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SOME NEW CONGRUENCE PROPERTIES FOR t-REGULAR BI-PARTITION FUNCTIONS WITH TAGGED SUMMAND

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Abstract Let $BPd_t(n)$ denotes the number of t-regular bi-partitions of n with tagged (designated) summands. In this paper, we prove infinite families of congruences modulo small powers of 2 and 3 for $BPd_t(n)$. For example, if $a \geq 0$ and $1 \leq m \leq \rho - 1$, then

$$BPd_2(24 \cdot \rho^{2a+1}(\rho n + m) + 15 \cdot \rho^{2a+2}) \equiv 0 \pmod{16}.$$

1 Introduction

A partition of a non-negative integer n is defined as a non-increasing sequence of positive integers, which are called the parts of the partitions, such that the sum of all the part is equals to n. The number of unrestricted partitions for a non-negative integer n is generally represented by $p(n)$ (with $p(0) = 1$) and its generating function is given by

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots , \tag{1.1}$$

where for $z \in \mathbb{C}$ and $|q| < 1$, we have

$$(z;q)_\infty = \prod_{n\geq 0}(1 - zq^n).$$

For simplicity, for any positive integer k, we denote $(q^k;q^k)_\infty$ as f_k. Let t be any positive integer, then t-regular partition can be defined as a partition in which no part is a multiple of t. If $b_t(n)$ represent the total number of t-regular partitions of n, then the generating function of $b_t(n)$ is given by

$$\sum_{n\geq 0} b_t(n)q^n = \frac{f_t}{f_1}. \tag{1.2}$$

Several mathematicians studied the congruence properties for $b_t(n)$. In this direction, see [6, 10, 17, 18, 19].

In 2004, Andrews *et al.* [1] studied the partitions with tagged (designated) summand which are defined as the partitions in which, accurately one part is tagged or designated out of the parts of the partitions having equal magnitude. If $Pd(n)$ represents the number of partitions with tagged summand then, the generating function of $Pd(n)$ is given by

$$\sum_{n\geq 0} Pd(n)q^n = \frac{f_6}{f_1 f_2 f_3}. \tag{1.3}$$

For example, there are 5 partitions of 3 with tagged summands with relevant partitions are

$$3', \quad 2' + 1', \quad 1' + 1 + 1, \quad 1 + 1' + 1, \quad 1 + 1 + 1'.$$

Chen et al. [7] and Xia [20] have proved several congruences modulo powers of 3 for $Pd(n)$ and found a few infinite families of congruence modulo powers of 3 for $Pd(n)$. Naika and Gireesh

[13] considered 3-regular partition with tagged summands, where none of the parts is a multiple of 3. The total number of 3-regular partitions of n with designated summand is represented by $Pd_3(n)$ and its generating function [13] is given by

$$\sum_{n\geq 0} Pd_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}. \tag{1.4}$$

They proved congruences modulo 4, 9, 12, 36, 48 and 144 for the partition function $Pd_3(n)$. A bi-partition (ν, ζ) of a non-negative integer n can be defined as a partition pair (ν, ζ) in which the sum of all the parts of ν and ζ is equal to n. Naika and Shivashankar [14] have discovered several congruences modulo 3 and powers of 2 for the partition function $BPd(n)$, which counts the total number of bi-partitions of n with tagged summands, where

$$\sum_{n\geq 0} BPd(n)q^n = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}. \tag{1.5}$$

Naika and Nayaka [15] proved congruences modulo for $BPd_3(n)$, which denotes the number of 3-regular bi-partitions of n with tagged summands, where

$$\sum_{n\geq 0} BPd_3(n)q^n = \frac{f_6^4 f_9^2}{f_1^2 f_2^2 f_{18}^2}. \tag{1.6}$$

They [15] established infinite families of congruences modulo 3, 4 and 6 of $BPd_3(n)$.

In recent time, Naika and Harishkumar [16] established infinite families of congruences modulo 3, 4, 8 and 9 for $BPt_t(n)$, which is the number of t-regular partitions triples of n with tagged summands. Its generating function is given by

$$\sum_{n\geq 0} BPt_t(n)q^n = \frac{f_6^3 f_t^3 f_{2t}^3 f_{3t}^3}{f_1^3 f_2^3 f_3^3 f_{6t}^3}. \tag{1.7}$$

Motivated from the above paper, in this paper, we study about the partition function $BPd_t(n)$, which counts the total number of t-regular bi-partitions of n with tagged summands. The generating function of $BPd_t(n)$ is given by

$$\sum_{n\geq 0} BPd_t(n)q^n = \frac{f_6^2 f_t^2 f_{2t}^2 f_{3t}^2}{f_1^2 f_2^2 f_3^2 f_{6t}^2}. \tag{1.8}$$

For example, $BPd_2(4) = 22$ and they are

$(3'+1', \emptyset)$, $(3', 1')$, $(1', 3')$, $(\emptyset, 3'+1')$, $(1'+1+1+1, \emptyset)$, $(1+1'+1+1, \emptyset)$, $(1+1+1'+1, \emptyset)$, $((1+1+1+1', \emptyset)$, $(1', 1'+1+1)$, $(1', 1+1'+1)$, $(1', 1+1+1')$, $(1'+1, 1'+1)$, $(1'+1, 1+1')$, $(1+1', 1'+1)$, $(1+1', 1+1')$, $(1'+1+1, 1')$, $(1+1'+1, 1')$ $(1+1+1', 1')$, $(\emptyset, 1'+1+1+1)$, $(\emptyset, 1+1'+1+1)$, $(\emptyset, 1+1+1'+1)$, $(\emptyset, 1+1+1+1')$.

In Section 3, 4 and 5, we prove infinite families of congruences modulo 3, 4, 6, 8 and 16 for $BPd_2(n)$, congruences modulo 16 for $BPd_4(n)$ and congruences modulo 8 and 16 for $BPd_6(n)$ respectively. To prove our results, we use some theta function and q-series identites which are listed in Section 2.

