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ON STRONGLY r-PRECIOUS RINGS

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Abstract. In this article the study of strongly r- precious ring is initiated. An element $r \in R$ is strongly r-precious if r is the sum of an idempotent element, a regular element, a nilpotent element and these elements are commute with each other. A ring R is considered to be a strongly r-precious ring if all of its elements are strongly r-precious. In this article, we discuss some of the fundamental properties of strongly r-precious rings and discuss the behaviour of strongly r-precious rings under finite direct product, homomorphic images, matrix ring and group rings.

1 Introduction

From last few decades researchers are studying various rings which are defined by properties of some special elements like unit, idempotent, nilpotent and regular elements etc. In this study first such ring is introduced by Nicholson [8] known as clean ring (every element is the sum of a unit and an idempotent), strongly clean ring (every element is sum of a unit and an idempotent and in addition to this they commute with each other) [10], nil clean (every element is the sum of nilpotent and an idempotent) [5], r-clean ring (every element is the sum of regular and an idempotent) [2] [3], r-precious ring(every element is the sum of regular element, nilpotent and an idempotent) [4].

The concept of strongly r-precious ring is studied in this article. The family of strongly r-precious rings is quite big that includes clean rings, strongly clean rings and strongly precious rings etc. Here we discuss the basic properties of strongly r-precious ring and the behaviour of strongly r-precious under the construction of homomorphic images, direct products,quotient rings and matrix rings etc.

In this article, rings are associative with unity. The notations $Ide(R), Un(R)$, $Ni(R)$ and $NReg(R)$ signify the set of idempotent elements, the group of unit elements, the set of nilpotent elements and the set of Von-Neumann regular elements respectively, in a ring R.

2 Strongly r-precious rings

We start by defining strongly precious ring and strongly r-precious ring .

Definition 2.1. We say that a ring R is 'strongly precious' if for given any $r \in R$, $r = u+m+n$ and $um = mu$, $un = nu, mn = nm$, where $u \in Un(R), n \in Ni(R)$ and $m \in Ide(R)$.

Definition 2.2. We say that a ring R is 'strongly r-precious' if for given any $r \in R$, $r = k+m+n$ and $km = mk$, $kn = nk$ and $mn = nm$, where $k \in NReg(R), n \in Ni(R)$ and $m \in Ide(R)$.

The example for strongly r-precious ring is $R = \mathbb{Z} \times \mathbb{Z}_4$.
From the above definitions it follows that the strongly precious rings and strongly clean rings are strongly r-precious. In general converse is not true. However, the following results hold.

Theorem 2.3. *For a strongly r-precious indecomposable ring following holds.*
(a) Ring R is strongly precious.
(b) Ring R is strongly 2-clean. (every element of a ring is sum of an idempotent element, two unit elements and these elements are commute with each other).

Proof. (a) Noting that in a strongly r-precious ring, each $r \in R$ has the form $r = k + m + n$, $km = mk$, $kn = nk$ and $mn = nm$, where $k \in NReg(R), n \in Ni(R)$ and $m \in Ide(R)$. If $k = 0$ then $r = m + n = (2m - 1) + (1 - m) + n$. Clearly $2m - 1 \in Un(R)$ and $1 - m \in Ide(R)$. Now, we need to prove that $(2m - 1), (1 - m)$ and n are commute with each other. Consider $(2m - 1)(1 - m) = (m - 1) = (1 - m)(2m - 1)$. Now, consider $(2m - 1)n = n(2m) - n = n(2m - 1)$. Similarly, we can show $(2m - 1)$ commute with n. So r is strongly precious and hence R is strongly precious. Now for $k \neq 0$ we have $kpk = k$ for some $p \in R$. So we have $kp \in Ide(R)$. But $kp = 1$ or $kp = 0$. If $kp = 0$ then $k = kpk = 0$ which gives a contradiction. Hence $kp = 1$ and similarly $pk = 1$. Thus $k \in Un(R)$. Hence R is strongly precious.

(b) From previous part R is strongly precious. Hence for any $r \in R$ we have $r = k + m + n$ and $km = mk$, $kn = nk$ and $mn = nm$, where $k \in Un(R), m \in Ide(R)$ and $n \in Ni(R)$. If $m = 0$, then $r = k + (n - 1) + 1$. Clearly $k, (n - 1), 1$ commute with each other and $(n - 1) \in Un(R)$. Now, if $m = 1$ then $r = k + (1 + n) + 0$, clearly $(1 + n)r = r(1 + n)$ and $(1 + n) \in Un(R)$. Thus r is strongly 2-clean element. This implies R is strongly 2-clean. □

Theorem 2.4. *For a ring R with no non-trivial zero divisors then following statements are equivalent.*

(a) R is strongly precious.

