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SOME NEW CONGRUENCE MODULO POWERS OF 2 FOR (j,k) - REGULAR OVERPARTITION

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Abstract Let $\overline{p}_{j,k}(n)$ denotes the number of (j,k)-regular overpartitions of a positive integer n such that none of the parts is congruent to j modulo k. Naika et al. (2021) proved infinite families of congruences modulo powers of 2 for $\overline{p}_{3,6}(n)$, $\overline{p}_{5,10}(n)$ and $\overline{p}_{9,18}(n)$. In this paper, we obtain infinite families of congruences modulo 4, 8, 16, 32 and 64 for $\overline{p}_{4,8}(n)$, modulo 4 and 8 for $\overline{p}_{6,12}(n)$, and modulo 16 for $\overline{p}_{8,16}(n)$. For example, we prove that for all integers $n \geq 0$ and $\alpha \geq 0$,

$$\overline{p}_{4,8}\left(5^{2\alpha+1}\cdot 7^{2\alpha}\left(16(5n+j)+14\right)\right) \equiv 0 \pmod{64}.$$

1 Introduction

A partition of a natural number n is a non-increasing sequence of natural number called parts, whose sum is equal to n. The number of partitions of a natural number n is usually denoted by $p(n)$ (with $p(0) = 1$) and the generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where, for any complex number a and q,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1. \tag{1.2}$$

Throughout the paper, we denote

$$f_k := (q^k;q^k)_\infty. \tag{1.3}$$

where k is any non-negative integer. An overpartition of a non-negative integer n is a partition of n in which the first occurrence of each parts may be overlined. For example, there are 14 overpartition of 4, namely

$$\overline{4},\quad 4,\quad \overline{3}+\overline{1},\quad 3+\overline{1},\quad \overline{3}+1,\quad 3+1,\quad \overline{2}+2,\quad 2+2,\quad \overline{2}+\overline{1}+1,\quad \overline{2}+1+1,\quad 2+\overline{1}+1,$$
$$2+1+1,\quad \overline{1}+1+1+1,\quad 1+1+1+1.$$

If $\overline{p}(n)$ denotes the number of overpartition of n, then the generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.4}$$

Again, for any positive integer ℓ, an ℓ-regular partition of n is a partition in which no part is divisible by ℓ. If $b_\ell(n)$ denotes the number of ℓ-regular partitions of n (with $b_\ell(0) = 1$), then the generating function of $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{1.5}$$

Naika et al. [5] defined a new overpartition functions known as (j,k)-regular overpartition. An overpartition of a non-negative integer n is said to be (j,k)-regular overpartition if none of the parts is congruent to $j \pmod{k}$. If $p_{j,k}(n)$ denotes the number of (j,k)-regular overpartition of n (with $p_{j,k}(0) = 1$), then its generating function is given by

$$\sum_{n=0}^{\infty} \overline{p}_{j,k}(n)q^n = \frac{(-q;q)_\infty (q^j;q^k)_\infty}{(q;q)_\infty(-q^j;q^k)_\infty}. \tag{1.6}$$

For example, the $(4,8)$-regular overpartition of 4 are given by

$$\overline{3}+\overline{1},\quad \overline{3}+1,\quad 3+\overline{1},\quad 3+1,\quad \overline{2}+2,\quad 2+2,\quad \overline{2}+\overline{1}+1,\quad \overline{2}+1+1,\quad 2+\overline{1}+1,$$
$$2+1+1,\quad \overline{1}+1+1+1,\quad 1+1+1+1.$$

Naika et al. [5] obtain many infinite families of congruences modulo powers of 2 for $\overline{p}_{3,6}(n)$, $\overline{p}_{5,10}(n)$ and $\overline{p}_{9,18}(n)$. In this paper, we prove many infinite families of congruences modulo 4, 8, 16, 32 and 64 for $\overline{p}_{4,8}(n)$, modulo 4 and 8 for $\overline{p}_{6,12}(n)$, and modulo 16 for $\overline{p}_{8,16}(n)$.

2 Some q-Series Identities

Lemma 2.1. *We have*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.1}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.2}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.3}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.4}$$

The equation (2.1) is the 2-dissection of $\phi(q)^2$ [4, (1.10.1)]. The equation (2.2) is the 2-dissection of $\phi(q)$ [4, (1.9.4)]. The equation (2.3) can be derived from the equations (2.2) by substituting $-q$ in place of q respectively. The equation (2.4) is obtained from [4, (22.1.14)]

Lemma 2.2. *We have*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{2.5}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{2.6}$$

The identity (2.5) is equivalent to the 3-dissection of $\phi(-q)$ (see [4, (Eq.14.3.2)]). The Identity (2.6) can be obtained from the first by replacing q with ωq and $\omega^2 q$ and then multiplying the two results, where ω is a primitive cube root of unity.

