Palestine Journal of Mathematics

Vol. 12(Special Issue III) (2023) , 73–78

ON MINIMAL RING EXTENSIONS

Atul Gaur and Rahul Kumar

MSC 2020 Classifications: Primary 13B99; Secondary 13B25, 13F30.

Keywords and phrases: Minimal ring extension, von Neumann regular ring, valuation ring, flat epimorphism.

The authors thank Prof. Alok K. Maloo for providing Example 3.2 and for the stimulating discussions. The first author was supported by the MATRICS grant from DST-SERB, No. MTR/2018/000707. The second author was supported by the Research Initiation Grant Scheme from Birla Institute of Technology and Science Pilani, Pilani, India.

Abstract Let R be a commutative ring with identity. The ring $R \times R$ can be viewed as an extension of R via the diagonal map $\Delta : R \hookrightarrow R \times R$, given by $\Delta(r) = (r, r)$ for all $r \in R$. It is shown that, for any $a, b \in R$, the extension $\Delta(R)[(a, b)] \subset R \times R$ is a minimal ring extension if and only if the ideal $< a - b >$ is a maximal ideal of R. A complete classification of maximal subrings of $R(+)R$ is also given. The minimal ring extension of a von Neumann regular ring R is either a von Neumann regular ring or the idealization $R(+)R/\mathfrak{m}$ where $\mathfrak{m} \in \text{Max}(R)$.

1 Introduction

All rings considered below are commutative with nonzero identity; all ring extensions, ring homomorphisms, and algebra homomorphisms are unital. For any ring R, let $\text{tq}(R)$ denotes the total quotient ring of R and $\text{Max}(R)$ denotes the set of all maximal ideals of R. By an overring of R, we mean any subring of $\text{tq}(R)$ which contains R. For any ring extension $R \subseteq S$, the conductor $(R : S) := \{s \in S \mid sS \subseteq R\}$. By a local ring, we mean a ring with a unique maximal ideal.

An injective ring homomorphism f that is not an isomorphism is called a minimal ring homomorphism if any factorization $f = g \circ h$ entails that one of the ring homomorphisms g, h is an isomorphism, see [9]. Let R be any proper subring of a ring T. Then T is called a minimal ring extension of R or equivalently, R is a maximal subring of T if the inclusion map $R \hookrightarrow T$ is a minimal ring homomorphism, that is, if there is no ring S such that $R \subset S \subset T$ where $\subset$ denotes proper inclusion. By a minimal overring of R, we mean any overring of R which is a minimal ring extension of R. Note that if $R \subset T$ is a minimal ring extension, then either $R \subset T$ is an integral ring extension or $R \hookrightarrow T$ is a flat epimorphism, see [9, Théorème 2.2].

If R is a ring, then R can be viewed as a subring of $R \times R$ via the diagonal map, that is, via the canonical injective ring homomorphism, $\Delta : R \hookrightarrow R \times R$, given by $\Delta(r) = (r, r)$ for all $r \in R$. It was shown in [5, Lemma 2.1] that $\Delta(R)[(r, s)] = R \times R$ for $r, s \in R$ if and only if $r - s \in U(R)$, where $U(R)$ denote the set of units of R. Dobbs [5, Proposition 2.2] also proved that $\Delta(R) \subset R \times R$ is a minimal ring extension if and only if R is a field. In Theorem 2.3, we show that, for any $r, s \in R$, $\Delta(R)[(r, s)] \subset R \times R$ is a minimal ring extension if and only if the ideal $< r - s >$ is a maximal ideal of R.

If R is a domain but not a field, then minimal ring extensions of R are the R-algebras that are isomorphic to one of the following three types of rings: a minimal overring of R; an idealization $R\,(+)\,R/\mathfrak{m}$ where $\mathfrak{m} \in \text{Max}(R)$; a direct product $R \times R/\mathfrak{m}$ where $\mathfrak{m} \in \text{Max}(R)$, see [6, Theorem 2.7]. This result is generalized by assuming that $\text{tq}(R)$ is a von Neumann regular ring and $\text{Max}(R) \cap \text{Min}(R) = \phi$, see [7, Corollary 2.5]. Dobbs and Shapiro also classified the integral minimal ring extensions of R, see [7, Proposition 2.12]. In Propositions 2.1 and 2.2, we classify the minimal ring extension of a von Neumann regular ring, and thereby settled the open problem posed by Dobbs in [8, p. 35].