(b) R is strongly r-precious.

Proof. $(a) \Rightarrow (b)$ Clearly R is indecomposable because R have only trivial zero divisors. Hence by Theorem[2.3] the result holds.

$(b) \Rightarrow (a)$ Follows from the definitions 2.1 and 2.2.□

Proposition 2.5. *For a 2-primal indecomposable ring R, R is strongly r-precious if and only if R is strongly clean.*

Proof. Let R be a strongly r-precious and 2-primal indecomposable. Hence for any $r \in R$, $r = k + m + n, mk = km, mn = nm, kn = nk$ where $k \in NReg(R), m \in Ide(R)$ and $n \in Ni(R)$. From the part (a) of Theorem [2.3], in an indecomposable ring any regular element k is either a unit element or 0. Hence we have either $k \in Un(R)$ or $k = 0$. If $k = 0$, then $r = k + n$. So r is strongly nil clean and this implies r is strongly clean. Now for $k \in Un(R)$, we have $r = (k + n) + m$. But R is 2-primal ring hence $k + n \in Un(R)$, and $(k + n)m = (km + nm) = mk + mn = m(k + n)$. So r is strongly clean element.
The second part follows from the definition of strongly clean ring. □

Now we study the behaviour of strongly r-precious rings on the basis of fundamental ring construction of direct product, homomorphic images and quotient rings.

Theorem 2.6. *If R is a strongly r-precious ring then a homomrphic image of R is strongly r-precious.*

Proof. We know that a homomorphic image of R is isomorphic with R/I for some ideal I in R. Hence we prove R/I is strongly r-precious.
As R is strongly r-precious ring therefore for any $r \in R$ has the form $r = k + m + n$, $km = mk$, $kn = nk$ and $mn = nm$, where $m \in Ide(R)$, $k \in NReg(R)$ and $n \in Ni(R)$. Let $\bar{r} = r + I = k + m + n + I = \bar{k} + \bar{m} + \bar{n} \in R/I$. We need to show that $\bar{k}$ is regular, $\bar{m}$ is idempotent, $\bar{n}$ is nilpotent and all these elements are commute with each other.
Since k is regular there exists $p \in R$ such that $kpk = k$, therefore $\bar{k}\bar{p}\bar{k} = \bar{k}$, so $\bar{k}$ is regular and clearly $\bar{m} = m + I \in Id(R/I)$ and $\bar{n} = n + I \in Ni(R)$. Now we need to prove all these elements are commute with each other. Consider $\bar{m}\bar{k} = (m + I)(k + I) = (mk + I) = (km + I) = (k + I)(m + I) = \bar{k}\bar{m}$. Now, consider $\bar{n}\bar{k} = (n + I)(k + I) = (nk + I) = (kn + I) = \bar{k}\bar{n}$. Similarly we can show $\bar{m}\bar{n} = \bar{n}\bar{m}$. This follows R/I is strongly r-precious ring. □

Theorem 2.7. *For $R = \prod_{j \in I} R_j$, where $(R_j)_{j \in I}$ is a family of rings.*

(a) If R is strongly r-precious then R_j is strongly r-precious,for all $j \in I$.

(b) If R_j is strongly r-precious $\forall j \in I$, with $\prod_{j\in I} N(R_j) \subseteq Ni(R)$ then ring R is strongly r-precious.

(c) For a finite index set I, R is strongly r-precious if and only if each R_j is strongly r-precious.

Proof.(a) Clearly R_j is homomorphic image of strongly r-precious ring R. Hence for each $j \in I, R_j$ is strongly r-precious.

(b)Noting that R_j is strongly r-precious ring. Hence for each $r_j \in R_j$ we have $r_j = k_j + m_j + n_j$ and k_j, m_j, n_j are commute with each other, where $(k_j) \in NReg(R_j), (m_j) \in Ide(R_j)$ and $(n_j) \in N_i(R_j)$. Clearly $(m_j) \in Ide(R), (k_j) \in NReg(R)$. Now $(n_j) \in Ni(R_j)$ and hence $(n_j) \in \prod_{j\in I} Ni(R_j) \subseteq Ni(R)$ and k_j, m_j, n_j commute with each other. Thus R is strongly r-precious ring.