Lemma 2.3. *[3, Theorem 2.2] Let $r \geq 5$ be any prime, then we have*

$$f_1 = \sum_{\substack{k=\frac{-(r-1)}{2} \\ k\neq \frac{\pm r-1}{6}}}^{(r-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3r^2+(6k+1)r)/2}, -q^{(3r^2-(6k+1)r)/2}\right)$$
$$+(-1)^{(\pm r-1)/6} q^{(r^2-1)/24} f_{r^2}. \tag{2.7}$$

where

$$\frac{\pm r-1}{6}=\begin{cases}\dfrac{(r-1)}{6}, & if\ r\equiv 1 \pmod 6,\\ \dfrac{(-r-1)}{6}, & if\ r\equiv -1 \pmod 6.\end{cases}$$

Furthermore, if

$$\frac{-(r-1)}{2}\le k\le\frac{(r-1)}{2} \quad and \quad k\neq\frac{(\pm r-1)}{2},$$

then

$$\frac{(3k^2+k)}{2}\not\equiv\frac{(r^2-1)}{24} \pmod r.$$

Lemma 2.4. *[1, Lemma 2.3] For any prime $r\ge 3$, we have*

$$f_1^3=\sum_{\substack{k=0\\ k\neq(\pm r-1)/2}}^{(r-1)}(-1)^k q^{(k(k+1))/2}\sum_{n=0}^{\infty}(-1)^n(2rn+2k+1)q^{rn\cdot(rn+2k+1)/2}$$

$$+r(-1)^{(r-1)/2}q^{(r^2-1)/8}f_{r^2}^3. \tag{2.8}$$

Furthermore, if $k\neq\frac{(r-1)}{2}$, $\quad 0\le k\le r-1$, then

$$\frac{(k^2+k)}{2}\not\equiv\frac{(r^2-1)}{8} \pmod r.$$

Lemma 2.5. *[4, Eq.(8.1.1)] We have*

$$f_1=f_{25}(R(q^5)-q-q^2R(q^5)^{-1}), \tag{2.9}$$

where

$$R(q)=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}.$$

Lemma 2.6. *[2, p. 303, Entry 17(v)] We have*

$$f_1=f_{49}\left(\frac{E(q^7)}{C(q^7)}-q\frac{D(q^7)}{E(q^7)}-q^2+q^5\frac{C(q^7)}{D(q^7)}\right), \tag{2.10}$$

where $D(q)=f(-q^3,-q^4), E(q)=f(-q^2,-q^5)$ and $C(q)=f(-q,-q^6)$.

In addition to above $q-$series identities, we will be using following congruence properties which follows from binomial theorem: For any positive integer k and m,

$$f_k^{2m}\equiv f_{2k}^m \pmod 2, \tag{2.11}$$

$$f_k^{4m}\equiv f_{2k}^{2m} \pmod 4, \tag{2.12}$$

$$f_k^{8m}\equiv f_{2k}^{4m} \pmod 8. \tag{2.13}$$

3 Congruences for $\overline{p}_{4,8}(n)$

Theorem 3.1. *If* $s \in \{1,2,3,4,5,6\}$ *and* $t \in \{0,2,3,4\}$ *. Then for all integers* $n \geq 0$ *and* $\alpha \geq 0$*, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(5^{2\alpha} \cdot 7^{2\alpha}(16n+6)\right) q^n \equiv 32 f_1 f_8 \pmod{64}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(5^{2\alpha+1} \cdot 7^{2\alpha}(16n+14)\right) q^n \equiv 32 q f_5 f_{40} \pmod{64}, \tag{3.2}$$

$$\overline{p}_{4,8}\left(5^{2\alpha+1} \cdot 7^{2\alpha}(16(5n+t)+14)\right) \equiv 0 \pmod{64}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(5^{2\alpha} \cdot 7^{2\alpha+1}(16n+10)\right) q^n \equiv 32 q^2 f_7 f_{56} \pmod{64}, \tag{3.4}$$

$$\overline{p}_{4,8}\left(5^{2\alpha} \cdot 7^{2\alpha+1}(16(7n+s)+14)\right) \equiv 0 \pmod{64}. \tag{3.5}$$

Proof. Setting $j=4$ and $k=8$ in (1.6), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(n) q^n = \frac{(-q;q)_\infty (q^4;q^8)_\infty}{(q;q)_\infty (-q^4;q^8)_\infty}. \tag{3.6}$$

Applying elementary q-operation and employing (1.3), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(n) q^n = \frac{f_2 f_4^2 f_{16}}{f_1^2 f_8^3}. \tag{3.7}$$

Using (2.2) in (3.7), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(n) q^n = \frac{f_4^2 f_8^2}{f_2^4 f_{16}} + 2q \frac{f_4^4 f_{16}^3}{f_2^4 f_8^4}. \tag{3.8}$$

Extracting the even and odd powers of q from both sides of (3.8), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(2n) q^n = \frac{f_2^2 f_4^2}{f_1^4 f_8} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(2n+1) q^n = 2 \frac{f_2^4 f_8^3}{f_1^4 f_4^4}, \tag{3.10}$$

respectively. Employing (2.1) in (3.9), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(2n) q^n = \frac{f_4^{16}}{f_2^{12} f_8^5} + 4q \frac{f_4^4 f_8^3}{f_2^8}. \tag{3.11}$$

Extracting the even and odd powers of q from both sides of (3.11), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n) q^n = \frac{f_2^{16}}{f_1^{12} f_4^5} \tag{3.12}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n+2) q^n = 4 \frac{f_2^4 f_4^3}{f_1^8}, \tag{3.13}$$

respectively. Using (2.1) in (3.13), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n+2) q^n = 4 \frac{f_4^{31}}{f_2^{24} f_8^8} + 32q \frac{f_4^{19}}{f_2^{20}} + 64q^2 \frac{f_4^7 f_8^8}{f_2^{16}}. \tag{3.14}$$

Extracting coefficients of the terms involving q^{2n+1} from (3.14), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(8n+6)q^n = 32q\frac{f_2^{19}}{f_1^{20}}. \tag{3.15}$$

