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DIRECT SUM AND HOMOMORPHIC IMAGE OF CCWRS MODULES

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Abstract: Our interest here is to study the properties of cofinitely closed weak Rad-supplemented module (briefly, $ccwrs$) which is the strict and simultaneous generalizations of extending, rs, wrs and cofinitely weak Rad-supplemented modules respectively. Some relevant counter examples are provided to show the distinction of above mentioned module structures. It observed that, $ccwrs-$module is not inherited by direct sum and homomorphic image, relevant examples are provided in Remark 2.13 and Remark 2.18 respectively. In this regard, we proved many result under some restrictions which showed that $ccwrs$ is closed under finite (direct) sum and homomorphic image.

1 Introduction

Recall from [6] and [10], for submodules A and B of M, B is called a supplement (weak supplement) of A, if $A + B = M$ and $A \cap B \ll B(A \cap B \ll M)$. A module M is said to be supplemented (weak supplemented (briefly, ws)), if each submodule of M has a supplement (weak supplement) in M. Obviously every supplement is weak supplement. Recently, many authors have detailed study on various generalizations of supplemented and $ws-$modules. A submodule A of M is said to be cofinite if M/A is fg (finitely generated). A right $R-$module M is said to be cofinitely supplemented (weak supplemented), if each cofinite submodule A of M has (is) a supplement (weak supplement) in M, i.e. if we get a $B \subseteq M$ such that $M = A + B$ and $A \cap B \ll B(A \cap B \ll M)$. Xue [16], weaken the concepts of supplement condition to introduce new concept generalized supplemented also known as Rad-supplemented (briefly, rs), if every $A \subseteq M$ has Rad-supplement in M, where submodule B is Rad-supplement of A, if $A + B = M$ and $A \cap B \subseteq RadB$. Cleary, lifting module and supplemented module lie in the class of $rs-$module, which are also studied by many authors in a series of papers [2], [3], [11] and [12]. An $R-$module M is said to be weak Rad-supplemented (briefly, wrs), if every $A \subseteq M$ has a weak Rad-supplement in M, where submodule B is weak Rad-supplement of A, if $M = A + B$ and $A \cap B \subseteq RadM$. Recall from [11], A right $R-$module M is said to be cofinitely Rad-supplemented (weak Rad-supplemented), if every cofinite submodule A of M has (is) a Rad-supplement (weak Rad-supplement) in M, i.e. if we get a $B \subseteq M$ with $M = A + B$ and $A \cap B \subseteq RadB(A \cap B \subseteq RadM)$.

A non zero $A \subseteq M$ is $A \subseteq_e M$, if every $0 \neq B \subseteq M$ has a nontrivial intersection with A, clearly every submodule of uniform module like, $M = \mathbb{Z}_{\mathbb{Z}}$ is essential in M. $P \subseteq M$ is $P \subseteq_c M$, if it has no proper essential extension in M i.e., if $P \subseteq_e Q$ for some submodule $Q \subseteq M \Rightarrow Q = P$. It is clear that direct summands are closed submodule in M. M is an extending (or CS) module, if every $A \subseteq_c M$ is a $A \subseteq^{\oplus} M$ or equivalently, every $A \subseteq M$ is $A \subseteq_e D$ then $D \subseteq^{\oplus} M$. The class of extending module includes uniform, semisimple, (quasi-) injective modules and finitely generated torsion-free $\mathbb{Z}-$modules. $A \subseteq M$ is said to be cofinite closed (represented by $A \subseteq_{cc} M$) if A is closed as well as cofinite submodule of M.

Motivated by above notions, we have already defined and studied the properties of a new notion of module as a proper generalization of $wrs-$module called closed weak Rad-supplemented (briefly, $cwrs$) [13]. In this article, we are interested to study the properties of another generalization of $wrs-$module and $cwrs-$module namely $ccwrs-$ module. The class of $ccwrs-$module includes hollow, local, uniform, semisimple, (quasi-) injective, extending, wrs and $cwrs-$module. We observe that, direct summands and factor modules of a $ccwrs$ modules is $ccwrs$, however it is not inherited by direct sum and homomorphic images, examples are provided in Remark 2.13 and Remark 2.18 respectively. Thus our main concern is to investigate the conditions which ensure the property of module being $ccwrs$ is preserve under direct sum and homomorphic images.

Throughout this article, all rings R are an associative ring having identity and all modules are unitary right $R-$modules. Consider M is an $R-$module, then we adopt the representation $A \subseteq M$ $(A \subset M)$, $A \subseteq_{fg} M$, $A \subseteq_e M$, $A \subseteq_c M$ and $A \subseteq_{cc} M$ means that A is a submodule (proper), finitely generated submodule, essential, closed (or complement) and cofinite closed submodule of M respectively. A is said to be small in M (represented by $A \ll M$) if there is no any $B \subset M$ with $A + B = M$. Recall [6], module M is hollow (semi-hollow), if every $A \subset M (A \subset_{fg} M)$ is small in M. The sum of every $A \ll M$ is known as radical of M denoted by $RadM$. Recall from [6], module M is said to be radical, if it has no proper maximal submodule i.e., $RadM = M$ and $P(M)$ will denote the sum of each radical submodules of M. Recall from [4], module M is said to be $w-$local, if it has a maximal submodule, which is unique. The right annihilator of any element $m \in M$, is defined as $r_R(m) = \{r \in R | m.r = 0\}$.