(c) Clearly finitness of I implies $\prod_{j\in I} Ni(R_j) = Ni(R)$. Hence from (a) and (b) the result follows. □

Now we study some more properties of strongly r-precious ring.

Proposition 2.8. *For a ring R any element $r \in R$ is strongly r-precious if and only if $1 - r$ is strongly r-precious.*

Proof. As $r \in R$ is a strongly r-precious, hence $r = k + m + n, mk = km, mn = nm, kn = nk$ where $k \in NReg(R), m \in Ide(R)$ and $n \in Ni(R)$. Thus, $(1 - r) = (-k) + (1 - m) + (-n)$. As $k \in NReg(R)$, therefore there exist $p \in R$ such that $kpk = k$. Therefore, $(-k) = (-k)(-p)(-k)$, hence $(-k) \in NReg(R)$. Also $(1-m)^2 = (1-m)$, therefore $(1-m) \in Ide(R)$ and clearly $(-n) \in Ni(R)$. Now, consider $(-k)(1 - m) = (-k + km) = (-k + mk) = (1 - m)(-k)$. Also, $(1 - m)(-n) = (-n + mn) = (-n + nm) = (-n)(1 - m)$, clearly $(-k)(-n) = (-n)(-k)$. Hence $(1 - r)$ is strongly r-precious.
Conversely, assume $(1-r)$ is strongly r-precious, then $1-r = k+m+n, mk = km, mn = nm$ and $kn = nk$ where $k \in NReg(R), m \in Ide(R)$ and $n \in Ni(R)$. Thus $r = -k+(1-m)-n$ like previous part $-k \in NReg(R), -n \in Ni(R)$ and $(1 - m) \in Ide(R)$ and all $(-k), (1 - m), (-n)$ commute with each other. □

Proposition 2.9. *Ring R is strongly r-precious if and only if $r = k - m + n, \forall r \in R$ where k, m, n are commute with each other and $k \in NReg(R), m \in Ide(R)$ and $n \in Ni(R)$.*

Proof. As R is a strongly r-precious ring and $-r \in R$, hence $-r = k+m+n, mk = km, mn = nm, kn = nk$ where $k \in NReg(R), m \in Ide(R)$ and $n \in Ni(R)$. Thus $r = (-k) - m + (-n)$ here $(-k) \in NReg(R), (-m) \in Ide(R)$ and $-n \in Ni(R)$. Clearly $(-k), (-m)$ and $-n$ are commute with each other.
Converse part follows like previous part. □

Proposition 2.10. *Let m be a central element of strongly r-precious ring then mRm is strongly r-precious.*

Proof. For $r \in R$, $r = k + m_1 + n$ and $m_1k = km_1, m_1n = nm_1, kn = nk$ where $k \in NReg(R), n \in Ni(R)$ and $m_1 \in Ide(R)$. Let $a \in mRm$, therefore for some $r \in R$ we have $a = mrm$, where m is central idempotent element. Thus, $a = m(k + m_1 + n)m = mkm + mm_1m + mnm = km + m_1m + nm$. Now, we prove km is regular, m_1m is an idempotent, nm is nilpotent and also they commute with each other.
Consider $(m_1m)^2 = (m_1m)(m_1m) = (m_1^2m^2) = m_1m$. As $k$is regular therefore $(km)(mpm)(km) = kmmpmmk = kmpmk = (km)p(mk) = kpkm^2 = km$. Hence km is regular. Clearly, nm is nilpotent. Now, consider $(km)(m_1m) = (km_1)m^2 = (m_1k)m^2 = (m_1m)(km)$. On similar line we can prove that $(nm)(km) = (km)(nm)$ and $(nm)(m_1m) = (m_1m)(nm)$. Hence mRm is strongly r-precious. □

Now we discuss the matrix ring in context of strongly r-precious ring. For this we require following result.

Proposition 2.11. *Let R be a ring with $m_1, m_2, ..., m_n$ be central orthogonal idempotents with $m_1 + m_2 + ... + m_n = 1$. Then $m_1Rm_1, m_2Rm_2, ...m_nRm_n$ are strongly r-precious if and only if R is strongly r-precious.*

Proof. Clearly $\prod_{j=1}^{n} m_jRm_j \cong R$. Hence by Theorem 2.7 the result holds. □

Proposition 2.12. *Suppose a ring R is strongly r-precious then $M_n(R)$ is strongly r-precious.*

Proof. It is easy to see that $I_n = M_{11}+M_{22}+....+M_{nn}$, where M_{jj} is elementary matrix of order $n \times n$ whose $(jj)^{th}$ entry is 1 and remaining entries are 0, and I_n is the identity matrix. It is easy to see that each M_{jj} is idempotent and M_{jj} are mutually orthogonal, $\forall j \in \{1, ...n\}$. Clearly $M_{jj}M_n(R)M_{jj} \cong R$. But each R is strongly r-precious ring which implies $M_{jj}M_n(R)M_{jj}$ is strongly r-precious. Consequently by $Proposition$ 2.11 it implies that $M_n(R)$ is strongly r-precious. □

Now we characterize strongly r-precious ring on the basis of quotient ring. For this we start with the following definition.