2 Preliminaries

Ramanujan's general theta function $f(\alpha, \beta)$ is defined by

$$f(\alpha, \beta) = \sum_{t=-\infty}^{\infty} \alpha^{t(t+1)/2}\beta^{t(t-1)/2}, \text{ where for complex numbers } \alpha \text{ and } \beta,\ |\alpha\beta| < 1. \tag{2.1}$$

Special cases of $f(\alpha,\beta)$ [5, p. 36, Entry 22 (ii)] is given by

$$\psi(q) := f(q,q^3) = \sum_{n\geq 0} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2{}^2}{f_1}. \tag{2.2}$$

Lemma 2.1. *[8, Theorem 2.1] If $\rho > 2$ be any prime, then*

$$\psi(q) = \sum_{s=0}^{(\rho-3)/2} q^{\frac{(s^2+s)}{2}} f\left(q^{\frac{(\rho^2+(2s+1)\rho)}{2}}, q^{\frac{(\rho^2-(2s+1)\rho)}{2}}\right) + q^{\frac{(\rho^2-1)}{8}}\psi(q^{\rho^2}). \tag{2.3}$$

Furthermore, $\frac{(s^2+s)}{2} \not\equiv \frac{(\rho^2-1)}{8} \pmod{\rho}$ *for* $0 \leq s \leq (\rho-3)/2$.

Lemma 2.2. *[8, Theorem 2.2]. If $\rho > 5$ be any prime, then*

$$f_1 = \sum_{\substack{j=-(\rho-1)/2 \\ j\neq(\pm\rho-1)/6}}^{(\rho-1)/2} (-1)^j q^{\frac{3j^2+j}{2}} f\left(-q^{\frac{3\rho^2+(6j+1)\rho}{2}}, -q^{\frac{3\rho^2-(6j+1)\rho}{2}}\right) + (-1)^{(\pm\rho-1)/6} q^{\frac{\rho^2-1}{24}} f_{\rho^2}, \tag{2.4}$$

where

$$\frac{\pm\rho-1}{6} = \begin{cases} \frac{(\rho-1)}{6}, & if\ \rho\equiv 1 \pmod{6} \\ \frac{(-\rho-1)}{6}, & if\ \rho\equiv -1 \pmod{6}. \end{cases}$$

Furthermore, if $\dfrac{-(\rho-1)}{2} \leq j \leq \dfrac{(\rho-1)}{2}$ *and* $j \neq \dfrac{(\pm\rho-1)}{6}$, *then*

$$\frac{3j^2+j}{2} \not\equiv \frac{\rho^2-1}{24} \pmod{\rho}.$$

Lemma 2.3. *[5, p. 303, Entry 17(v)] We have*

$$f_1 = f_{49}\left(\frac{A(q^7)}{E(q^7)} - q\frac{D(q^7)}{A(q^7)} - q^2 + q^5\frac{E(q^7)}{D(q^7)}\right), \tag{2.5}$$

where $D(q) = f(-q^3,-q^4), A(q) = f(-q^2,-q^5)$ *and* $E(q) = f(-q,-q^6)$.

Lemma 2.4. *[9] We have*

$$f_1 = f_{25}(G(q^5) - q - q^2G(q^5)^{-1}), \tag{2.6}$$

where

$$G(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}.$$

Lemma 2.5. *We have*

$$f_1^3 = \frac{f_6f_9^6}{f_3f_{18}^3} - 3qf_9^3 + 4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}, \tag{2.7}$$

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2} \tag{2.8}$$

and

$$\frac{f_1}{f_4} = \frac{f_6f_9f_{18}}{f_{12}^3} - q\frac{f_3f_{18}^4}{f_{12}^3f_9^2} - q^2\frac{f_6^2f_9f_{36}^3}{f_{12}^4f_{18}^2}. \tag{2.9}$$

For the proof of equation (2.7), see [5]. Equation (2.8) can be found in [11]. For the proof of (2.9), see Baruah and Ojah [4, Lemma 2.6].

Lemma 2.6. *We have*

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}, \tag{2.10}$$

$$\frac{f_1^2}{f_3^2}=\frac{f_2f_4^2f_{12}^4}{f_6^5f_8f_{24}}-2q\frac{f_2^2f_8f_{12}f_{24}}{f_4f_6^4}, \tag{2.11}$$

$$f_1^2=\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}, \tag{2.12}$$

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}, \tag{2.13}$$

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}} \tag{2.14}$$

and

$$\frac{1}{f_1f_3}=\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}. \tag{2.15}$$

For the proof of identity (2.10) and (2.11), see Xia and Yao [21]. Identity (2.13) was given by Hirschhorn [12, p. 40]. One can arrive at (2.12) by substituting q with $-q$ in (2.13). Identity (2.14) is the 2-dissection of $\varphi(q)^2$, see Hirschhorn[12, (1.10.1)]. Equation (2.15) was proved by Baruah and Ojha [3].

We conclude this section with the following congruences.

Lemma 2.7. *If δ be any prime, then*

$$f_\delta\equiv f_1{}^\delta \pmod{\delta}. \tag{2.16}$$

Using binomial theorem, we obtain (2.16).

Lemma 2.8. *[2, Lemma 1.4] If δ be any prime, then*

$$f_1^{\delta^2}\equiv f_\delta{}^\delta \pmod{\delta^2}. \tag{2.17}$$

3 Congruences modulo 3, 4, 8 and 16 for $BPd_2(n)$

Theorem 3.1. *Let ρ be any prime such that $(\frac{-4}{\rho})=-1$ and $1\le m\le\rho-1$, then for any $a\ge 0$, we have*

$$BPd_2(24n+7)\equiv 0 \pmod{16}, \tag{3.1}$$

$$\sum_{n\ge0}BPd_2(24\cdot\rho^{2a}n+15\cdot\rho^{2a})q^n\equiv 8\psi(q)\psi(q^4) \pmod{16} \tag{3.2}$$

and

$$BPd_2(24\cdot\rho^{2a+1}(\rho n+m)+15\cdot\rho^{2a+2})\equiv 0 \pmod{16}. \tag{3.3}$$