Employing (2.11) in (3.15), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(8n+6)q^n = 32f_2^9 \pmod{64}. \tag{3.16}$$

Extracting even powers of q from both sides of (3.16), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(16n+6)q^n = 32f_1^8 f_1 \pmod{64}. \tag{3.17}$$

Again, using (2.11) in (3.17), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(16n+6)q^n = 32f_8 f_1 \pmod{64}. \tag{3.18}$$

The equation (3.18) is the case $\alpha = \beta = 0$ of equation (3.1). Assume that the congruence (3.1) is true for any integer $\alpha \geq 0$ with $\beta = 0$. Utilising (2.9) in (3.1) with $\beta = 0$ and then extracting the coefficients of q^{5n+4}, we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(5^{2\alpha+1}(16n+14)\right) q^n \equiv 32q f_5 f_{40} \pmod{64}. \tag{3.19}$$

Extracting the coefficients of q^{5n+1} from both side of (3.19), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(5^{2(\alpha+1)}(16n+6)\right) q^n \equiv 32f_1 f_8 \pmod{64}, \tag{3.20}$$

which implies that (3.1) is true for $\alpha+1$ with $\beta = 0$. By principle of mathematical induction, (3.1) is true for all non negative integers $\alpha \geq 0$ with $\beta = 0$. Assume that the congruence (3.1) holds for $\alpha, \beta \geq 0$. Utilising (2.10) in (3.1) and then extracting the coefficients of q^{7n+4}, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(5^{2\alpha}\cdot 7^{2\alpha+1}(16n+10)\right) q^n \equiv 32q^2 f_7 f_{56} \pmod{64}, \tag{3.21}$$

which proves (3.4). Now extracting coefficients of the term q^{7n+2} from both sides of (3.21), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(5^{2\alpha}\cdot 7^{2(\alpha+1)}(16n+6)\right) q^n \equiv 32f_1 f_8 \pmod{64}, \tag{3.22}$$

which implies that (3.1) is true for all $\beta+1$. By principle of mathematical induction (3.1) is true for all positive integers α, β.

Using (2.9) in (3.1), then extracting coefficients of the term q^{5n+4}, we arrive at (3.2). Again utilising (2.9) in (3.2) , then extracting coefficients of the term q^{5n+t} for $t \in \{0,2,3,4\}$ from (3.2), we arrive at (3.3). Employing (2.10) in (3.21) and then extracting coefficients of the term q^{7n+s} for $s \in \{0,1,3,4,5,6\}$, we arrive at (3.5). □

Theorem 3.2. *If* $1 \leq j \leq r-1$, *then for all integers* $\alpha \geq 0$ *and* $n \geq 0$, *we have*

$$\overline{p}_{4,8}(16n+4c+2) \equiv 0 \pmod{32}; \quad c \in (1,2,3), \tag{3.23}$$

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(2\cdot r^{2\alpha}(8n+1)\right)q^n \equiv 4f_1^3 \pmod{32}, \tag{3.24}$$

$$\overline{p}_{4,8}\left(2\cdot r^{2\alpha+1}(8(rn+j)+r)\right) \equiv 0 \pmod{32}. \tag{3.25}$$

Proof. From (3.13), we note that

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}(4n+2)q^n \equiv \frac{4f_2^4f_4^3}{f_1^8} \pmod{32}. \tag{3.26}$$

Employing (2.13) in (3.26), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}(4n+2)q^n \equiv 4f_4^3 \pmod{32}. \tag{3.27}$$

Extracting the terms involving q^{4n+c} for $c\in\{1,2,3\}$ from (3.27), we arrive at (3.23). Again, extracting the terms involving q^{4n} from (3.27), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}(16n+2)q^n \equiv 4f_1^3 \pmod{32}. \tag{3.28}$$

Congruence (3.28) is $\alpha=0$ case of equation (3.24). Suppose that the congruence (3.24) is true for any integer $\alpha\geq 0$. Utilising (2.8) in (3.24), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\Big(2{\cdot}r^{2\alpha}\,(8n+1)\Big)q^n \equiv \sum_{\substack{k=0\\ k\neq(\pm r-1)/2}}^{(r-1)}(-1)^kq^{(k(k+1))/2}\sum_{n=0}^{\infty}(-1)^n(2rn{+}2k{+}1)q^{rn\cdot(rn+2k+1)/2}$$
$$+r(-1)^{(r-1)/2}q^{(r^2-1)/8}f_{r^2}^3 \pmod{32}. \tag{3.29}$$

Extracting the term involving $q^{rn+(r^2-1)/8}$ from both sides of (3.29), dividing throughout by $q^{(r^2-1)/8}$ and then replacing q^r by q, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\Big(2\cdot r^{2\alpha+1}\,(8n+r)\Big)q^n \equiv f_r^3 \pmod{32}. \tag{3.30}$$

Extracting the terms involving q^{rn} from (3.30) and replacing q^r by q, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\Big(2\cdot r^{2(\alpha+1)}\,(8n+1)\Big)q^n \equiv f_1^3 \pmod{32}, \tag{3.31}$$

which is the α + 1 case of (3.24). Thus, by the principle of mathematical induction, we arrive at (3.24). Extracting the coefficients of terms involving q^{rn+j} for $1\leq j\leq r-1$, from both sides of (3.30), we complete the proof of (3.25). □

