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ON APPLICATIONS OF SEMI LOCAL FUNCTIONS

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Abstract In this paper, firstly, semi-$\mathcal{I}$-convergence and semi-$\mathcal{I}$-clustering of filters are introduced using the concept of semi local function. Further, semi-T_2 mod $\mathcal{I}$ spaces are introduced and investigated. Finally, characterizations of semi-T_2 mod $\mathcal{I}$ spaces and semi-$\mathcal{I}$-compact spaces in terms of semi local functions are explored.

1 Introduction

Semi open sets were firstly introduced and investigated by N. Levine in [8]. In [7], R. M. Latif introduced semi-convergence and semi-clustering of filters and gave related characterizations. Further, in [9], S. N. Maheshwari and R. Prasad introduced semi-T_2 spaces. As, it is very convenient to study the topology of a space by specifying which nets, equivalently which filters, converge to which points of the space. Ideals, which are dual to the concept of filters (i.e. "the collection of complements of members of a proper ideal will form a filter") are very well suited for such a study as they simultaneously generalize such important convergence concepts as closure point, ω-accumulation point, condensation point etc. Ideals were introduced by Kuratowski [6] to study various topological properties. Also a new topology $\sigma^*(\mathcal{I}, \sigma)$ finer than the original topology whose closure is given by $cl^*(E) = E \cup E^*(\mathcal{I}, \sigma)$ [11] was obtained. Further, semi-local functions were studied using semi-open sets in [5]. Thus it motivates to introduce some topological concepts (semi-$\mathcal{I}$-convergence and semi-$\mathcal{I}$-clustering of a filter etc.) and characterize using semi-local function in ideal topological spaces.

In Section 2, semi-$\mathcal{I}$-convergence and semi-$\mathcal{I}$-clustering of a filter are introduced and relationship between them is investigated. In Section 3, we introduce semi-T_2 mod $\mathcal{I}$ spaces and give its characterizations in terms of semi-local function (Theorem 3.4) and in terms of semi-$\mathcal{I}$-convergence of a filter (Theorem 3.7). Further, we obtain the sufficient condition for an image (inverse image) of T_2 mod $\mathcal{I}$ space to be semi-T_2 mod $\mathcal{I}$ (Theorems 3.5 and 3.6). Finally, we obtain the characterization of semi-$\mathcal{I}$-compact space in terms of filter using semi-local function (Theorem 3.8).

Throughout this paper in a topological space (Y, σ), $SO(Y)$ will denote semi-open subsets of Y and for an element e in Y, $SO(e)$ will denote the collection of semi-open sets containing e and $\mathcal{R}$ will denote the set of real numbers.

Preliminaries

Definition 1.1. [8] A mapping $g : (Y, \sigma) \to (Z, \nu)$ is called semi open (semi continuous) if image (inverse image) of open set in Y (Z) is semi open in Z (Y).

Definition 1.2. [5] Let $(Y, \sigma, \mathcal{I})$ be an ideal space then for any subset E of Y, the semi-local function of E with respect to $\mathcal{I}$ and $SO(Y)$, denoted by E_* is given by $E_* = \{e \in Y | U \cap E \notin \mathcal{I}$ for all $U \in SO(e)\}$.

Definition 1.3. [2] "An ideal space $(Y, \sigma, \mathcal{I})$ is known as semi-$\mathcal{I}$-compact if for every cover $\{S_\alpha : \alpha \in \Delta\}$ of Y, where S_α are semi-open subsets of Y, there exists a finite subset Δ_0 of Δ such that $Y - \bigcup\{S_\alpha : \alpha \in \Delta_0\} \in \mathcal{I}$".

Lemma 1.4. *[5] For an ideal space $(Y, \sigma, \mathcal{I})$ and two subsets E and B of it, the following are true:*

(a) $E_* \subseteq E^*$.

(b) If $E \subseteq B$ *then* $E_* \subseteq B_*$.

(c) $(E_*)_* \subseteq E_*$.

(d) $(E \cup B)_* = E_* \cup B_*$.

Remark 1.5. In Lemma 1.4(d), M. Khan and T. Noiri proved that $(E \cup B)_* = E_* \cup B_*$, which is not true always as the Example 1.6 shows below.