Proof. Putting $t=2$ in (1.8), we have

$$\sum_{n\ge0}BPd_2(n)q^n=\frac{f_6^4f_4^2}{f_1^2f_3^2f_{12}^2}=\frac{f_6^4f_4^2}{f_{12}^2}\left(\frac{1}{f_1^2f_3^2}\right). \tag{3.4}$$

Substituting (2.15) in (3.4) and then equating the coefficients of all the terms that contain q^{2n+1}, we obtain

$$\sum_{n\ge0}BPd_2(2n+1)q^n=2\frac{f_6^2f_2^6}{f_3^2f_1^6}. \tag{3.5}$$

With the aid of (2.17), (3.5) can be written as

$$\sum_{n\geq 0} BPd_2(2n+1)q^n \equiv 2f_2^2 f_6^2\Big(\frac{f_1^2}{f_3^2}\Big) \pmod{16}. \tag{3.6}$$

Utilising (2.11) in (3.6) and then extracting all the terms that involve q^{2n+1}, $n \geq 0$, we have

$$\sum_{n\geq 0} BPd_2(4n+3)q^n \equiv 12\frac{f_1^4 f_4 f_6 f_{12}}{f_2}\Big(\frac{1}{f_3^2}\Big) \pmod{16}. \tag{3.7}$$

Utilising (2.13) in (3.7), we get

$$\sum_{n\geq 0} BPd_2(4n+3)q^n \equiv 12\frac{f_1^4 f_4 f_6 f_{12}}{f_2}\Big(\frac{f_{24}^5}{f_6^5 f_{48}^2} + 2q^3\frac{f_{12}^2 f_{48}^2}{f_6^5 f_{24}}\Big) \pmod{16}. \tag{3.8}$$

Extracting all the terms that involve q^{2n+1}, $n \geq 0$, from (3.8), we obtain

$$\sum_{n\geq 0} BPd_2(8n+7)q^n \equiv 8q\Big(\frac{f_1 f_2 f_3 f_6^3 f_{24}^2}{f_3^5 f_{12}}\Big) \pmod{16}. \tag{3.9}$$

Utilising (2.16) in (3.9), we have

$$\sum_{n\geq 0} BPd_2(8n+7)q^n \equiv 8q f_3^2 f_{12}^3\Big(f_1 f_2\Big) \pmod{16}. \tag{3.10}$$

Utilising (2.8) in (3.10), we obtain

$$\sum_{n\geq 0} BPd_2(8n+7)q^n \equiv 8q f_3^2 f_{12}^3\Big(\frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}\Big) \pmod{16}. \tag{3.11}$$

Extracting all the terms that involve q^{3n} from (3.11), we obtain the desired result (3.1). Next, extracting all the terms that involve q^{3n+1} from (3.11) and with the aid of (2.2), we obtain

$$\sum_{n\geq 0} BPd_2(24n+15)q^n \equiv 8\psi(q)\psi(q^4) \pmod{16}. \tag{3.12}$$

The congruence (3.12) is the case $a = 0$ of (3.2). Assume that the result (3.2) exists for some integer $a \geq 0$. Utilising (2.3) in (3.2), we deduce that

$$\sum_{n\geq 0} BPd_2\Big(24\cdot\rho^{2a}n + 15\cdot\rho^{2a}\Big)q^n$$

$$\equiv 8\left(\sum_{k=0}^{(\rho-3)/2} q^{\frac{(k^2+k)}{2}} f\left(q^{\frac{(\rho^2+(2k+1)\rho)}{2}}, q^{\frac{(\rho^2-(2k+1)\rho)}{2}}\right) + q^{\frac{(\rho^2-1)}{8}}\psi(q^{\rho^2})\right)\times$$

$$\left(\sum_{j=0}^{(\rho-3)/2} q^{2(j^2+j)} f\left(q^{2(\rho^2+(2j+1)\rho)}, q^{2(\rho^2-(2j+1)\rho)}\right) + q^{(\rho^2-1)/2}\psi(q^{4\rho^2})\right) \pmod{16}. \tag{3.13}$$

Consider the congruence

$$\frac{(k^2+k)}{2} + 2(j^2+j) \equiv \frac{5(\rho^2-1)}{8} \pmod{\rho},$$

which is similar to

$$(2k+1)^2 + 4(2j+1)^2 \equiv 0 \pmod{\rho}. \tag{3.14}$$

For $\left(\frac{-4}{\rho}\right) = -1$, the only solution of congruence (3.14) is $k = j = \frac{\rho - 1}{2}$. Therefore, extracting all the terms that involve $q^{\rho n+5(\rho^2-1)/8}$ from (3.13), dividing throughout by $q^{5(\rho^2-1)/8}$ and then replacing q^ρ with q, we have

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot \rho^{2a+1} n + 15 \cdot \rho^{2a+2}\Big) q^n \equiv 8\psi(q^\rho)\psi(q^{4\rho}) \pmod{16}. \tag{3.15}$$

Extracting all the terms that involve $q^{\rho n}$ from (3.15) and replacing q^ρ with q, we have

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot \rho^{2(a+1)} n + 15 \cdot \rho^{2a+2}\Big) q^n \equiv 8\psi(q)\psi(q^4) \pmod{16}. \tag{3.16}$$

Thus, equation (3.16) is the case $a + 1$ of (3.2). Hence, we complete the proof of (3.2) by using induction method. Finally, extracting all the terms that involve $q^{\rho n+m}$, for $1 \leq m \leq \rho - 1$, from (3.15), we get the result (3.3). □

Theorem 3.2. *Let* $u_1 \in \{33, 57\}$, $u_2 \in \{51, 99\}$, $u_3 \in \{39, 63, 87, 111\}$ *and* $u_4 \in \{45, 69, 93, 117, 141, 165\}$. *Then for all integers* $\alpha, \beta, \gamma \geq 0$, *we have*

$$BPd_2(8n + 7) \equiv 0 \pmod 8, \tag{3.17}$$

$$BPd_2(24n + 19) \equiv 0 \pmod 8, \tag{3.18}$$

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} \cdot n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big) q^n \equiv 4f_1^3 \pmod 8, \tag{3.19}$$