Theorem 3.3. *Let $j\in\{0,2,3,4\}$ and $k\in\{0,1,3,4,5,6\}$. Then for all integers $\alpha\geq 0$ and $\beta\geq 0$, we have*

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\Big(8\cdot 5^{2\alpha}\cdot 7^{2\beta}(n)+3\cdot 5^{2\alpha}\cdot 7^{2\beta}\Big)q^n \equiv 8f_1^9 \pmod{16}, \tag{3.32}$$

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\Big(8\cdot 5^{2\alpha+1}\cdot 7^{2\beta}(n)+7\cdot 5^{2\alpha+1}\cdot 7^{2\beta}\Big)q^n \equiv 8qf_5^9 \pmod{16}, \tag{3.33}$$

$$\overline{p}_{4,8}\Big(8\cdot 5^{2\alpha+1}\cdot 7^{2\beta}(5n+j)+7\cdot 5^{2\alpha+1}\cdot 7^{2\beta}\Big) \equiv 0 \pmod{16}, \tag{3.34}$$

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\Big(8\cdot 5^{2\alpha}\cdot 7^{2\beta+1}(n)+5\cdot 5^{2\alpha}\cdot 7^{2\beta+1}\Big)q^n \equiv 8q^2f_7^9 \pmod{16}, \tag{3.35}$$

$$\overline{p}_{4,8}\Big(8\cdot 5^{2\alpha}\cdot 7^{2\beta+1}(7n+k)+5\cdot 5^{2\alpha}\cdot 7^{2\beta+1}\Big) \equiv 0 \pmod{16}. \tag{3.36}$$

Proof. From (3.10), we have

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(2n+1)q^n = 2\frac{f_2^4 f_8^3}{f_1^4 f_4^4}. \tag{3.37}$$

Using (2.1) in (3.37), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(2n+1)q^n = 2\frac{f_4^{10}}{f_2^{10} f_8} + 8q\frac{f_8^7}{f_2^6 f_4^2}. \tag{3.38}$$

Now extracting the odd powers of q from both sides of (3.38), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n+3)q^n = 8\frac{f_4^7}{f_1^6 f_2^2}. \tag{3.39}$$

Utilising (2.11) in (3.39), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n+3)q^n \equiv 8f_2^9 \pmod{16}. \tag{3.40}$$

Extracting the even powers of q from both sides of (3.40), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(8n+3)q^n \equiv 8f_1^9 \pmod{16}. \tag{3.41}$$

The equation (3.41) is the case $\alpha = \beta = 0$ of equation (3.32). Assume that the congruence (3.32) is true for any integer $\alpha \geq 0$ with $\beta = 0$. Utilising (2.9) in (3.32) with $\beta = 0$ and then extracting coefficients of the term q^{5n+4}, we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 5^{2\alpha+1}(n) + 7 \cdot 5^{2\alpha+1}\right)q^n \equiv 8qf_5^9 \pmod{16}. \tag{3.42}$$

Extracting coefficients of the term q^{5n+1} from both sides of (3.42), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 5^{2(\alpha+1)}(n) + 3 \cdot 5^{2(\alpha+1)}\right)q^n \equiv 8f_1^9 \pmod{16}, \tag{3.43}$$

which implies that (3.32) is true for $\alpha + 1$ with $\beta = 0$. By principle of mathematical induction, (3.32) is true for all positive integers $\alpha \geq 0$ with $\beta = 0$. Assume that the congruence (3.32) holds for $\alpha, \beta \geq 0$. Utilising (2.10) in (3.32), then extracting the terms involving q^{7n+4}, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 5^{2\alpha} \cdot 7^{2\beta+1}(n) + 5 \cdot 5^{2\alpha} \cdot 7^{2\beta+1}\right)q^n \equiv 8q^2 f_7^9 \pmod{16}, \tag{3.44}$$

which proves (3.35). Now extracting coefficients of the term q^{7n+2} from both sides of (3.44), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 5^{2\alpha} \cdot 7^{2(\beta+1)}(n) + 3 \cdot 5^{2\alpha} \cdot 7^{2(\beta+1)}\right)q^n \equiv 8f_1^9 \pmod{16}, \tag{3.45}$$

which implies that (3.32) is true for all $\beta + 1$. By principle of mathematical induction (3.32) is true for all positive integers α, β.

Utilising (2.9) in (3.32), then extracting coefficients of the term q^{5n+4}, we arrive at (3.33). Again employing (2.9) in (3.33) and extracting coefficients of the term q^{5n+j} for $j \in \{0, 2, 3, 4\}$ from (3.33), we arrive at (3.34). Employing (2.10) in (3.44) and then extracting coefficients of the term q^{7n+k} for $k \in \{0, 1, 3, 4, 5, 6\}$, we arrive at (3.36). □

Theorem 3.4. *Let $r \geq 5$ be a prime with $\left(\frac{-2}{r}\right) = -1$ and $1 \leq t \leq (r-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(2 \cdot r^{2\alpha}(8n+1)\right) q^n \equiv f_1 f_2 \pmod{8}, \tag{3.46}$$

$$\overline{p}_{4,8}\left(2 \cdot r^{2(\alpha+1)}(8(rn+t)+1)\right) \equiv 0 \pmod{8}. \tag{3.47}$$

Proof. From (3.13), we note that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n+2)q^n = 4\frac{f_2^4 f_4^3}{f_1^8}. \tag{3.48}$$

Using (2.12) in (3.59), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n+2)q^n \equiv 4f_4^3 \pmod{8}. \tag{3.49}$$