Definition 2.13. A ring R with an ideal I is said to be modulo abelian ring if $x + I$ and $y + I$ are elements of R/I and $(x + I)(y + I) = (y + I)(x + I)$ then $xy = yx$ in R.

From the above definition it is easy to see that if R is modulo abelian ring then the elements of ideal I commute with all elements of R.

Theorem 2.14. *Let I be a central nil ideal of modulo abelian ring R. Then R is strongly r-precious ring if and only if R/I is an strongly r-precious.*

Proof. : Assume $\bar{R} = R/I$ is a strongly r-precious ring. For $r \in R, \bar{r} = r + I \in R/I$ and it is given that R/I is strongly r-precious, so we can write $\bar{r} = \bar{k} + \bar{m} + \bar{n}$, here $\bar{k} \in Reg(\bar{R}), \bar{n} \in N(\bar{R}), \bar{m} \in Id(\bar{R})$ and $\bar{m}\bar{k} = \bar{k}\bar{m}, \bar{k}\bar{n} = \bar{n}\bar{k}$ and $\bar{m}\bar{n}$=$\bar{n}\bar{m}$. Since idempotent can be lift modulo every nil ideal this implies $m \in Ide(R)$. Also by [7], regular element can be lift modulo every nil ideal I, so $r \in NReg(R)$. Hence, $r = k + m + n + p$ where $p \in I$. Since $\bar{n} \in N(\bar{R})$ so $n^a \in I$ for some $a \in \mathbb{N}$. This gives $(n+p)^a \in I$, hence $(n+p)$ is a nilpotent. Since R is modulo abelian ring, this implies $km = mk, k(n + p) = (n + p)k, (n + p)m = m(n + p)$. Hence r is strongly r-precious.
The second part is trivial. □

Proposition 2.15. *Let R be a ring and $dig(r_1, r_2, \dots, r_n)$ be a $n \times n$ diagonal matrix with r_i is entry on the main diagonal. If $r_1, r_2, \dots, r_n \in R$ is strongly r-precious then $dig(r_1, r_2, \dots, r_n)$ is strongly r-precious in $M_n(R)$.*

Proof. :Consider $r_i = k_i + m_i + n_i, k_im_i = m_ik_i, k_in_i = n_ik_i$ and $m_in_i = n_im_i$ where $k_i \in NReg(R), m_i \in Ide(R)$ and $n_i \in Ni(R)$ for every $i, 1 \leq i \leq n$. Hence $diag(r_1, r_2, \dots, r_n) = diag(k_1, k_2, \dots, k_n)+diag(m_1, m_2, \dots, m_n)+diag(n_1, n_2, \dots, n_n)$. But there exist $p_i \in R$ such that $k_ip_ik_i = k_i$ for every $i, 1 \leq i \leq n$. Thus $dig(k_1, \dots, k_n)dig(p_1, \dots, p_n)dig(k_1, \dots, k_n) = dig(k_1, \dots, k_n)$. Hence $dig(k_1, \dots, k_n)$ is regular. Also, as n_i is nilpotent so there exist some b_i such that $(n_i)^{b_i} = 0 \; \forall i, 1 \leq i \leq n$. Let $b = max(b_1, \dots, b_n)$, thus $(diag(n_1, \dots, n_n))^b = diag(n_1^b, \dots, n_n^b) = (0, \dots, 0)$. Hence $dig(n_1, \dots, n_n)$ is nilpotent. Clearly, $diag(m_1, \dots, m_n)$ is idempotent and $dig(k_1, \dots, k_n), dig(m_1, \dots, m_n), dig(n_1, \dots, n_n)$ are commute with each other, so $diag(r_1, \dots, r_n)$ is strongly r-precious. □

For a ring R , X be an $R-R$ bimodule which form a ring with $(xy)r = x(yr), (xr)y = x(ry)$ and $(rx)y = r(xy)$ where $r \in R$, and $x, y \in X$. For such module X, the ideal extention $\{I(R; X) = (r, x) : r \in R$ and $x \in X\}$ with usual addition, and the multiplication given by $(r, x)(r', x') = (rr', rx' + xr' + xx')$ form a ring.