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} \cdot n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\Big) q^n \equiv 4f_7^3 \pmod 8, \tag{3.20}$$

$$BPd_2\Big(24 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} \cdot n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big) \equiv \begin{cases} 2 \pmod 8, & if\ n\ is\ a\ pentagonal\ number \\ 0 \pmod 8, & otherwise, \end{cases} \tag{3.21}$$

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} \cdot n + 3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big) q^n \equiv 4f_3^3 \pmod 8, \tag{3.22}$$

$$BPd_2\Big(24 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} \cdot n + 17 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big) \equiv 0 \pmod 8, \tag{3.23}$$

$$BPd_2\Big(24 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} \cdot n + u_1 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big) \equiv 0 \pmod 8, \tag{3.24}$$

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} \cdot n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}\Big) q^n \equiv 4f_5^3 \pmod 8, \tag{3.25}$$

$$BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} \cdot n + u_2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big) \equiv 0 \pmod 8, \tag{3.26}$$

$$BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} \cdot n + u_3 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\Big) \equiv 0 \pmod 8, \tag{3.27}$$

$$BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} \cdot n + u_4 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}\Big) \equiv 0 \pmod 8. \tag{3.28}$$

Proof. With the aid of (2.17), (3.5) can be written as

$$\sum_{n\geq 0} BPd_2(2n + 1) q^n \equiv 2f_2^2 f_6^2 \Big(\frac{f_1^2}{f_3^2}\Big) \pmod 8. \tag{3.29}$$

Utilising (2.11) in (3.29) and then extracting all the terms that involve q^{2n+1}, $n \geq 0$, we have

$$\sum_{n\geq 0} BPd_2(4n+3)q^n \equiv 4f_2^3 f_{12} \pmod 8. \tag{3.30}$$

Extracting all the terms that involve q^{2n+1}, we obtain the desired congruence (3.17). Next, extracting all the terms that involve q^{2n}, we have

$$\sum_{n\geq 0} BPd_2(8n+3)q^n \equiv 4f_1^3 f_6 \pmod 8. \tag{3.31}$$

Utilising (2.7) in (3.31) and then extracting all the terms that involve q^{3n+2}, we get the desired result (3.18). Next, extracting all the terms that involve q^{3n} from resultant equation, we obtain

$$\sum_{n\geq 0} BPd_2(24n+3)q^n \equiv 4f_1^3 \pmod 8. \tag{3.32}$$

The result (3.32) is the case $\alpha = \beta = \gamma = 0$ of equation (3.19). Assume that the result (3.19) exists for any integer $\alpha \geq 0$ with $\beta = \gamma = 0$. Utilising (2.7) in (3.19) with $\beta = \gamma = 0$ and then extracting all the terms that involve q^{3n+1}, we obtain

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha+1}n + 3 \cdot 3^{2\alpha+2}\Big)q^n \equiv 4f_3^3 \pmod 8. \tag{3.33}$$

Extracting all the terms that involve q^{3n} from (3.33), we deduce that

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha+2}n + 3 \cdot 3^{2\alpha+2}\Big)q^n \equiv 4f_1^3 \pmod 8, \tag{3.34}$$

which shows, the congruence (3.19) exists for integer $\alpha + 1$ with $\beta = \gamma = 0$. Thus, by using the method of induction, (3.19) exist for all integer α. Suppose that the equation (3.19) exists for $\alpha, \beta \geq 0$ with $\gamma = 0$. Utilising (2.6) in (3.19) and then equating q^{5n+3} terms, we obtain

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta+1}n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta+2}\Big)q^n \equiv 4f_5^3 \pmod 8. \tag{3.35}$$

Extracting all the terms that involve q^{5n}, from (3.35), we obtain

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta+2}n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta+2}\Big)q^n \equiv 4f_1^3 \pmod 8. \tag{3.36}$$

Equation (3.36) shows that the result (3.19) exists for integers $\beta + 1$ with $\gamma = 0$. Thus, by induction method, the congruence (3.19) exists for all positive integers α, β with $\gamma = 0$. Assume that the result (3.19) exists for $\alpha, \beta, \gamma \geq 0$. Utilising (2.5) in (3.19) and then equating the q^{7n+6} terms, we obtain

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\Big)q^n \equiv 4f_7^3 \pmod 8, \tag{3.37}$$

that proves (3.20). Extracting all the terms that involve q^{7n} from (3.37), we obtain

$$\sum_{n\geq 0} BPd_2\Big(24 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}n + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\Big)q^n \equiv 4f_1^3 \pmod 8. \tag{3.38}$$

Equation (3.38) shows that (3.19) exists for all integers $\gamma + 1$. By using induction method, the result (3.19) exists for all non-negative integers α, β, γ.

Utilising (2.7) in (3.19) and then equating coefficients of q^{3n}, q^{3n+1} and q^{3n+2} terms, we arrive at (3.21), (3.22) and (3.23) respectively. Extracting all the terms that involve q^{3n+r}, where $r \in \{1, 2\}$ from (3.22), we get the result (3.24).