Example 1.6. Consider $Y = \mathcal{R}$ with usual topology and $\mathcal{I} = \{\emptyset\}$. Then $SO(Y) = [e, b], (e, b), (e, b], [e, b)$ where $e, b \in \mathcal{R}$. For the subsets $E = (0, 1)$ and $B = (1, 2)$, $(E \cup B)_* = (0, 2)$ and $E_* = (0, 1)$, $B_* = (1, 2)$. Therefore, $E_* \cup B_* = (0, 1) \cup (1, 2)$ and so we can see that $(E \cup B)_* \neq E_* \cup B_*$. Hence $(E \cup B)_* \neq E_* \cup B_*$ in general.

Definition 1.7. [3] Let $(Y, \sigma, \mathcal{I})$ be an ideal space. Then for any map $g : (Y, \sigma, \mathcal{I}) \to (Z, \nu)$, $g(\mathcal{I})$ is an ideal for the topological space (Z, ν). Also if g is injective and $\mathcal{J}$ is an ideal for the topological space (Z, ν) then $g^{-1}(\mathcal{J})$ is an ideal for the topological space (Y, σ).

2 "Semi-$\mathcal{I}$-convergence and semi-$\mathcal{I}$-clustering of a filter"

Definition 2.1. In an ideal space $(Y, \sigma, \mathcal{I})$, for the filter $\mathcal{H}$ on Y having empty intersection with $\mathcal{I}$ and an element $e \in Y$, $\mathcal{H}$ is said to be Semi-$\mathcal{I}$-converges to e, denoted by $\mathcal{H} \to_{s_{\mathcal{I}}} e$ if for every $S \in SO(e)$, there exists $H \in \mathcal{H}$ such that $H - S \in \mathcal{I}$.

The example below shows that $\mathcal{H} \to_{s_{\mathcal{I}}} e$ but $\mathcal{H} \nrightarrow_s e$.

Example 2.2. Let $Y = \{e, b\}$, $\sigma = \{\emptyset, \{e\}, Y\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $\mathcal{H} = \{Y\}$. Then $\mathcal{H} \to_{s_{\mathcal{I}}} e$, but $\mathcal{H} \nrightarrow_s e$.

Remark 2.3. In an ideal space $(Y, \sigma, \mathcal{I})$ where $\mathcal{I}$ is empty ideal, $\mathcal{H} \to_s e$ if and only if $\mathcal{H} \to_{s_{\mathcal{I}}} e$.

The following theorem gives the sufficient condition for the equivalence of above concept.

Theorem 2.4. *In an ideal space $(Y, \sigma, \mathcal{I})$, for the ultrafilter $\mathcal{U}$ disjoint from an ideal $\mathcal{I}$, $\mathcal{U} \to_{s_{\mathcal{I}}} e$ if and only if $\mathcal{U} \to_s e$.*

Proof. Let S be semi open subset containing e then $\mathcal{U} \to_{s_{\mathcal{I}}} e$ implies that $U - S \in \mathcal{I}$ for some $U \in \mathcal{U}$. Therefore, $U - S = J$ for some $J \in \mathcal{I}$ and so $U \subset S \cup J$ and hence $S \cup J \in \mathcal{U}$. It follows that $S \in \mathcal{U}$ as $J \notin \mathcal{U}$, since $\mathcal{U}$ is ultrafilter. Hence $\mathcal{U} \to_s e$. □

Next we define semi-$\mathcal{I}$-clusterence of a filter and utilize it in our next section.

Definition 2.5. In an ideal space $(Y, \sigma, \mathcal{I})$, for the filter $\mathcal{H}$ having disjoint intersection with $\mathcal{I}$ and for an element $e \in Y$, e is semi-$\mathcal{I}$-cluster point of $\mathcal{H}$ denoted as $\mathcal{H} \rtimes_{s_{\mathcal{I}}} e$ if $e \in \bigcap_{H \in \mathcal{H}} H_*$.

It can be seen easily that $\mathcal{H} \rtimes_{s_{\mathcal{I}}} e$ implies $\mathcal{H} \rtimes_s e$, but semi-clustering does not imply semi-$\mathcal{I}$-clustering always as the example below shows:

Example 2.6. Let $Y = \{e, b\}$, $\sigma = \{\emptyset, \{e\}, Y\}$, $\mathcal{I} = \{\emptyset, \{e\}\}$ and $\mathcal{H} = \{Y\}$. Then $e \in scl(Y)$ but $e \notin Y_*$.

Further, it can be seen easily that semi-$\mathcal{I}$-convergence implies $\mathcal{I}$-convergence and semi-$\mathcal{I}$-clustering implies $\mathcal{I}$-clustering but converse is not true.