Recall [10, cf. Nagata, 1962, p.2] that if R is a ring and E is an R-module, then the idealization $R(+)E$ is the ring defined as follows: Its additive structure is that of the abelian group $R \oplus E$, and its multiplication is defined by $(r_1, e_1)\,(r_2, e_2) := (r_1r_2, r_1e_2 + r_2e_1)$ for all $r_1, r_2 \in R$

and $e_1, e_2 \in E$. It will be convenient to view R as a subring of $R(+)E$ via the canonical injective ring homomorphism that sends r to $(r, 0)$. Note that every ring has a minimal ring extension, see [5]. However, $\mathbb{Z}$ has no maximal subring, that is, maximal subrings need not always exist. In Corollary 2.6, we show that for any ring R, the ring $R(+)R$ has maximal subrings. In Proposition 2.5, we prove that $R(+)Rb$ is a maximal subring of $R(+)R$ if and only if Rb is a maximal ideal of R.

Let $R \subset T$ be a minimal ring extension. By [9, Théorème 2.2(i)] and [9, Lemme 1.3], there exists a unique maximal ideal J of R such that $R_J \hookrightarrow T_J := T_{R\setminus J}$ is not an isomorphism; moreover, $R_J \hookrightarrow T_J$ is then a minimal ring extension, and $R_P \hookrightarrow T_P$ is an isomorphism for all $P \in \text{Spec}(R) \setminus \{J\}$. The maximal ideal J appearing in the above statement is called the crucial maximal ideal [4, Definition 2.9].

The Proposition 2.11 of [4] states that if $R \subset T$ is a minimal ring extension, then the crucial maximal ideal is the only maximal ideal which contains $(R : T)$. In [4, Corollary 2.14], the author states that if $R \subset T$ is a minimal ring extension and T is an integral domain, then $(R : T)$ = 0 if and only if R is a field and T is a field extension of prime degree over R, or R is a valuation ring of altitude one and T is its quotient field. We give an example which shows the above mentioned proposition and corollary are not true.

2 Maximal subrings of certain commutative rings

The problem of classifying the minimal ring extensions of a von Neumann regular ring was posed by Dobbs in [8]. In our first result, we present a complete classification of minimal ring extensions of a von Neumann regular ring.

Proposition 2.1. *Let $R \subset T$ be a minimal ring extension where R is a von Neumann regular ring. Then either T is a von Neumann regular ring or $T \cong R(+)R/\mathfrak{m}$ (as R-algebra) for some maximal ideal $\mathfrak{m}$ of R.*

Proof. Since R is von Neumann regular, R is reduced. First assume that T is not reduced. Then by [7, Proposition 2.3], $T \cong R(+)R/\mathfrak{m}$ (as R-algebra) for some maximal ideal $\mathfrak{m}$ of R. Now, assume that T is a reduced ring. Then T is a von Neumann regular ring, by the proof of [2, Proposition 3.20]. □

The next result further characterizes the minimal ring extensions of a von Neumann regular ring.

Proposition 2.2. *Let R be a von Neumann regular ring. Then T is a minimal ring extension of R if and only if there exists a maximal ideal $\mathfrak{m}$ of R such that one of the following three conditions holds:*

(i) $\mathfrak{m}$ is a maximal ideal of T and $T/\mathfrak{m}$ is a minimal field extension of $R/\mathfrak{m}$;

(ii) There exists $q \in T \setminus R$ such that $T = R[q]$, $q^2 - q \in \mathfrak{m}$, and $\mathfrak{m}q \subseteq R$;

(iii) There exists $q \in T \setminus R$ such that $T = R[q]$, $q^2 \in R$, $q^3 \in R$, and $\mathfrak{m}q \subseteq R$.

If any of the above three conditions holds, then $\mathfrak{m}$ is uniquely determined as $(R : T)$. Also (i)-(iii) are mutually exclusive.