Utilising (2.6) in (3.19) and then extracting all the terms that involve q^{5n+3}, we achieve the result (3.25). Again, utilising (2.6) in (3.19) and then extracting all the terms that involve q^{5n+u}, where $u \in \{2, 4\}$, yields the result (3.26). Extracting all the terms that involve q^{5n+a}, where $a \in \{1, 2, 3, 4\}$ from (3.25), yields (3.27). Finally, extracting all the terms that involve q^{7n+c}, where $c \in \{1, 2, 3, 4, 5, 6\}$ from (3.20), yields (3.28). □

Theorem 3.3. *If ρ be any prime with $(\frac{-9}{\rho})=-1$ and $1\leq m\leq \rho-1$, then for any $a\geq 0$, we have*

$$BPd_2(4n+3)\equiv 0 \pmod 4, \tag{3.39}$$

$$BPd_2(12n+9)\equiv 0 \pmod 4, \tag{3.40}$$

$$\sum_{n\geq 0} BPd_2(12\cdot\rho^{2a}n+5\cdot\rho^{2a})q^n\equiv 2f_1\psi(q^3) \pmod 4 \tag{3.41}$$

and

$$BPd_2(12\cdot\rho^{2a+1}(\rho n+m)+5\cdot\rho^{2a+2})\equiv 0 \pmod 4. \tag{3.42}$$

Proof. Utilising (2.16) in (3.5), we have

$$\sum_{n\geq 0} BPd_2(2n+1)q^n\equiv 2f_2^3f_6 \pmod 4. \tag{3.43}$$

Extracting all the terms that involve q^{2n+1} from (3.43), we get the desired result (3.39). Again, extracting all the terms that involve q^{2n} from (3.43), we obtain

$$\sum_{n\geq 0} BPd_2(4n+1)q^n\equiv 2f_1^3f_3 \pmod 4. \tag{3.44}$$

Utilising (2.7) in (3.44) and then extracting all the terms that involve q^{3n+2}, we get the desired result (3.40). Next, extracting all the terms that involve q^{3n+1}, we obtain

$$\sum_{n\geq 0} BPd_2(12n+5)q^n\equiv 2f_1\psi(q^3) \pmod 4. \tag{3.45}$$

Equation (3.45) is the case $a=0$ of (3.41). Let us assume, the result (3.41) exists for any integer $a\geq 0$. Utilising (2.3) and (2.4) in (3.41), we get

$$\sum_{n\geq 0} BPd_2\Big(12\cdot\rho^{2a}n+5\cdot\rho^{2a}\Big)q^n$$

$$\equiv 2\Bigg(\sum_{\substack{j=-(\rho-1)/2\\ j\neq(\pm\rho-1)/6}}^{(\rho-1)/2} q^{\frac{(3j^2+j)}{2}} f\left(q^{\frac{(3\rho^2+(6j+1)\rho)}{2}}, q^{\frac{(3\rho^2-(6j+1)\rho)}{2}}\right)+q^{\frac{(\rho^2-1)}{24}}f_{\rho^2}\Bigg)\times$$

$$\left(\sum_{m=0}^{(\rho-3)/2} q^{\frac{3(m^2+m)}{2}} f\left(q^{\frac{3(\rho^2+(2m+1)\rho)}{2}}, q^{\frac{3(\rho^2-(2m+1)\rho)}{8}}\right)+q^{\frac{3(\rho^2-1)}{8}}\psi(q^{3\rho^2})\right) \pmod 4. \tag{3.46}$$

Consider the congruence

$$\frac{(3j^2+j)}{2}+3\frac{(m^2+m)}{2}\equiv\frac{10(\rho^2-1)}{24} \pmod\rho,$$

which is similar to

$$(6j+1)^2+9(2m+1)^2\equiv 0 \pmod\rho. \tag{3.47}$$

For $\left(\frac{-9}{\rho}\right)=-1$, the only possible solution of equation (3.47) is $j=\frac{(\pm\rho-1)}{6}$ and $m=\frac{(\rho-1)}{2}$. Therefore, extracting all the terms that involve $q^{\rho n+10(\rho^2-1)/24}$ terms from (3.46), dividing throughout by $q^{10(\rho^2-1)/24}$ and then replacing q^ρ with q, we get

$$\sum_{n\geq 0} BPd_2\Big(12\cdot\rho^{2a+1}n+5\cdot\rho^{2a+2}\Big)q^n\equiv 2f_\rho\psi(q^{3\rho}) \pmod 4. \tag{3.48}$$

Extracting all the terms that involve $q^{\rho n}$ from (3.48) and then replacing q^{ρ} with q, we obtain

$$\sum_{n\geq 0} BPd_2\Big(12\cdot\rho^{2(a+1)}n+5\cdot\rho^{2a+2}\Big)q^n \equiv 2f_1\psi(q^3) \pmod 4. \tag{3.49}$$

The congruence (3.49) is the case $a+1$ of (3.41). Thus, by induction method, we prove the congruence (3.41). Next, extracting all the terms that involve $q^{\rho n+m}$ for $1\leq m\leq \rho-1$ from (3.48), we get the desired result (3.42). □

Theorem 3.4. *For all $n,\alpha,\beta\geq 0$, we have*

$$\sum_{n\geq 0} BPd_2\Big(12\cdot 5^{2\alpha}\cdot 7^{2\beta}n+5^{2\alpha}\cdot 7^{2\beta}\Big)q^n \equiv 2f_1^2 \pmod 4, \tag{3.50}$$

$$\sum_{n\geq 0} BPd_2\Big(12\cdot 5^{2\alpha+1}\cdot 7^{2\beta}n+5^{2\alpha+2}\cdot 7^{2\beta}\Big)q^n \equiv 2f_5^2 \pmod 4, \tag{3.51}$$

$$BPd_2\Big(12\cdot 5^{2\alpha+1}\cdot 7^{2\beta}(5n+i)+5^{2\alpha+2}\cdot 7^{2\beta+2}\Big)\equiv 0 \pmod 4,\quad i=1,2,3,4, \tag{3.52}$$

$$\sum_{n\geq 0} BPd_2\Big(12\cdot 5^{2\alpha}\cdot 7^{2\beta+1}n+5^{2\alpha}\cdot 7^{2\beta+2}n\Big)q^n \equiv 2f_7^2 \pmod 4 \tag{3.53}$$

and

$$BPd_2\Big(12\cdot 5^{2\alpha+1}\cdot 7^{2\beta+1}(7n+j)+5^{2\alpha}\cdot 7^{2\beta+1}\Big)\equiv 0 \pmod 4,\quad j=1,2,3,4,5,6. \tag{3.54}$$