Now, we prove some elementary results on Ideal extention.

Proposition 2.16. *Let $I(R;X)$ be strongly r-precious then R is strongly r-precious.*

Proof. Let $P = I(R;X)$ and $r \in R$, then $(r,0) \in P$. Thus, there exist $(m_1,m_2) \in Id(P)$, and $(k_1,k_2) \in NReg(P), (n_1,n_2) \in N(P)$ such that $(r,0) = (m_1,m_2) + (k_1,k_2) + (n_1,n_2)$. But $(k_1,k_2)(m_1,m_2) = (m_1,m_2)(k_1,k_2)$, $(n_1,n_2)(m_1,m_2) = (m_1,m_2)(n_1,n_2)$ and $(n_1,n_2)(k_1,k_2) = (k_1,k_2)(n_1,n_2)$. This implies $m_1k_1 = k_1m_1, m_1n_1 = n_1m_1$ and $k_1n_1 = n_1k_1$. Also, $(k_1,k_2)(p_1,p_2)(k_1,k_2) = (k_1,k_2)$ for some $(p_1,p_2) \in P$ and this implies $k_1p_1k_1 = k_1$, so $k_1 \in NReg(R)$. Similarly we can show $n_1 \in Ni(R)$ and $m_1 \in Ide(R)$. This implies $r = m_1 + k_1 + n_1$ and hence r is strongly r-precious. □

Proposition 2.17. *For a strongly r-precious ring R with $Ide(R)$ and $Ni(R)$ are subset of $Z(R)$ (centre of the ring R) and $xpk + xpx + kpx = x \; \forall x \in X, \; k,p \in R$, then $I(R;X)$ is strongly r-precious.*

Proof. Noting that in a strongly r-precious ring R for $\forall r \in R$ we have $r = k + m + n, km = mk, mn = nm$ and $kn = nk$, where $k \in NReg(R), \; m \in Ide(R)$ and $n \in Ni(R)$. Consider $(r,x) \in I(R;X)$, this implies $(r,x) = (k,x) + (m,0) + (n,0)$. But $k = kpk$ for some $p \in R$, so $(k,x)(p,0)(k,x) = (kpk, xpk + xpx + kxp) = (k,x)$ because for any $x \in X$ and $k,p \in R$, we have $xpk + xpx + kxp = x$. Therefore $(k,x) \in NReg(I(R;V))$. Now we need to prove that $(k,x),(m,0)$ and $(n,0)$ are commute with each other. Here $(k,x)(m,0) = (km,xm) = (mk,mx) = (m,0)(k,x)$. Similarly, $(n,0)(k,x) = (k,x)(n,0)$ and $(m,0)(n,0) = (n,0)(m,0)$. Hence $I(R;X)$ is strongly r-precious. □

For a given ring R and group G, the group ring over R is denoted by RG. By $Theorem$ 2.6 it is easy to see that if group ring RG is strongly r-precious then fundamental ring R is also strongly r-precious.

Now we give some groups and rings for which the corresponding group ring become a strongly r-precious.

Proposition 2.18. *For a group G, if R is a semiperfect commutative ring and $(mRm)G$ is a strongly r-precious for each local idempotent m in R then RG is strongly r-precious ring.*

Proof. By [1, Theorem 2.1], semiperfect ring R has a complete orthogonal set of idempotents $m_1.m_2, m_3, ...m_n$ with each m_jRm_j is a local ring $\forall j \in \{1,2,\ldots,n\}$. So idempotent m_j is local for each j. Now, by assumption $(m_jRm_j)G$ is a strongly r-precious ring. Since $(m_jRm_j)G \cong m_j(RG)m_j$, $\forall j \in \{1,2...,n\}$, this implies $m_j(RG)m_j$ is strongly r-precious ring. Thus, RG is strongly r-precious ring by $Proposition$ 2.10. □

Proposition 2.19. *For a group $G = \{1,g\}$ and a ring R with 2 is invertible, the group ring RG is strongly r-precious if and only if R is strongly r-precious.*

Proof. From $Theorem$ 2.6, one direction is obvious. Conversely in a strongly r-precious ring where 2 is invertible, we have $RG \cong R \times R$ ([3, Proposition 3]). This implies RG is strongly r-precious. □

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