Proof. Note that by the proof of [2, Proposition 3.20], any minimal ring extension of R is an integral extension of R. Now, the result follows by [7, Proposition 2.12]. □

In [7, Theorem 2.4], a characterization of minimal ring extension of a reduced ring R such that the total quotient ring of R, is a von Neumann regular ring, is given. However, till now we do not know any minimal ring extension of a non-reduced ring R other than $R(+)R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R. In the next theorem, we have shown that $R \times R$ is a minimal ring extension of its subring which may not be reduced.

Theorem 2.3. *For any ring R, let $\Delta : R \hookrightarrow R \times R$ be the diagonal map, given by $\Delta(r) = (r, r)$ for all $r \in R$. Then for any $a, b \in R$, $\Delta(R)[(a, b)] \subset R \times R$ is a minimal ring extension if and only if the ideal $< a - b >$ is a maximal ideal of R.*

Proof. First, we claim that

$$\Delta(R)[(a, b)] = \{(c, d) \in R \times R \mid c - d \in < a - b >\}. \tag{2.1}$$

Let $(c, d) \in R \times R$ such that $c - d \in < a - b >$. Then $c - d = (a - b)t$ for some $t \in R$. As

$$(c, d) = (c - ta, c - ta) + (t, t)(a, b),$$

we conclude that $(c, d) \in \Delta(R)[(a, b)]$. Now, assume that $(e, f) \in \Delta(R)[(a, b)]$. So,

$$(e, f) = (a_0, a_0) + (a_1, a_1)(a, b) + (a_2, a_2)(a, b)^2 + \cdots + (a_n, a_n)(a, b)^n,$$

where $(a_i, a_i) \in \Delta(R)$ for all i. This gives,

$$e = a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n, \tag{2.2}$$

$$f = a_0 + a_1 b + a_2 b^2 + \cdots + a_n b^n. \tag{2.3}$$

On subtracting (2.3) from (2.2), we have

$$e - f = a_1(a - b) + a_2(a^2 - b^2) + \cdots + a_n(a^n - b^n).$$

This gives $e - f \in < a - b >$. So, the claim holds. Now, suppose that $< a - b >$ is a maximal ideal of R. We assert that $\Delta(R)[(a, b)] \subset R \times R$. If possible, suppose $\Delta(R)[(a, b)] = R \times R$. Then $(1, 0) \in \Delta(R)[(a, b)]$. Therefore, by (2.1), we have $1 \in < a - b >$, which is a contradiction. Therefore, $\Delta(R)[(a, b)] \neq R \times R$. Now, to show that $\Delta(R)[(a, b)] \subset R \times R$ is a minimal ring extension, enough to show that $(\Delta(R)[(a, b)])[(e, f)] = R \times R$ for any $(e, f) \in (R \times R) \setminus \Delta(R)[(a, b)]$.

Note that $e - f \notin < a - b >$, by (2.1). Therefore, $< a - b > + < e - f > = R$ and hence

$$1 = (a - b)t_1 + (e - f)t_2 \text{ for some } t_1, t_2 \in R.$$

This gives,

$$(1, 0) = ((a - b)t_1, 0) + ((e - f)t_2, 0).$$

Now, by (2.1), we have

$$((a - b)t_1, 0) \in \Delta(R)[(a, b)] \subseteq (\Delta(R)[(a, b)])[(e, f)]$$

and

$$((e - f)t_2, 0) \in \Delta(R)[(e, f)] \subseteq (\Delta(R)[(a, b)])[(e, f)].$$

Thus, $(1, 0) \in (\Delta(R)[(a, b)])[(e, f)]$. Similarly, $(0, 1) \in (\Delta(R)[(a, b)])[(e, f)]$ and hence the claim holds.