Proof. Utilising (2.7) in (3.44) and then extracting all the terms that involve q^{3n}, we obtain

$$\sum_{n\geq 0} BPd_2(12n+1)q^n \equiv 2f_1^2 \pmod 4. \tag{3.55}$$

Equation (3.55) is the case $\alpha=\beta=0$ of (3.50). Suppose that the congruence (3.50) exists for any integer $\alpha\geq 0$ with $\beta=0$. Utilising (2.6) in (3.50) with $\beta=0$ and then extracting all the terms that involve q^{5n+2}, we get

$$\sum_{n\geq 0} BPd_2\Big(12\cdot 5^{2\alpha+1}\cdot 7^{2\beta}n+5^{2\alpha+2}\cdot 7^{2\beta}\Big)q^n \equiv 2f_5^2 \pmod 4, \tag{3.56}$$

which proves (3.51). Extracting all the terms that involve q^{5n+i} with $i\in\{1,2,3,4\}$ from (3.56), we arrive at the result (3.52). Next, extracting all the terms that involve q^{5n} from (3.56), we deduce

$$\sum_{n\geq 0} BPd_2\Big(12\cdot 5^{2(\alpha+1)}\cdot 7^{2\beta}n+5^{2(\alpha+1)}\cdot 7^{2\beta}\Big)q^n \equiv 2f_1^2 \pmod 4. \tag{3.57}$$

Thus, the result (3.50) exists for integers $\alpha+1$ with $\beta=0$. Thus, by induction method, (3.19) exists for all integer α. Assume that the result (3.50) exist for $\alpha,\beta\geq 0$. Utilising (2.5) in (3.50) and then extracting all the terms that involve q^{7n+4}, we obtain

$$\sum_{n\geq 0} BPd_2\Big(12\cdot 5^{2\alpha}\cdot 7^{2\beta+1}n+5^{2\alpha}\cdot 7^{2\beta+2}\Big)q^n \equiv 2f_7^2 \pmod 4, \tag{3.58}$$

which proves (3.53). Extracting all the terms that involve q^{7n+j} with $j\in\{1,2,3,4,5,6\}$ from (3.58), we obtain the desired result (3.54). Next, extracting all the terms that involve q^{7n} from (3.58), we obtain

$$\sum_{n\geq 0} BPd_2\Big(12\cdot 5^{2\alpha}\cdot 7^{2(\beta+1)}n+5^{2\alpha}\cdot 7^{2\beta+2}\Big)q^n \equiv 2f_1^2 \pmod 4, \tag{3.59}$$

which shows that (3.50) exists for all integers $\beta+1$. By mathematical induction, (3.50) exists for all integers $\alpha,\beta\geq 0$. □

Theorem 3.5. *If $\rho \geq 5$ be any prime with $\left(\frac{-3}{\rho}\right) = -1$ and $1 \leq m \leq \rho - 1$, then for all integers $a \geq 0$, we have*

$$BPd_2\Big(6n+3\Big) \equiv 0 \pmod 3, \tag{3.60}$$

$$BPd_2\Big(12n+i\Big) \equiv 0 \pmod 3, \quad i = 5, 8, 11, \tag{3.61}$$

$$\sum_{n\geq 0} BPd_2\Big(12\cdot\rho^{2a}n + 2\cdot\rho^{2a}\Big)q^n \equiv 2f_1f_3 \pmod 3 \tag{3.62}$$

and

$$BPd_2\Big(12\cdot\rho^{2a+1}(\rho n+m) + 2\cdot\rho^{2a+2}\Big) \equiv 0 \pmod 3. \tag{3.63}$$

Proof. Utilising (2.16) in (3.4), we obtain

$$\sum_{n\geq 0} BPd_2(n)q^n \equiv \frac{f_6^4}{f_3^3 f_{12}}\left(\frac{f_1}{f_4}\right) \pmod 3. \tag{3.64}$$

Utilising (2.9) in (3.64), we obtain

$$\sum_{n\geq 0} BPd_2(n)q^n \equiv \frac{f_6^4}{f_3^3 f_{12}}\left(\frac{f_6f_9f_{18}}{f_{12}^3} - q\frac{f_3f_{18}^4}{f_{12}^3f_9^2} - q^2\frac{f_6^2f_9f_{36}^3}{f_{12}^4f_{18}^2}\right) \pmod 3. \tag{3.65}$$

Extracting all the terms that involve q^{3n} from (3.65), we obtain

$$\sum_{n\geq 0} BPd_2(3n)q^n \equiv \frac{f_2^5f_6}{f_4^4} \pmod 3. \tag{3.66}$$

Extracting all the terms that involve q^{2n+1} from (3.66), we get the desired result (3.60). Again, extracting all the terms that involve q^{3n+2} from (3.65) and utilising (2.16), we obtain

$$\sum_{n\geq 0} BPd_2(3n+2)q^n \equiv 2\frac{f_{12}^3}{f_4^5} \pmod 3. \tag{3.67}$$

Extracting all the terms that involve q^{4n+u}, with $u \in \{1, 2, 3\}$ from (3.67), we obtain the desired result (3.61). Next, extracting all the terms that involve q^{4n} from (3.67), we arrive at

$$\sum_{n\geq 0} BPd_2(12n+2) \equiv 2\frac{f_3^3}{f_1^5} \equiv 2f_1f_3 \pmod 3. \tag{3.68}$$

The equation (3.68) is the case $a = 0$ of (3.62). Suppose that the result (3.62) exists for some integer $a \geq 0$. Utilising (2.4) in (3.62), we obtain

$$\sum_{n\geq 0} BPd_2\Big(12\cdot\rho^{2a}n + 2\cdot\rho^{2a}\Big)q^n$$

$$\equiv 2\left(\sum_{\substack{s=-(\rho-1)/2\\ s\neq(\pm\rho-1)/6}}^{(\rho-1)/2} (-1)^s q^{\frac{(3s^2+s)}{2}} f\left(q^{\frac{(3\rho^2+(6s+1)\rho)}{2}}, q^{\frac{(3\rho^2-(6s+1)\rho)}{2}}\right) + (-1)^{\frac{(\pm\rho-1)}{6}} q^{\frac{(\rho^2-1)}{24}} f_{\rho^2}\right)\times$$

$$\left(\sum_{\substack{t=-(\rho-1)/2\\ t\neq(\pm\rho-1)/6}}^{(\rho-1)/2} (-1)^t q^{\frac{3(3t^2+t)}{2}} f\left(q^{\frac{3(3\rho^2+(6t+1)\rho)}{2}}, q^{\frac{3(3\rho^2-(6t+1)\rho)}{2}}\right) + (-1)^{\frac{(\pm\rho-1)}{6}} q^{\frac{(\rho^2-1)}{8}} f_{3\rho^2}\right) \pmod 3. \tag{3.69}$$