Extracting coefficients of the term q^{4n} from (3.49), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(16n+2)q^n \equiv 4f_1 f_2 \pmod{8}. \tag{3.50}$$

Congruence (3.50) is the $\alpha = 0$ case of (3.46). Assume that congruence (3.46) is true for all $\alpha \geq 0$. Utilising (2.7) in (3.46), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(2 \cdot r^{2\alpha}(8n+1)\right) q^n \equiv &\Bigg\{ \sum_{\substack{k=-(r-1)/2 \\ k\neq(\pm r-1)/6}}^{k=(r-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3r^2+(6k+1)r)/2}, -q^{(3r^2-(6k+1)r)/2}\right) \\ &+(-1)^{(\pm r-1)/6} q^{(r^2-1)/24} f_{r^2} \Bigg\} \\ &\times \Bigg\{ \sum_{\substack{m=-(r-1)/2 \\ m\neq(\pm r-1)/6}}^{m=(r-1)/2} (-1)^m q^{(3m^2+m)} f\left(-q^{(3r^2+(6m+1)r)}, -q^{(3r^2-(6m+1)r)}\right) \\ &+(-1)^{(\pm r-1)/6} q^{(r^2-1)/12} f_{2r^2} \Bigg\} \pmod{8}. \end{aligned} \tag{3.51}$$

Now consider the congruence

$$(3m^2+m) + \frac{(3k^2+k)}{2} \equiv \frac{(r^2-1)}{8} \pmod{r},$$

which is equal to

$$2(6m+1)^2 + (6k+1)^2 \equiv 0 \pmod{r}.$$

For $\left(\frac{-2}{r}\right) = -1$, the above congruence has only solution $k = m = \left(\frac{\pm r-1}{6}\right)$. Therefore, extracting the terms involving $q^{rn+(r^2-1)/8}$ from (3.51), dividing throughout by $q^{(r^2-1)/8}$ and then replacing q^r by q, we obtain

$$\sum_{n=0}^{\infty} \overline{r}_{4,8}\left(2 \cdot r^{2\alpha+1}(8n+r)\right) q^n \equiv f_r f_{2r} \pmod{8}. \tag{3.52}$$

Extracting coefficients of the term q^{rn} from both sides of (3.52) and substituting q^r by q, we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(2 \cdot r^{2(\alpha+1)}(8n+1)\right) q^n \equiv f_1 f_2 \pmod{8}, \tag{3.53}$$

which is the $\alpha+1$ case of (3.46). Therefore, by mathematical induction, we arrive at (3.46). Equating the coefficients of terms q^{rn+t} for $1 \le t \le r-1$, from both sides of (3.52), we complete the proof of (3.47). □

Theorem 3.5. *For all integers $n \ge 0$, $\alpha \ge 0$ and $k \in \{1,2\}$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 3^{2\alpha} n\right) q^n \equiv \frac{f_1^2}{f_2} \pmod 4, \tag{3.54}$$

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 3^{2\alpha+1} n\right) q^n \equiv \frac{f_3^2}{f_6} \pmod 4, \tag{3.55}$$

$$\overline{p}_{4,8}\left(8 \cdot 3^{2\alpha+1} n + 16 \cdot 3^{2\alpha}\right) \equiv 0 \pmod 4, \tag{3.56}$$

$$\overline{p}_{4,8}\left(8 \cdot 3^{2\alpha}(3n+k)\right) \equiv 0 \pmod 4. \tag{3.57}$$

Proof. From (3.10), we have

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n) q^n = \frac{f_2^{16}}{f_1^{12} f_4^5}. \tag{3.58}$$

Using (2.12) in (3.59), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(4n) q^n = \frac{f_2^2}{f_4} \pmod 4. \tag{3.59}$$

Extracting the even powers of q from both sides of (3.59), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(8n) q^n = \frac{f_1^2}{f_2} \pmod 4. \tag{3.60}$$

Congruence (3.60) is the $\alpha = 0$ case of (3.54). Suppose that (3.54) is true for all $\alpha \ge 0$. Using (2.6) in (3.54), then extracting coefficients of the term q^{3n} from both sides, dividing throughout by q^3 and substituting q^3 by q, we find that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 3^{2\alpha+1} n\right) q^n \equiv \frac{f_3^2}{f_6} \pmod 4, \tag{3.61}$$

which proves (3.55). Again extracting coefficients of the term q^{3n} from both sides and replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(8 \cdot 3^{2(\alpha+1)} n\right) q^n \equiv \frac{f_1^2}{f_2} \pmod 4, \tag{3.62}$$

which is the $\alpha+1$ case of (3.54). Hence, by mathematical induction, we arrive at (3.54). Now using (2.6) in (3.54), then extracting coefficients of the term q^{3n+2} from both sides, dividing throughout by q^2 and substituting q^3 by q we prove (3.56). Again, extracting coefficients of the term q^{3n+k} for $k \in \{1,2\}$ from both sides of (3.61) and replacing q^3 by q we prove (3.57). □

Theorem 3.6. *For all integers $n \ge 0$ and $\alpha \ge 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24 \cdot 2^{2\alpha} n + 8 \cdot 2^{2\alpha}) q^n \equiv 2\frac{f_3^3}{f_1} \pmod 4, \tag{3.63}$$