Conversely, suppose that $\Delta(R)[(a, b)] \subset R \times R$ is a minimal ring extension. First we assert that $< a - b >$ is a proper ideal of R. If possible, suppose that $1 \in < a - b >$. Then $(1, 0), (0, 1) \in \Delta(R)[(a, b)]$ by (2.1). It follows that $\Delta(R)[(a, b)] = R \times R$, a contradiction. Thus, $< a - b >$ is a proper ideal of R. Now, let I be any ideal of R properly containing the ideal $< a - b >$. Choose $e \in I \setminus < a - b >$. Then by (2.1), $(e, 0) \notin \Delta(R)[(a, b)]$. By minimality, we conclude that $(\Delta(R)[(a, b)])\, [(e, 0)] = R \times R$. Thus,

$$(1, 0) = (a_0, b_0) + (a_1, b_1)(e, 0) + (a_2, b_2)(e, 0)^2 + \cdots + (a_n, b_n)(e, 0)^n,$$

where $(a_i, b_i) \in \Delta(R)[(a, b)]$ for all i.

This gives,

$$1 = a_0 + a_1 e + \cdots + a_n e^n \text{ and } b_0 = 0.$$

Now, by (2.1), $a_0 - b_0 \in < a - b > \subset I$. As $a_i e \in I$ for all i, we must have $1 \in I$. Therefore, $< a - b >$ is a maximal ideal of R. □

Remark 2.4. Note that [5, Proposition 2.2] is a particular case of Theorem 2.3 with $a = b$.

Note that a maximal subring of a ring R may not exists. For example, the ring of integers $\mathbb{Z}$ does not admit any maximal subring. However, $R(+)R$ always admits a maximal subring as we have in the next result. In fact, in the next proposition, we present a complete classification of maximal subrings of $R(+)R$.

Proposition 2.5. *For any ring R, let $R \hookrightarrow R(+)R$ be the canonical injective ring homomorphism, given by $r \mapsto (r, 0)$ for all $r \in R$. Then for any $a, b \in R$, $R[(a, b)] \subset R(+)R$ is a minimal ring extension if and only if the ideal $< b >$ is a maximal ideal of R.*

Proof. Note that $R[(a, b)] = R(+) < b >$, by [5, Lemma 2.3]. First suppose that $R[(a, b)] \subset R(+)R$ is a minimal ring extension. Thus, $< b >$ is a proper ideal of R. Let I be any ideal of R properly containing $< b >$. Then we have $R(+) < b > \subset R(+)I$. It follows that $R(+)I = R(+)R$ and so $I = R$. Therefore, $< b >$ is a maximal ideal of R.

Conversely, assume that $< b >$ is a maximal ideal of R. Thus, $R[(a, b)] \subset R(+)R$ as $R[(a, b)] = R(+) < b >$. Let T be a subring of $R(+)R$ containing $R[(a, b)]$ properly. Then by [5, Remark 2.9], $T = R(+)I$ for some ideal I of R. It follows that $< b > \subset I$ and so $I = R$. Therefore, $R[(a, b)] \subset R(+)R$ is a minimal ring extension. □

The following corollaries can be deduced immediately from the above proposition.

Corollary 2.6. *Let R be any ring and M be a maximal ideal of R. Then $R(+)M$ is a maximal subring of $R(+)R$. In particular, $R(+)R$ has maximal subrings for any ring R.*

Corollary 2.7. *Let R be a ring. Then R is a maximal subring (upto isomorphism) of $R(+)R$ if and only if R is a field.*

We end this section with the following remark.

Remark 2.8. In [1, Corollary 2.8], Azarang proved that every finitely generated algebra over a commutative ring has a maximal subring. The result does not seem to be correct as by [3, Example 3.19], there are rings with no maximal subring but any such ring is a finitely generated algebra over itself.

3 Correction to some known results

We assume throughout that J denote the crucial maximal ideal of minimal ring extension $R \subset T$ unless otherwise stated. For completeness, we first list the results which we are going to discuss in this section.

(1) [4, Proposition 2.11] Let $R \subset T$ be a minimal ring extension. Then $(R : T) \in \text{Spec}(R)$ and J is the only maximal ideal in R which contains $(R : T)$. Moreover, if no maximal ideal in T lies over J, then the following statement holds: $(R : T) \subset J$, $T_J = R_{(R:T)}$ is local, $(R_J : T_J) = (R : T)R_J$ is the maximal ideal in T_J, height$(J/(R : T)) = 1$, and $(R : T)T \in \text{Max}(T)$.