Example 2.7. Let $Y = \{e, b, c\}$, $\sigma = \{\emptyset, \{e\}, Y\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $\mathcal{H} = \{Y\}$. Then $\mathcal{H} \to_{\mathcal{I}} b$ but $\mathcal{H} \nrightarrow_{s_{\mathcal{I}}} b$, since $\{e, b\}$ is semi-open and $Y - \{e, b\} = \{c\} \notin \mathcal{I}$.

Example 2.8. Let $Y = \{e, b, c\}$, $\sigma = \{\emptyset, \{e\}, Y\}$, $\mathcal{I} = \{\emptyset, \{e\}, \{b\}, \{e, b\}\}$ and $\mathcal{H} = \{Y\}$. Then $b \in Y^*$ but $b \notin Y_*$.

Theorem 2.9. *In an ideal space $(Y, \sigma, \mathcal{I})$, consider the filter $\mathcal{H}$ having disjoint intersection with $\mathcal{I}$. Then $\mathcal{H} \rightarrow_{s_{\mathcal{I}}} e$ implies $\mathcal{H} \rtimes_{s_{\mathcal{I}}} e$ for an element $e \in Y$.*

Proof. Let $\mathcal{H} \not\rtimes_{s_{\mathcal{I}}} e$ then there exists $E \in \mathcal{H}$ such that $e \notin E_*$. Therefore, $E \cap S \in \mathcal{I}$ for some semi open subset S containing e in Y. Now $\mathcal{H} \rightarrow_{s_{\mathcal{I}}} e$, so there exists $B \in \mathcal{H}$ such that $B - S \in \mathcal{I}$. This implies that $(E \cap B) - S, E \cap B \cap S \in \mathcal{I}$ and so $E \cap B \in \mathcal{I}$, which contradicts the fact that $\mathcal{H} \cap \mathcal{I} = \emptyset$. Thus $E \cap S \notin \mathcal{I}$, so $e \in E_*$. Hence $\mathcal{H} \rtimes_{s_{\mathcal{I}}} e$. □

3 Semi-T_2 mod $\mathcal{I}$ spaces

In this section, we will introduce semi-T_2 mod $\mathcal{I}$ spaces and give its characterization using the concepts of semi-$\mathcal{I}$-convergence points of a filter and semi-local function.

Definition 3.1. An ideal space $(Y, \sigma, \mathcal{I})$ is known as semi-T_2 mod $\mathcal{I}$ if for two distinct points e, b in Y there are semi open subsets S and T containing e and b respectively such that $S \cap T \in \mathcal{I}$.

It can be seen easily that every semi T_2 space is semi T_2 mod $\mathcal{I}$, since $\emptyset \in \mathcal{I}$ but the converse is not true.

Example 3.2. Let $Y = \{e, b, c\}$, $\sigma = \{\emptyset, \{e\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{e\}\}$. Then $SO(Y) = \{\emptyset, \{e\}, \{e, b\}, \{e, c\}, Y\}$. Therefore, Y is semi-T_2 mod $\mathcal{I}$ but not semi T_2.

Theorem 3.3. *Let $(Y, \sigma, \mathcal{I})$ be semi-T_2 mod $\mathcal{I}$ and $\mathcal{I} \subseteq \mathcal{J}$, where $\mathcal{J}$ is an ideal then Y is semi-T_2 mod $\mathcal{J}$.*

The following Theorem 3.4, gives the characterization of semi-T_2 mod $\mathcal{I}$ space using the concept of semi-local function.

Theorem 3.4. *Let $(Y, \sigma, \mathcal{I})$ be an ideal space. Then the following are equivalent:*

(a) Y is semi-T_2 mod $\mathcal{I}$ space.

(b) For every $y \neq e$ in Y, where e is some fixed element in Y there exists semi open set S containing e such that $y \notin S_*$.

(c) For every e in Y, $\bigcap\{S_{e*} | S_e \in SO(e)\} = \{\emptyset\}$ or $\{e\}$.