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24 \cdot 2^{2\alpha+1} n + 8 \cdot 2^{2(\alpha+1)}) q^n \equiv 2\frac{f_6^3}{f_2} \pmod 4, \tag{3.64}$$

$$\overline{p}_{4,8}(24 \cdot 2^{2(\alpha+1)} n + 20 \cdot 2^{2(\alpha+1)})) q^n \equiv 0 \pmod 4. \tag{3.65}$$

Proof. Employing (2.6) in (3.60), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(8n)q^n = \left(\frac{f_9^2}{f_{18}} + 2q\frac{f_3 f_{18}^2}{f_6 f_9}\right) \pmod 4. \tag{3.66}$$

Extracting coefficients of the term q^{3n+1} from both sides of (3.66), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24n+8)q^n = 2\frac{f_1 f_6^2}{f_2 f_3} \pmod 4. \tag{3.67}$$

Using (2.11) in (3.67), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24n+8)q^n = 2\frac{f_3^3}{f_1} \pmod 4. \tag{3.68}$$

Congruence (3.68) is the $\alpha = 0$ case of (3.63). Assume that (3.63) is true for all $\alpha \geq 0$. Using (2.4) in (3.63), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24 \cdot 2^{2\alpha} n + 8 \cdot 2^{2\alpha})q^n \equiv 2\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod 4. \tag{3.69}$$

Then extracting coefficients of the term q^{2n+1} from both sides, dividing throughout by q and substituting q^2 by q, we find that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24 \cdot 2^{2\alpha+1} n + 8 \cdot 2^{2\alpha+2})q^n \equiv 2\frac{f_6^3}{f_2} \pmod 4, \tag{3.70}$$

which proves (3.64). Again extracting coefficients of the term q^{2n} from both sides, dividing throughout by q and substituting q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24 \cdot 2^{2(\alpha+1)} n + 8 \cdot 2^{2(\alpha+1)})q^n \equiv 2\frac{f_3^3}{f_1} \pmod 4, \tag{3.71}$$

which is the α + 1 case of (3.63). Hence, by mathematical induction, we arrive at (3.63). Then extracting the even powers of q from both sides of (3.70), dividing throughout by q and substituting q^2 by q, we prove (3.65). □

Theorem 3.7. *If* $t \in \{1, 2, 3, 4, 5, 6, 7\}$ *and* $1 \leq k \leq r-1$, *then for all integers* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$\overline{p}_{4,8}\left(48(8n+t)+8\right)q^n = 0 \pmod 4, \tag{3.72}$$

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}\left(384 \cdot r^{2\alpha} n + 8(2r^{2\alpha} - 1)\right) q^n \equiv f_1 \pmod 4, \tag{3.73}$$

$$\overline{p}_{4,8}\left(384 \cdot r^{2\alpha+1}(rn+k) + 8(2r^{2\alpha} - 1)\right) q^n \equiv 0 \pmod 4. \tag{3.74}$$

Proof. Utilising (2.4) in (3.68), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(24n+8)q^n = 2\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod 4. \tag{3.75}$$

Extracting coefficients of the term q^{2n} from both sides of (3.75), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{4,8}(48n+8)q^n = 2\frac{f_4^3 f_6^2}{f_2^2 f_{12}} \pmod 4. \tag{3.76}$$

Employing (2.11) in (3.76), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}(48n+8)q^n = 2f_8 \pmod 4. \tag{3.77}$$

Extracting coefficients of the term q^{8n+t} for $t \in \{1,2,3,4,5,6,7\}$ from both sides of (3.77), we arrive at (3.72). Again extracting coefficients of the term q^{8n} from both sides of (3.77), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}(384n+8)q^n = 2f_1 \pmod 4, \tag{3.78}$$

which is the $\alpha = 0$ case of (3.73). Assume (3.73) is true for any $\alpha \geq 0$. Employing (2.7) in (3.78), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\left(384\cdot r^{2\alpha}n + 8(2r^{2\alpha}-1)\right)q^n$$

$$\equiv \Bigg\{ \sum_{\substack{k=-(r-1)/2 \\ k\neq(\pm r-1)/6}}^{(r-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3r^2+(6k+1)r)/2}, -q^{(3r^2-(6k+1)r)/2}\right)$$

$$+(-1)^{(\pm r-1)/6} q^{(r^2-1)/24} f_{r^2}\Bigg\} \pmod 4. \tag{3.79}$$

Extracting coefficients of the term $q^{rn+(r^2-1)/24}$ from both sides of (3.79), dividing by $q^{(r^2-1)/24}$ and then substituting q^r by q, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\left(384\cdot r^{2\alpha+1}(n) + 8(2r^{2(\alpha+1)}-1)\right)q^n \equiv f_r \pmod 4. \tag{3.80}$$

Extracting coefficients of the term q^{rn} from both sides of (3.80) and substituing q^r by q, we find that

$$\sum_{n=0}^{\infty}\overline{p}_{4,8}\left(384\cdot r^{2(\alpha+1)}n + 8(2r^{2(\alpha+1)}-1)\right)q^n \equiv f_1 \pmod 4. \tag{3.81}$$

which is the α + 1 case of (3.73). Therefore, by mathematical induction, the proof of (3.73) is complete. Extracting coefficients of the term q^{rn+k}, for $1 \leq k \leq r-1$, from both sides of (3.80), we arrive at (3.74). □

4 Congruences for $\overline{p}_{6,12}(n)$

Theorem 4.1. *For all integers $n \geq 0$, $\alpha \geq 0$ and $1 \leq t \leq (r-1)$, we have*

