are vector spaces over B/n.

Let $y_1,\dots,y_r \in K$ and define the map $\left(\dfrac{K'^t}{nK'^t}\right)^s \xrightarrow{f} \dfrac{K'^{t+1}}{nK'^{t+1}}$ such that $f(\overline{z_1'},\dots,\overline{z_s'}) = \overline{y_1'z_1' + y_2'z_2' + \dots + y_s'z_s'} = y_1'z_1' + \dots + y_s'z_s' + nK'^{t+1}$. we observe that that f is a well defined map.

Now $f(\overline{z_1'},\dots,\overline{z_s'}) = \overline{y_1'z_1' + y_2'z_2' + \dots + y_s'z_s'} = \overline{y_1'z_1'} + \dots + \overline{y_s'z_s'} = (\overline{a_{11}}\ \overline{x_1'} + \dots + \overline{a_{1m}}\ \overline{x_m'})\overline{x_1'^{s_1}x_2'^{s_2}.\dots.x_m'^{s_m}} + \dots + (\overline{a_{s1}}\ \overline{x_1'} + \dots + \overline{a_{sm}}\ \overline{x_m'})\overline{x_1'^{s_1}x_2'^{s_2}.\dots.x_m'^{s_m}} = (\overline{a_{11}} + \overline{a_{21}} + \dots + \overline{a_{s1}})\overline{x_1'^{r_1+1}x_2'^{s_2}.\dots.x_m'^{s_m}} + \dots + (\overline{a_{1m}} + \overline{a_{2m}} + \dots + \overline{a_{sm}})\overline{x_1'^{s_1}x_2'^{s_2}.\dots.x_m'^{s_m+1}}$, where $y_i = a_{i1}x_1 + \dots, a_{im}x_m$ by assumption. We can see that there are $\overline{x_1'^{s_1+1}x_2'^{s_2}.\dots.x_m'^{s_m}}, \overline{x_1'^{s_1}x_2'^{s_2+1}.\dots.x_m'^{s_m}}, \dots, \overline{x_1'^{s_1}x_2'^{s_2}.\dots}$ m- monomials with coefficients $\overline{a_{ij}}$. These coefficients are linear.

We have to show that $(y_1',\dots,y_s')K'^t = K'^{t+1}$ if and only if f is onto. We have to show that T is onto. If $\overline{x'} \in \dfrac{k'^{t+1}}{nK'^{t+1}}$, where $\overline{x'} = x' + nK'^{t+1}$ and $x \in K'^{t+1}$, then $x \in (y_1',\dots,y_s')K'^t$, for $K'^{t+1} = (y_1',\dots,y_s')K^t$. Thus $x' = (y_1'a_1 + \dots + y_s'a_s)y' = y_1'a_1y' + \dots + y_s'a_sy'$. Pick $z_1' =$

$a_1y', z_2' = a_2y', \ldots, z_s' = a_sy'$, where $a_i \in B/ann(H)$ and $y \in K'^t$. This implies that if $\overline{x'} = y_1'z_1' + \cdots + y_s'z_s' + nK'^{t+1}$ and $(\overline{z_1'}, \ldots, \overline{z_s'}) \in \left(\frac{K'^t}{mK'^t}\right)^s$, then $f(\overline{z_1'}, \ldots, \overline{z_s'}) = \overline{x'}$. This shows that map f is onto. Conversely suppose that f is onto. Then for any $\overline{x'} \in K'^{t+1}/nK'^{t+1}$ there exists $(\overline{z_1'}, \ldots, \overline{z_s'}) \in \left(\frac{K'^t}{nK'^t}\right)^s$ such that $f(\overline{z_1'}, \ldots, \overline{z_s'}) = \overline{x'}$. Thus $\overline{x'} = y_1'z_1' + \cdots + y_s'z_s' + nK'^{t+1}$ and $K'^{t+1} \subseteq (y_1', \ldots, y_s')K'^t + nK'^{t+1}$. Going modulo $(y_1', \ldots, y_s')K'^t$, $\frac{K'^{t+1}}{(y_1', \ldots, y_s')\ K'^t} \subseteq \frac{(y_1', \ldots, y_s')\ K'^t + nK'^{t+1}}{(y_1', \ldots, y_s')\ K'^t} = n\left(\frac{K'^{t+1}}{(y_1', \ldots, y_s')\ K'^t}\right)$. By Nakayama lemma, $\frac{K'^{t+1}}{(y_1', \ldots, y_s')\ K'^t} = 0$. Therefore, $K'^{t+1} = (y_1', \ldots, y_s')\ K'^t$ and $K^{t+1} \subseteq (y_1, \ldots, y_s)K^t + ann(H)$. In this case we have $(y_1, \ldots, y_s)$ is a reduction of K in $B/ann(H)$. By Lemma 3.1 $(y_1, \ldots, y_s)$ is a reduction of K for H. Defining the set W,