Consider the congruence

$$\frac{(3s^2+s)}{2}+3\frac{(3t^2+t)}{2}\equiv\frac{(\rho^2-1)}{6}\pmod{\rho},$$

which is analogous to

$$(6s+1)^2+3(6t+1)^2\equiv 0\pmod{\rho}. \tag{3.70}$$

For $\left(\frac{-3}{\rho}\right)=-1$, the only possible solution of the congruence (3.70) is $s=t=\frac{(\pm\rho-1)}{6}$. Therefore, equating the coefficients of $q^{\rho n+(\rho^2-1)/6}$ terms from (3.69), dividing throughout by $q^{(\rho^2-1)/6}$ and then replace q^{ρ} with q, we obtain

$$\sum_{n\geq 0}BPd_2(12\cdot\rho^{2a+1}n+2\cdot\rho^{2a+2})q^n\equiv 2f_{\rho}f_{3\rho}\pmod{3}. \tag{3.71}$$

Extracting all the terms that involve $q^{\rho n}$ from (3.71) and replacing q^{ρ} with q, we obtain

$$\sum_{n\geq 0}BPd_2(12\cdot\rho^{2(a+1)}n+2\cdot\rho^{2a+2})q^n\equiv 2f_1f_3\pmod{3}. \tag{3.72}$$

The congruence (3.72) is the case $a+1$ of (3.62). Thus, we complete the proof of congruence (3.62) by induction method. Next, extracting all the terms that involve $q^{\rho n+m}$ for $1\leq r\leq\rho-1$ from (3.71), we achieve the result (3.63). □

Theorem 3.6. *For all $n\geq 0$, we have*

$$BPd_2(6n+3)\equiv 0\pmod{6} \tag{3.73}$$

and

$$BPd_2(6n+5)\equiv 0\pmod{6}. \tag{3.74}$$

Proof. Utilising (2.16) in (3.5), we obtain

$$\sum_{n\geq 0}BPd_2(2n+1)q^n\equiv 2\frac{f_6^4}{f_3^4}\pmod{6}. \tag{3.75}$$

Extracting all the terms that involve q^{3n+u} with $u\in\{1,2\}$, from (3.75), we arrive at the result (3.73) and (3.74) respectively. □

4 Congruences modulo 16 for $BPd_4(n)$

Theorem 4.1. *If ρ be any prime with $\left(\frac{-6}{\rho}\right)=-1$ and $1\leq m\leq\rho-1$, then for all integers $a\geq 0$, we have*

$$BPd_4\Big(8n+3\Big)\equiv 0\pmod{16}, \tag{4.1}$$

$$BPd_4\Big(24n+23\Big)\equiv 0\pmod{16}, \tag{4.2}$$

$$\sum_{n\geq 0}BPd_4\Big(24\cdot\rho^{2a}n+7\cdot\rho^{2a}\Big)q^n\equiv 8f_1\psi(q^2)\pmod{16} \tag{4.3}$$

and

$$BPd_4\Big(24\cdot\rho^{2a+1}(\rho n+m)+7\cdot\rho^{2a+2}\Big)\equiv 0\pmod{16}. \tag{4.4}$$

Proof. Putting $t = 4$ in (1.8), we obtain

$$\sum_{n\geq 0} BPd_4(n)q^n = \frac{f_4^2 f_6^2 f_8^2 f_{12}^2}{f_2^2 f_{24}^2}\left(\frac{1}{f_1^2 f_3^2}\right). \tag{4.5}$$

Utilising (2.15) in (4.5) and then extracting all the terms that involve q^{2n+1} and utilising (2.17), we obtain

$$\sum_{n\geq 0} BPd_4(2n+1)q^n \equiv 2\frac{f_2^2 f_4^2 f_6^6}{f_{12}^2}\left(\frac{1}{f_3^4}\right) \pmod{16}. \tag{4.6}$$

Utilising (2.14) in (4.6) and then extracting all the terms that involve q^{2n+1} from (4.6), we arrive at

$$\sum_{n\geq 0} BPd_4(4n+3)q^n \equiv 8q f_2^3 f_6^6 \pmod{16}. \tag{4.7}$$

Extracting all the terms that involve q^{2n} from (4.7), we obtain the desired result (4.1). Next, extracting coefficients of q^{2n+1} terms from (4.7), we get

$$\sum_{n\geq 0} BPd_4(8n+7)q^n \equiv 8 f_1^3 f_3^6 \pmod{16}. \tag{4.8}$$

Utilising (2.7) in (4.8) and then extracting all the terms that involve q^{3n+2} terms, we achieve the desired congruence (4.2). Next, extracting all the terms that involve q^{3n} and using (2.2), we arrive at

$$\sum_{n\geq 0} BPd_4(24n+7)q^n \equiv 8 f_1 \psi(q^2) \pmod{16}. \tag{4.9}$$

The congruence (4.9) is the case $a = 0$ of (4.3). Utilising (2.4) in (4.3) and continuing like the proof of (3.41), we obtain

$$\sum_{n\geq 0} BPd_4\Big(24\cdot\rho^{2a+1}n + 7\cdot\rho^{2a+2}\Big)q^n \equiv 8 f_\rho \psi(q^{2\rho}) \pmod{16}. \tag{4.10}$$

Extracting all the terms that involve $q^{\rho n}$ from (4.10) and then replacing q^ρ with q, we arrive at

$$\sum_{n\geq 0} BPd_4\Big(24\cdot\rho^{2(a+1)}n + 7\cdot\rho^{2(a+1)}\Big)q^n \equiv 8 f_1 \psi(q^2) \pmod{16}. \tag{4.11}$$

The congruence (4.11) is $a + 1$ case of (4.3). Thus, the proof of (4.3) is complete.