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}\left(24\cdot r^{2\alpha}n + r^{2\alpha})\right)q^n \equiv f_1 \pmod 4, \tag{4.1}$$

$$\overline{p}_{6,12}\left(r^{2\alpha+1}(24(rn+t)+r)\right) \equiv 0 \pmod 4. \tag{4.2}$$

Proof. Setting $j = 6$ and $k = 12$ in (1.6), we note that

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(n)q^n = \frac{(-q;q)_\infty(q^6;q^{12})_\infty}{(q;q)_\infty(-q^6;q^{12})_\infty}.$$

Applying elementary q-operation and using (1.3), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(n)q^n = \frac{f_2 f_6^2 f_{24}}{f_1^2 f_{12}^3}. \tag{4.3}$$

Utilising (2.5) in (4.3), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(n)q^n=\frac{f_6^2f_{24}}{f_{12}^3}\left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right). \tag{4.4}$$

Extracting coefficients of the term q^{3n+1} from both sides of (4.4), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(3n+1)q^n=2\frac{f_2^5f_3^3f_8}{f_1^7f_4^3}. \tag{4.5}$$

Using (2.11) in (4.5), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(3n+1)q^n\equiv 2\frac{f_3^3}{f_1} \pmod 4. \tag{4.6}$$

Utilising (2.4) in (4.6), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(3n+1)q^n\equiv 2\frac{f_4^3f_6^2}{f_2^2f_{12}}+2q\frac{f_{12}^3}{f_4} \pmod 4. \tag{4.7}$$

Extracting the even powers of q from both sides of (4.7) and using (2.11), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(6n+1)q^n\equiv 2f_2^2 \pmod 4. \tag{4.8}$$

Again, extracting the even powers of q from both sides of (4.8) and using (2.11), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(12n+1)q^n\equiv 2f_2 \pmod 4. \tag{4.9}$$

Again, extracting even powers of q from both sides of (4.9), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(24n+1)q^n\equiv 2f_1 \pmod 4, \tag{4.10}$$

which is the $\alpha=0$ case of (4.1). Suppose (4.1) is true for any $\alpha\geq 0$. Employing (2.7) in (4.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}&\overline{p}_{6,12}\left(r^{2\alpha}(24n+1)\right)q^n\\ &\equiv\Bigg\{\sum_{\substack{k=-(r-1)/2\\ k\neq(\pm r-1)/6}}^{(r-1)/2}(-1)^kq^{(3k^2+k)/2}f\left(-q^{(3r^2+(6k+1)r)/2},-q^{(3r^2-(6k+1)r)/2}\right)\\ &\quad+(-1)^{(\pm r-1)/6}q^{(r^2-1)/24}f_{r^2}\Bigg\} \pmod 4.\end{aligned} \tag{4.11}$$

Extracting coefficients of the term $q^{rn+(r^2-1)/24}$ from both sides of (4.11), dividing by $q^{(r^2-1)/24}$ and then substituting q^r by q, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}\left(r^{2\alpha+1}(24n+r)\right)q^n\equiv f_r \pmod 4. \tag{4.12}$$

Extracting coefficients of the term q^{rn} from both sides of (4.12) and substituting q^r by q, we find that

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}\left(r^{2(\alpha+1)}(24n+1)\right)q^n\equiv f_1 \pmod 4, \tag{4.13}$$

which is the $\alpha+1$ case of (4.1). Hence, by mathematical induction, the proof of (4.1) is complete. Extracting coefficients of the term q^{rn+t} for $1\leq t\leq r-1$, from both sides of (4.12), we arrive at (4.2). □

Theorem 4.2. *For all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}\left(3 \cdot 2^{2\alpha} n + 2^{2\alpha}\right) q^n \equiv 2\frac{f_3^3}{f_1} \pmod 4, \tag{4.14}$$

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}\left(3 \cdot 2^{2\alpha+1} n + 2^{2(\alpha+1)}\right) q^n \equiv 2\frac{f_6^3}{f_2} \pmod 4, \tag{4.15}$$

$$\overline{p}_{6,12}\left(3 \cdot 2^{2(\alpha+1)})n + 5 \cdot 2^{2(\alpha+1)}\right) \equiv 0 \pmod 4. \tag{4.16}$$

Proof. From (4.6), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}(3n+1) q^n \equiv 2\frac{f_3^3}{f_1} \pmod 4. \tag{4.17}$$

The remaining part of the proof is similar to proofs of the identities (3.63)-(3.65).

□

Theorem 4.3. *For all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}\left(2^{2\alpha+1}(3n+1)\right) q^n \equiv 4\frac{f_3^3}{f_1} \pmod 8, \tag{4.18}$$

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}\left(2^{2\alpha+2}(3n+2)\right) q^n \equiv 4\frac{f_6^3}{f_2} \pmod 8, \tag{4.19}$$

$$\overline{p}_{6,12}\left(2^{2\alpha+2}(6n+5)\right) \equiv 0 \pmod 8. \tag{4.20}$$

Proof. Extracting the terms involving q^{3n+2} from (4.4), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}(3n+2) q^n = 4\frac{f_2^4 f_6^3 f_8}{f_1^6 f_4^3}. \tag{4.21}$$

Utilising (2.11) in (4.21), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}(3n+2) q^n \equiv 4\frac{f_6^3}{f_2} \pmod 8. \tag{4.22}$$