$$W = \left\{ \begin{bmatrix} \overline{a_{11}} & . & . & \overline{a_{1m}} \\ \overline{a_{21}} & . & . & \overline{a_{2m}} \\ . & . & . & . \\ . & . & . & . \\ \overline{a_{s1}} & . & . & \overline{a_{sm}} \end{bmatrix} \in B_{s\times m}(k) \mid \begin{bmatrix} a_{11} & . & . & a_{1m} \\ a_{21} & . & . & a_{2m} \\ . & . & . & . \\ . & . & . & . \\ a_{s1} & . & . & a_{sm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_s \end{bmatrix} \right.$$
$$\left. \text{and} \quad f \text{ is onto} \right\}.$$

By (Lemma 3.7, [16]) the theorem is proved. □

Theorem 3.5. *Suppose H is a finitely generated B-module and (B, n) is a local Noetherian ring with infinite residue field $B/n = k$. Let K be an ideal of B. Assume that $m \geq 1$ and $s \geq 0$ are integers such that $\mu_B(K^mH) < \binom{m+s}{s}$. Then there exists of $x_i \in K$ for $i = 1, \ldots, s$ such that $(x_1, \ldots, x_s)K^{m-1}H = K^mH$.*

Proof. The proof is based on double induction m and s. If $s = 0$, then $\binom{m}{0} = 1$ and assumption $dim_k\left(\frac{K^mH}{nK^mH}\right) = \mu_B(K^mH) = 0$. This implies that $K^mH = nK^mH$ and Nakayama lemma $K^mH = 0$. Hence $K^mH = 0 = 0K^{m-1}H$ and 0 is a reduction of K for H.

Therefore, result is proved for $s = 0$. If $m = 1$, then $\mu_B(KH) = dim_k\left(\frac{KH}{nKH}\right) < \binom{s+1}{1} = s + 1$. This implies that $dim_k\left(\frac{KH}{nKH}\right) \leq s$ and there exists $a_1 + nKM, \ldots, a_s + nKH$ such that $\frac{KH}{nKH} =< a_1 + nKH + \cdots + a_s + nKH >$, where $a_i = x_iy_i$, $x_i \in K$ and $y_i \in H$. This implies that $KH = (x_1, \ldots, x_s)H + nKH$. We go through module $(x_1, \ldots, x_s)H$, we have $\frac{KH}{(x_1, \ldots, x_s)H} = n(\frac{KH}{(x_1, \ldots, x_s)H})$. By Nakayama lemma, $KH = (x_1, \ldots, x_s)H$. Thus result is proved for $m = 1$.

Now assume that the result is false. Pick a counter example for the case $\mu_B(K^mH) < \binom{m+s}{s}$ and there does not exist of $x_i \in K$ for $i = 1, \ldots, s$ such that $(x_1, \ldots, x_s)K^{m-1}H = K^mH$, where s is minimal and m is minimal for this s. Now, we may suppose that $m \geq 2$ and $s \geq 1$.

There are two cases.

Case 1. Suppose there exists $y \in K \setminus nK$ such that $dim_k\left(\frac{yK^{m-1}H + nK^mH}{nK^mH}\right) \geq \binom{m-1+s}{s}$.

If $s = 1$, then hypothesis of the theorem $\mu(K^mH) = dim_k(\frac{K^mH}{nK^mH})$ is generated by at most

m elements and also minimum dimension of $\dfrac{yK^{m-1}H+nK^mH}{nK^mH}$ over B/n is m. So that $\dfrac{K^mH}{nK^mH}=\dfrac{yK^{m-1}H+nK^mH}{nK^mH}$ and $K^mH=yK^{m-1}H+nK^mH$. We go through modulo $yK^{m-1}H$ and $\dfrac{K^mH}{yK^{m-1}H}=n(\dfrac{K^mH}{yK^{m-1}H})$. By Nakayama lemma $\dfrac{K^mH}{yK^{m-1}H}=0$. This implies that $K^mH=yK^{m-1}H$. We can say that y is a reduction of K for H. Therefore, this is a contradiction because m is minimal for the given $s=1$. Now we may suppose that $s>1$. $\overline{F(H)}=\dfrac{F(H)}{yF(H)}$. Note that

$$\overline{F(H)_m}=\frac{\dfrac{K^mH}{nK^mH}}{\dfrac{yK^{m-1}H+nK^mH}{nK^mH}} \text{ and } \binom{m+s}{s}-\binom{m+s-1}{s}=\binom{m+s-1}{s-1}.$$