Extracting all the terms that involve $q^{\rho n+m}$, for $1 \leq m \leq \rho - 1$, from (4.10), we arrive at (4.4). □

5 Congruences modulo 8 and 16 for $BPd_6(n)$

Theorem 5.1. *If $\rho \geq 5$ be any prime with $\left(\frac{-3}{\rho}\right) = -1$ and $1 \leq m \leq \rho - 1$, then for all integers $a \geq 0$, we have*

$$BPd_6\Big(6n+3\Big) \equiv 0 \pmod{16}, \tag{5.1}$$

$$BPd_6\Big(6n+5\Big) \equiv 0 \pmod{16}, \tag{5.2}$$

$$\sum_{n\geq 0} BPd_6\Big(24\cdot\rho^{2a}n + 19\cdot\rho^{2a}\Big)q^n \equiv 8 f_3 f_{16} \pmod{16} \tag{5.3}$$

and

$$BPd_6\Big(24\cdot\rho^{2a+1}(\rho n + m) + 19\cdot\rho^{2a+2}\Big) \equiv 0 \pmod{16}. \tag{5.4}$$

Proof. Putting $t = 6$ in (1.8), we arrive at

$$\sum_{n\geq 0} BPd_6(n)q^n = \frac{f_6^4 f_{12}^2 f_{18}^2}{f_2^2 f_{36}^2}\left(\frac{1}{f_1^2 f_3^2}\right). \tag{5.5}$$

Utilising (2.15) in (5.5) and then extracting all the terms that involve q^{2n+1} and utilising (2.17), we get

$$\sum_{n\geq 0} BPd_6(2n+1)q^n \equiv 2\frac{f_6^6 f_9^2}{f_3^2 f_{18}^2} \pmod{16}. \tag{5.6}$$

Extracting the coefficients of the terms q^{3n+a}, where $a \in \{1, 2\}$ from (5.6), we obtain the desired result (5.1) and (5.2) respectively. Next, extracting all the terms that involve q^{3n} from (5.6) and utilising (2.17), we arrive at

$$\sum_{n\geq 0} BPd_6(6n+1)q^n \equiv 2\frac{f_2^6}{f_6^2}\left(\frac{f_3^2}{f_1^2}\right) \pmod{16}. \tag{5.7}$$

Utilising (2.10) in (5.7) and then extracting all the terms that involve q^{2n+1}, we obtain

$$\sum_{n\geq 0} BPd_6(12n+7)q^n \equiv 4\frac{f_2 f_4 f_{12}}{f_6}\left(f_1^2\right) \pmod{16}. \tag{5.8}$$

Utilising (2.12) in (5.8), then extracting all the terms that involve q^{2n+1} and utilising (2.16), we obtain

$$\sum_{n\geq 0} BPd_6(24n+19)q^n \equiv 8 f_3 f_{16} \pmod{16}. \tag{5.9}$$

Equation (5.9) is the case $a = 0$ of the congruence (5.3). Utilising (2.4) in (5.3) and continuing like the proof of (3.62), we obtain

$$\sum_{n\geq 0} BPd_6\left(24\cdot\rho^{2\alpha+1}n + 19\cdot\rho^{2\alpha+2}\right)q^n \equiv f_{3\rho} f_{16\rho} \pmod{16}. \tag{5.10}$$

Extracting all the terms that involve $q^{\rho n}$ from (5.10) and then replacing q^ρ with q, we obtain

$$\sum_{n\geq 0} BPd_6\left(24\cdot\rho^{2(a+1)}n + 19\cdot\rho^{2(a+1)}\right)q^n \equiv f_3 f_{16} \pmod{16}, \tag{5.11}$$

Equation (5.11) is the case $a+1$ of (5.3). Thus, we complete the proof of result (5.3).

Next, extracting all the terms that involve $q^{\rho n+m}$, for $1 \leq m \leq \rho - 1$, from (5.10), we obtain the desired congruence (5.4). □

Theorem 5.2. *If $\rho \geq 5$ be any prime with $\left(\frac{-3}{\rho}\right) = -1$ and $1 \leq m \leq \rho - 1$, then for all integers $a \geq 0$, we have*

$$\sum_{n\geq 0} BPd_6\left(24\cdot\rho^{2a}n + 7\cdot\rho^{2a}\right)q^n \equiv 4 f_3 f_4 \pmod{8} \tag{5.12}$$

and

$$BPd_6\left(24\cdot\rho^{2a+1}(\rho n + m) + 7\cdot\rho^{2a+2}\right) \equiv 0 \pmod{8}. \tag{5.13}$$

Proof. Utilising (2.12) in (5.8), then extracting all the terms that involve q^{2n} and utilising (2.16), we obtain

$$\sum_{n\geq 0} BPd_6(24n+7)q^n \equiv 4 f_3 f_4 \pmod{8}. \tag{5.14}$$

which is the case $a = 0$ of (5.12). Using (2.4) in (5.12) and proceeding as in the proof of (3.62), we obtain

$$\sum_{n\geq 0} BPd_6\left(24\cdot\rho^{2a+1}n + 7\cdot\rho^{2a+2}\right)q^n \equiv 4 f_{3\rho} f_{4\rho} \pmod{8}. \tag{5.15}$$

Extracting all the terms that involve $q^{\rho n}$ from (5.15) and then replacing q^{ρ} with q, we deduce

$$\sum_{n\geq 0} BPd_6\Big(24\cdot\rho^{2(a+1)}n+7\cdot\rho^{2(a+1)}\Big)q^n \equiv 4f_3f_4 \pmod 8, \tag{5.16}$$

Equation (5.16) is the case $a+1$ of (5.12). Thus, we complete the proof of (5.12). Extracting all the terms that involve $q^{\rho n+m}$ with $1\leq m\leq \rho-1$ from (5.15), we obtain the congruence (5.13). □

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