Extracting the even powers of q from both sides of (4.22), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}(6n+2) q^n \equiv 4\frac{f_3^3}{f_1} \pmod 8. \tag{4.23}$$

The remaining part of the proof is similar to proofs of the identities (3.63)-(3.65). □

Theorem 4.4. *For all integers $n \geq 0$, $\alpha \geq 0$ and $1 \leq j \leq (r-1)$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}(2 \cdot r^{2\alpha}(24n+1)) q^n \equiv 4f_1 \pmod 8, \tag{4.24}$$

$$\overline{p}_{6,12}(2 \cdot r^{2\alpha+1}(24(rn+j)+1)) \equiv 0 \pmod 8. \tag{4.25}$$

Proof. Extracting coefficients of the term q^{3n+2} from both sides of (4.4), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{6,12}(3n+2) q^n = 4\frac{f_2^4 f_6^3 f_8}{f_1^6 f_4^3}. \tag{4.26}$$

Utilising (2.12) in (4.26), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(3n+2)q^n \equiv 4\frac{f_6^3}{f_2} \pmod{8}. \tag{4.27}$$

Extracting coefficients of the term q^{2n} from both sides of (4.27), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(6n+2)q^n \equiv 4\frac{f_3^3}{f_1} \pmod{8}. \tag{4.28}$$

Employing (2.4) in (4.28), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(6n+2)q^n \equiv 4\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod{8}. \tag{4.29}$$

Extracting the even powers of q from both sides of (4.29) and using (2.11), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(12n+2)q^n \equiv 4f_4 \pmod{8}. \tag{4.30}$$

Again, extracting coefficients of the term q^{4n} from both sides of (4.30), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{6,12}(48n+2)q^n \equiv 4f_1 \pmod{8}. \tag{4.31}$$

The remaining part of the proof is similar to the proof of the identities (4.1)-(4.2). □

5 Congruences for $\overline{p}_{8,16}(n)$

Theorem 5.1. *Let $j \in \{0,2,3,4\}$ and $k \in \{0,1,3,4,5,6\}$. Then for all integers $\alpha \geq 0$ and $\beta \geq 0$, we have*

$$\sum_{n=0}^{\infty}\overline{p}_{8,,16}\Big(8\cdot 5^{2\alpha}\cdot 7^{2\beta}(n)+3\cdot 5^{2\alpha}\cdot 7^{2\beta}\Big)q^n \equiv 8f_1^9 \pmod{16}, \tag{5.1}$$

$$\sum_{n=0}^{\infty}\overline{p}_{8,16}\Big(8\cdot 5^{2\alpha+1}\cdot 7^{2\beta}(n)+7\cdot 5^{2\alpha+1}\cdot 7^{2\beta}\Big)q^n \equiv 8qf_5^9 \pmod{16}, \tag{5.2}$$

$$\overline{p}_{8,16}\Big(8\cdot 5^{2\alpha+1}\cdot 7^{2\beta}(5n+j)+7\cdot 5^{2\alpha+1}\cdot 7^{2\beta}\Big) \equiv 0 \pmod{16}, \tag{5.3}$$

$$\sum_{n=0}^{\infty}\overline{p}_{8,16}\Big(8\cdot 5^{2\alpha}\cdot 7^{2\beta+1}(n)+5\cdot 5^{2\alpha}\cdot 7^{2\beta+1}\Big)q^n \equiv 8q^2 f_7^9 \pmod{16}, \tag{5.4}$$

$$\overline{p}_{8,16}\Big(8\cdot 5^{2\alpha}\cdot 7^{2\beta+1}(7n+k)+5\cdot 5^{2\alpha}\cdot 7^{2\beta+1}\Big) \equiv 0 \pmod{16}. \tag{5.5}$$

Proof. Setting $j=8$ and $k=16$ in (1.6), we note that

$$\sum_{n=0}^{\infty}\overline{p}_{8,16}(n)q^n = \frac{(-q;q)_\infty(q^8;q^{16})_\infty}{(q;q)_\infty(-q^8;q^{16})_\infty}.$$

Applying elementary q-operation and utilising (1.3), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_{8,16}(n)q^n = \frac{f_2 f_8^2 f_{32}}{f_1^2 f_{16}^3}. \tag{5.6}$$

Using (2.2) in (5.6), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{8,16}(n)q^n = \frac{f_8^7 f_{32}}{f_2^4 f_{16}^5} + 2q\frac{f_4^2 f_8 f_{32}}{f_2^4 f_{16}}. \tag{5.7}$$

Extracting the odd powers of q from both sides of (5.7), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{8,16}(2n+1)q^n = 2\frac{f_2^2 f_4 f_{16}}{f_1^4 f_8}. \tag{5.8}$$

Utilising (2.1) in (5.8), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{8,16}(2n+1)q^n = 2\frac{f_2^2 f_4 f_{16}}{f_8}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right). \tag{5.9}$$

Extracting the odd powers of q from both sides of (5.9), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{8,16}(4n+3)q^n = 8\frac{f_2^3 f_4^3 f_8}{f_1^8}. \tag{5.10}$$

Utilising (2.11) in (5.10), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{8,16}(4n+3)q^n = 8f_2^9 \pmod{16}. \tag{5.11}$$

Extracting the even powers of q from both sides of (5.11), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{8,16}(8n+3)q^n = 8f_1^9 \pmod{16}. \tag{5.12}$$

The remaining part of the proof is similar to proofs of the identities (3.32)-(3.36). □

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