Proof. $(a) \Rightarrow (b)$: Let Y be semi-T_2 mod $\mathcal{I}$ space and $e \in Y$ be any element. Then for any $y \neq e$ in Y, there exist semi open sets S and T containing e and y respectively such that $S \cap T \in \mathcal{I}$ which implies $y \notin S_*$. Hence (b) proved.
$(b) \Rightarrow (c)$: Let $y \neq e$ be any element in Y then by (b), there exists semi open set S_e containing e such that $y \notin S_{e*}$ and so $y \notin \bigcap\{S_{e*} | S_e \in SO(e)\}$. Hence $\bigcap S_{e*} = \{\emptyset\}$ or $\{e\}$, which proves (c).
$(c) \Rightarrow (a)$: Let $e \neq y$ be any two elements in Y then by (c), $y \notin \bigcap\{S_{e*} | S_e \in SO(e)\}$. Therefore, there exists $S_e \in SO(e)$ such that $y \notin S_{e*}$ and so there exists semi open set S_y containing y such that $S_e \cap S_y \in \mathcal{I}$. Hence Y is semi T_2 mod $\mathcal{I}$. □

Theorem 3.5. *Let $g : (Y, \sigma, \mathcal{I}) \rightarrow (Z, \nu)$ be semi open bijection and Y is T_2 mod $\mathcal{I}$ then Z is semi-T_2 mod $f(\mathcal{I})$.*

Proof. Let $c \neq d$ be any two elements in Z then there exist $a \neq b$ in Ysuch that $g(a) = c$ and $g(b) = d$, since g is bijective. Now, Y is T_2 mod $\mathcal{I}$ implies that there are open sets S and T containing a and b respectively such that $S \cap T \in \mathcal{I}$. Therefore, $g(S \cap T) \in g(\mathcal{I})$ and so $g(S) \cap g(T) \in g(\mathcal{I})$. Further, g is semi open implies that $g(S)$ and $g(T)$ are semi open subsets of Z containing c and d respectively such that $g(S) \cap g(T) \in g(\mathcal{I})$. Hence Z is semi-T_2 mod $g(\mathcal{I})$. □

The following Theorem 3.6 gives the sufficient condition for an inverse image of T_2 mod $\mathcal{J}$ space to be semi-T_2 mod $g^{-1}(\mathcal{J})$.

Theorem 3.6. *Let $g : (Y, \sigma, g^{-1}(\mathcal{J})) \rightarrow (Z, \nu, \mathcal{J})$ be semi continuous injection and Z be T_2 mod $\mathcal{J}$ space then Y is semi-T_2 mod $g^{-1}(\mathcal{J})$.*

Proof. Let $e \neq b$ be any two elements in Y then $g(e) \neq g(b)$ in Z, since g is injective. Now, Z is T_2 mod $\mathcal{J}$ implies that there are open subsets U and W containing $g(e)$ and $g(b)$ respectively such that $U \cap W \in \mathcal{J}$. Therefore, $g^{-1}(U \cap W) \in g^{-1}(\mathcal{J})$ and so $g^{-1}(U) \cap g^{-1}(W) \in g^{-1}(\mathcal{J})$. Further, $g^{-1}(U)$ and $g^{-1}(W)$ are semi open subsets of Y containing e and b respectively such that $g^{-1}(U) \cap g^{-1}(W) \in g^{-1}(\mathcal{J})$. Hence Y is semi-T_2 mod $g^{-1}(\mathcal{J})$. □

The next Theorem 3.7 characterizes semi-T_2 mod $\mathcal{I}$ space using semi-$\mathcal{I}$-convergence.

Theorem 3.7. *Let $(Y, \sigma, \mathcal{I})$ be semi-T_2 mod $\mathcal{I}$ space then for any filter $\mathcal{H}$ with $\mathcal{H} \cap \mathcal{I} = \emptyset$, $\mathcal{H}$ semi-$\mathcal{I}$-converges to at most one element of Y.*

Proof. Firstly, let Y be semi-T_2 mod $\mathcal{I}$ space and $\mathcal{H}$ be any semi-$\mathcal{I}$-convergent filter with $\mathcal{H} \cap \mathcal{I} = \emptyset$ such that $\mathcal{H} \rightarrow_{s_\mathcal{I}} e$, where $e \in Y$ be any element. Then for any $y \neq e$ in Y, there are semi open sets S and T containing e and y respectively such that $S \cap T \in \mathcal{I}$, since Y is semi-T_2 mod $\mathcal{I}$. Now if possible, $\mathcal{H} \rightarrow_{s_\mathcal{I}} y$ then there exist $E \in \mathcal{H}$ such that $E - T \in \mathcal{I}$. Further, $\mathcal{H} \rightarrow_{s_\mathcal{I}} e$ implies that there exists $B \in \mathcal{H}$ such that $B - S \in \mathcal{I}$. Therefore, $(E \cap B) - S \in \mathcal{I}$, $(E \cap B) - T \in \mathcal{I}$ and so $(E \cap B) - (S \cap T) \in \mathcal{I}$. Also $S \cap T \in \mathcal{I}$ implies that $E \cap B \in \mathcal{I}$, which is contradiction to the fact that $\mathcal{H} \cap \mathcal{I} = \emptyset$. Hence $\mathcal{H}$ semi-$\mathcal{I}$-converges to at most one element of Y. □

The next Theorem 3.8 characterizes semi-$\mathcal{I}$-compact spaces using the concept of semi-local function.