(2) [4, Corollary 2.14] If $R \subset T$ is a minimal ring extension and T is an integral domain, then $(R : T) = 0$ if and only if R is a field and T is a field extension of prime degree over R, or R is a valuation ring of altitude one and T is its quotient field.

(3) [11, Proposition 3.2(3)] Let $f : R \hookrightarrow T$ be a minimal ring homomorphism. If $f : R \hookrightarrow T$ is a flat epimorphism, then $R/(R : T)$ is a one-dimensional local domain, $(R : T) \in \text{Max}(T)$ and $T_J = R_{(R:T)}$.

(4) [11, Proposition 3.5] Let $R \hookrightarrow T$ be an injective ring homomorphism. Then $R \hookrightarrow T$ is minimal and a flat epimorphism if and only if $R/(R : T)$ is a one-dimensional valuation ring and $T/(R : T)$ is its quotient field.

We now present a counter example to show that (1) is not fully correct. More precisely, we show that J may not be the only maximal ideal containing $(R : T)$ and $(R : T)T$ may not belong to $\text{Max}(T)$. In fact, there may be infinitely many maximal ideals containing $(R : T)$. The example also proves that (2) is completely incorrect. On page 310 of [12], the authors mentioned that the assumption of R to be local in above results (3) and (4) is missing due to printing mistake. Our next example shows that why this extra assumption is needed in above results (3) and (4).

Example 3.1. Let $R = \mathbb{Z}$, $T = \mathbb{Z}[1/2]$. We assert that $R \subset T$ is a minimal ring extension. Suppose there is a ring S such that $R \subset S \subseteq T$. Choose $f(1/2) = \sum_{i=0}^{n} \alpha_i (1/2)^i \in S \setminus R$. Then $f(1/2) = m/2^k$ for some $k \in \mathbb{N}$ and $m \in R$. Thus, $m/2 = 2^{k-1}(m/2^k) \in S$, which gives $1/2 \in S$. Therefore, T is a minimal ring extension of R. Note that $(R : T) = 0$, as for every $\alpha \in R$, there exists $n \in \mathbb{N}$ such that $\alpha/2^n$ is not an integer. Now crucial maximal ideal J of the extension $R \subset T$ is $2\mathbb{Z}$ as $R_J \hookrightarrow T_J$ is not an isomorphism and $R_P \hookrightarrow T_P$ is an isomorphism for all $P \in \text{Spec}(R) \setminus \{J\}$. This counters (1) as every maximal ideal of R contains $(R : T)$. Also $0 = (R : T)T \notin \text{Max}(T)$. As R is not a field and neither R is a valuation ring nor T is its quotient field, this counters (2) completely. Now, observe that R is integrally closed in T. So, Ferrand's dichotomy [9, Théorème 2.2] gives that the inclusion map $f : R \hookrightarrow T$ is a flat epimorphism. This shows that the assumption of R to be local is needed in (3) and (4).

Though the above example shows that there may be infinitely many maximal ideals in R containing $(R : T)$ and $(R : T)T$ may not belong to $\text{Max}(T)$, however, the remaining statement of [4, Proposition 2.11] is correct, which is as follows: Let $R \subset T$ be a minimal ring extension and J be the crucial maximal ideal. Then $(R : T) \in \text{Spec}(R)$. Moreover, if no maximal ideal in T lies over J, then $(R : T) \subset J$, $T_J = R_{(R:T)}$ is local, $(R_J : T_J) = (R : T)R_J$ is the maximal ideal in T_J, and $\text{height}(J/(R : T)) = 1$.

We give one more example to counter (2). More precisely, the next example shows that if $R \subset T$ is a minimal ring extension and T is an integral domain with $(R : T) = 0$, then degree of T over R may not be prime.

Example 3.2. Let $n \geq 4$. Then there exist field extension K of $\mathbb{Q}$ such that $Gal_{\mathbb{Q}}(K) = S_n$. In fact, choose $f(X) \in \mathbb{Q}[X]$ irreducible of degree 4 such that $|Gal_{\mathbb{Q}}(K)| = 24$. Let α be a root of $f(X)$. Then $dim_{\mathbb{Q}}\mathbb{Q}(\alpha) = 4$ and $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ does not have any intermediate ring.