Consider the following exact sequence

$$0\longrightarrow\frac{yK^{m-1}H+nK^mH}{nK^mH}\longrightarrow\frac{K^mH}{nK^mH}\longrightarrow\frac{\dfrac{K^mH}{nK^mH}}{\dfrac{yK^{m-1}H+nK^mH}{nK^mH}}\longrightarrow 0.$$

Then $dim_k\left(\dfrac{yK^{m-1}H+nK^mH}{nK^mH}\right)-dim_k\left(\dfrac{K^mH}{nK^mH}\right)+dim_k\left(\dfrac{\dfrac{K^mH}{nK^mH}}{\dfrac{yK^{m-1}H+nK^mH}{nK^mH}}\right)=0.$

Since, $dim_k\left(\dfrac{K^mH}{nK^mH}\right)<\binom{m+s}{s}$ and $dim_k\left(\dfrac{yK^{m-1}H+nK^mH}{nK^mH}\right)\geq\binom{m+s-1}{s}$, it implies that $dim_k\left(\dfrac{\dfrac{K^mH}{nK^mH}}{\dfrac{yK^{m-1}H+nK^mH}{nK^mH}}\right)<\binom{m+s-1}{s-1}$. Therefore assumption of the theorem is satisfied for $\overline{F(H)}$ and by the minimality of s, there exist $x_1,\ldots,x_s\in K$ such that $\dfrac{(x_1,\ldots,x_s)K^{m-1}H+nK^mH}{nK^mH}=\dfrac{K^mH}{nK^mH}$ and $(x_1,\ldots,x_s)K^{m-1}H+nK^mH=K^mH$. Again by Nakayama lemma, $(x_1,\ldots,x_s)K^{m-1}H=K^mH$ which is again a contradiction.

Case 2. Now assume that for all $y\in K\setminus nK$, $dim_k\left(\dfrac{yK^{m-1}H+nK^mH}{nK^mH}\right)<\binom{m-1+s}{s}$ and define a multiplication map $f_y:\dfrac{K^{m-1}H}{nK^{m-1}H}\to\dfrac{K^mH}{nK^mH}$ such that $f_y(x+nK^{m-1}H)=yx+nK^mH$. Note that $ker(f_y)=\dfrac{(nK^mH:y)+nK^{m-1}H}{nK^{m-1}H}$. Fundamental theorem of B/n-module homomorphism $\dfrac{\dfrac{K^{m-1}H}{nK^{m-1}H}}{ker(f_y)}\simeq\dfrac{yK^{m-1}H+nK^mH}{nK^mH}$. Since $dim_k\left(\dfrac{yK^{m-1}H+nK^mH}{nK^mH}\right)<\binom{m-1+s}{s}$, $dim_k\left(\dfrac{\dfrac{K^{m-1}H}{nK^{m-1}H}}{ker(f_y)}\right)<\binom{m-1+s}{s}$.

Therefore, hypothesis of the theorem is satisfied for $\dfrac{\dfrac{K^{m-1}H}{nK^{m-1}H}}{ker(f_y)}$ and by induction on m, there exist $y_1,\ldots,y_s$ in K such that $\dfrac{K^{m-1}H}{nK^{m-1}H}=\dfrac{(y_1,\ldots,y_s)K^{m-2}H+(nK^mH:y)}{nK^{m-1}H}$. Let $K=(x_1,\ldots,x_s)$. Then by Theorem 3.4 there exists a non empty open subset W_i of k^{st} such that $(x_1,\ldots,x_s)K^{n-2}H+(nK^mH:x_i)=K^{m-1}H$ with reduction number at most $m-2$ for each x_i. Let $W=\cap_{i=1}^{t}W_i$. Since residue field is infinite, W is a non empty open set.

Choose $y_1, \dots, y_s \in K$ such that $(y_1, \dots, y_s)K^{m-2}H + (nK^mH : x_i) = K^{m-1}H$. For each $i = 1, \dots, s$ $x_iK^{m-1}H \subseteq (y_1, \dots, y_s)K^{m-1}H + nK^mH$. Therefore $K^mH \subseteq (y_1, \dots, y_s)K^{m-1}H + nK^mH \subseteq K^nH$. Equality holds through out and $(y_1, \dots, y_s) + nK$ is a reduction of K for H. By Lemma 3.1, $(y_1, \dots, y_s)$ is a reduction of K for H which is again a contradiction to the minimality of s. This proves the theorem.

□

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