Theorem 3.8. *An ideal space $(Y, \sigma, \mathcal{I})$ is semi-$\mathcal{I}$-compact if and only if for every filter $\mathcal{H}$ having empty intersection with $\mathcal{I}$, $\bigcap_{H \in \mathcal{H}} H_* \neq \emptyset$.*

Proof. Firstly, let Y be semi-$\mathcal{I}$-compact space and $\mathcal{H}$ be any filter on Y such that $\mathcal{H} \cap \mathcal{I} = \emptyset$. Let if possible, $\bigcap_{H \in \mathcal{H}} H_* = \emptyset$ then for every element a of Y, there exists a member H_a of $\mathcal{H}$ such that $a \notin H_{a*}$. Therefore, for every element a of Y, there exists semi open subset U_a containing a such that $U_a \cap H_a \in \mathcal{I}$. This implies that $\{U_a\}_{a \in Y}$ is a family of semi open cover for Y i.e. $Y = \bigcup_{a \in Y} U_a$. Further, Y is semi-$\mathcal{I}$-compact and so there exist finite subset of Y such that $Y - \bigcup_{i=1}^{n} U_{a_i} \in \mathcal{I}$. Now, for all $i = 1, 2, ..., n$; $U_{a_i} \cap H_{a_i} \in \mathcal{I}$ and so $U_{a_i} \cap (\bigcap_{i=1}^{n} H_{a_i}) \in \mathcal{I}$ and so $(\bigcup_{i=1}^{n} U_{a_i}) \cap (\bigcap_{i=1}^{n} H_{a_i}) \in \mathcal{I}$. Consider, $M = \bigcup_{i=1}^{n} U_{a_i}$ and $H = \bigcap_{i=1}^{n} H_{a_i}$. Then $H \in \mathcal{H}$ and $M \cap H \in \mathcal{I}$. Also $Y - M \in \mathcal{I}$ implies that $(M \cap H) \cup (Y - M) \in \mathcal{I}$. But $H \subset (M \cap H) \cup (Y - M)$ and so $H \in \mathcal{I}$, which is a contradiction that $\mathcal{H} \cap \mathcal{I} = \emptyset$. Hence $\bigcap_{H \in \mathcal{H}} H_* \neq \emptyset$.

Now, let $\bigcap_{H \in \mathcal{H}} H_* \neq \emptyset$. We have to prove that Y is semi-$\mathcal{I}$-compact. If possible, let Y be not semi-$\mathcal{I}$-compact. Then there exist semi open cover $\{M_\alpha\}_{\alpha \in \Lambda}$ of Y such that for every finite subset Λ_0 of Λ, $Y - \bigcup_{\alpha \in \Lambda_0} M_\alpha \notin \mathcal{I}$ i.e. for all $n \in \mathbb{N}$, $Y - \bigcup_{i=1}^{n} M_{\alpha_i} \notin \mathcal{I}$ and so $\bigcap_{i=1}^{n} (M_{\alpha_i})^C \notin \mathcal{I}$. Consider the filterbase $\mathcal{H}(\mathcal{B}) = \{\bigcap_{i=1}^{n} (M_{\alpha_i})^C | n \in \mathbb{N}\}$ and $\mathcal{H}$ be the filter generated by the filterbase $\mathcal{H}(\mathcal{B})$. Then it can be easily checked that $\mathcal{H} \cap \mathcal{I} = \emptyset$. Now, $\bigcap_{H \in \mathcal{H}} H_* \neq \emptyset$, so there exists an element e of Y such that $e \in \bigcap_{H \in \mathcal{H}} H_*$. But $Y = \bigcup_{\alpha \in \Lambda} M_\alpha$ and so there exists $\alpha \in \Lambda$ such that $e \in M_\alpha$. Further, $(M_\alpha)^C \in \mathcal{H}$ and $M_\alpha \cap (M_\alpha)^C = \emptyset$ implies that $e \notin (M_\alpha^C)_*$ which contradicts the fact that $e \in \bigcap_{H \in \mathcal{H}} H_*$. Hence Y is semi-$\mathcal{I}$-compact. □

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