We now present correct and modified versions of above discussed results (2), (3), and (4) which are easy consequences of [9, Proposition 3.3].

(1) If $R \subset T$ is a minimal ring extension and T is an integral domain, then $(R : T) = 0$ if and only if R is a field and T is a minimal field extension of R, or R_J is a valuation ring of altitude one and T_J is its quotient field.

(2) Let $f : R \hookrightarrow T$ be a minimal ring homomorphism. If $f : R \hookrightarrow T$ is a flat epimorphism, then $R_J/(R_J : T_J)$ is a one-dimensional local domain and $T_J = R_{(R:T)}$.

(3) Let $R \hookrightarrow T$ be a minimal ring homomorphism. Then $R \hookrightarrow T$ is a flat epimorphism if and only if $R_J/(R_J : T_J)$ is a one-dimensional valuation ring and $T_J/(R_J : T_J)$ is its quotient field.

Remark 3.3. There is an error in the proof of [7, Theorem 3.7]. Note that R is not local in [7, Theorem 3.7] but the proof of [7, Theorem 3.7] is citing [11, Proposition 3.5] which is true for local rings only. The error in the proof arises because the authors used [11, Proposition 3.5] to prove that $(R/P)_{M/P}$ is a valuation domain in (1) $\Rightarrow$ (3). But as we have seen earlier, [11, Proposition 3.5] is valid for local rings only. Thus, the proof of [7, Theorem 3.7] is not correct. Note that in [7, Theorem 3.7], we have $(R/P)_{M/P} \cong R_M/PR_M$ where $P = (R : T)$ and M is the crucial maximal ideal of the minimal ring extension $R \subset T$. Thus, by the above corrected version, we have $(R/P)_{M/P}$ is a valuation domain and hence [7, Theorem 3.7] holds.

References

[1] A. Azarang, On maximal subrings, *Far East J. Math. Sci. (FJMS)* **32**, 107–118 (2009).

[2] A. Azarang and O. A. S. Karamzadeh, On maximal subrings of commutative rings, *Algebra Colloq.* **19(Spec 1)**, 1125–1138 (2012).

[3] A. Azarang and O. A. S. Karamzadeh, Most commutative rings have maximal subrings, *Algebra Colloq.* **19(Spec 1)**, 1139–1154 (2012).

[4] L. I. Dechéne, *Adjacent extensions of rings*, PhD dissertation, University of California, Riverside (1978).

[5] D. E. Dobbs, Every commutative ring has a minimal ring extension, *Comm. Algebra* **34**, 3875–3881 (2006).

[6] D. E. Dobbs and J. Shapiro, A classification of the minimal ring extensions of an integral domain, *J. Algebra* **305**, 185–193 (2006).

[7] D. E. Dobbs and J. Shapiro, A classification of the minimal ring extensions of certain commutative rings, *J. Algebra* **308**, 800–821 (2007).

[8] D. E. Dobbs, *Recent progress on minimal ring extensions and related concepts*, Commutative Rings: New Research, Nova Science, New York, pp. 13–38 (2009).

[9] D. Ferrand and J.-P. Olivier, Homomorphismes minimaux d'anneaux, *J. Algebra* **16**, 461–471 (1970).

[10] M. Nagata, *Local rings*, Interscience, New York (1962).

[11] G. Picavet and M. Picavet-L' Hermitte, *About minimal morphisms*, in: Multiplicative Ideal Theory: A Tribute to the Work of Robert Gilmer, Springer-Verlag, New York, pp. 369–386 (2006).

[12] G. Picavet and M. Picavet-L' Hermitte, *Quasi-Prüfer extensions of rings*, in: Fontana M., Frisch S., Glaz S., Tartarone F., Zanardo P. (eds) Rings, Polynomials, and Modules, Springer, Cham, pp. 307–336 (2017).

Author information

Atul Gaur, Department of Mathematics, University of Delhi, Delhi 110007, India.
E-mail: gaursatul@gmail.com

Rahul Kumar, Department of Mathematics, Birla Institute of Technology and Science Pilani, Pilani 333031, Rajasthan, India.
E-mail: rahulkmr977@gmail.com