Now we are listing some well known property of closed submodule (Lemma 1.1), small submodule (Lemma 1.2), radical of module M i.e. $RadM$ (Lemma 1.3) and cofinite closed submodule (Lemma 1.4) which will use in later.

Lemma 1.1. *[7, 1.10] Let submodules D, E and F of an $R-$module N such that $D \subseteq E$, then we have;*
(1) If $K \subseteq_c N_1 \oplus N_2$, then $K \cap N_1 \subseteq_c N_1$ and $K \cap N_2 \subseteq_c N_2$, where $N_1 \oplus N_2$ is duo module.
(2) Transitivity: $D \subseteq_c E$ and $E \subseteq_c N$, then $D \subseteq_c N$.
(3) Let P be non-singular, $F \subseteq_c P$ and an epimorphism $g : N \longrightarrow P$, then $g^{-1}(F) \subseteq_c N$.

Lemma 1.2. *[6, 2.2 & 2.3] Let submodules D, E of an $R-$module N such that $D \subseteq E$, then we have;*
(1) If $D \ll N$, then N is fg if and only if N/D is fg.
(2) $D_1 \oplus D_2 \ll E_1 \oplus E_2$ if and only if $D_1 \ll E_1$ and $D_2 \ll E_2$.
(3) If $D \ll N$ and module homomorphism $g : N \longrightarrow P$, then $g(D) \ll P$.

Lemma 1.3. *[10, 21.6] For an $R-$module N, we have;*
(1) $RadA = A \cap RadN$ holds only for supplement (or rs) submodule A of N.
(2) If $N = \oplus_{i \in I} N_i$, then $RadM = \oplus_{i \in I} RadN_i$ and $N/RadN = \oplus_{i \in I} N_i/RadN_i$.
(3) $g(RadN) \subset RadP$ for any module homomorphism $g : N \longrightarrow P$.

Lemma 1.4. *Let D and E be submodules of an $R-$module N with $D \subseteq E \subseteq N$, then we have;*
(1) Transitivity: $D \subseteq_{cc} E$ and $E \subseteq_{cc} N$ then $D \subseteq_{cc} N$.
(2) $E \subseteq_{cc} N$ if and only if $E/D \subseteq_{cc} N/D$.

2 Cofinitely closed weak Rad-supplemented modules

Definition 2.1. A right $R-$module N is said to be cofinitely closed weak Rad-supplemented (briefly, $ccwrs$) if every $C \subseteq_c N$ with N/C is fg has (is) a weak Rad-supplement in N i.e., if there exists a $D \subseteq N$ with $N = C + D$ and $C \cap D \subseteq RadN$, corresponding to any $C \subseteq_{cc} N$.

Example 2.2. (1) Every rs and $wrs-$modules are $ccwrs$.
(2) Every local and hollow module are $ccwrs$, because every local is hollow and every hollow is $rs-$module.
(3) The set of rational number $\mathbb{Q}$ is rs as well as wrs but not supplemented module over the set of integers $\mathbb{Z}$.
(4) Every cofinitely weak supplemented module and every $cwrs-$ module are $ccwrs$.
(5) $M = \mathbb{Z}_{\mathbb{Z}}$ is CS and hence it will be $ccwrs$, but M is not wrs as for a cofinite submodule for all $n \geq 2$, $n\mathbb{Z}$ has not wrs in $\mathbb{Z}$. Consider $A = 2\mathbb{Z}$, then $B = 3\mathbb{Z}$ is only submodule of $\mathbb{Z}_{\mathbb{Z}}$ such that $M = 2\mathbb{Z} + 3\mathbb{Z}$ and $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z} \not\subset RadM = 0$.
(6) Every extending (or CS) module is $ccwrs$, however converse is not true. Consider $R =$

$\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$, where $\mathbb{Z}$ denotes the ring of integers. Then clearly R_R is not an extending right $R-$module. However the right ideals of R is of the form $I = \{\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} | A, B, C$ are ideals of $\mathbb{Z}$ and $A \subseteq B\}$. As $\mathbb{Z}_{\mathbb{Z}}$ is uniform module, then all closed right ideals of R listed as $0, R, I_1 = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}, I_2 = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}, I_3 = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}, I_4 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$. Clearly R is closed weak supplemented and hence a $ccwrs-$module over itself.
(7) Every fg module is wrs if and only if it is a cofinitely weak Rad-supplemented.

Thus we have the following implications, whose converse is not necessarily true, some relevant counter examples are provided in above Example 2.2:

Rad-supplemented ⇒ Weak Rad-supplemented ⇒ Closed weak Rad-supplemented

Weak supplemented   Cofinitely Rad-supplemented   ↓
↓   ↓

Cofinite supplemented ⇒ Cofinite weak supplemented ⇒ Cofinite weak Rad-supplemented ⇒ CCWRS

Remark 2.3. (1) Every quotient module of a $ccwrs-$module is again a $ccwrs$.
(2) Every direct summands of a $ccwrs-$module is again a $ccwrs$.

Theorem 2.4. *Let for any $C \subseteq N$, we get a wrs D of some maximal submodule E of N with $C + D \subseteq_{cc} N$. Then N is $ccwrs$ if and only if N is a wrs.*

Proof. Proof is obvious in the light of the Lemma 2.22 [13]. □

An $R-$module M is said to be amply weak Rad-supplemented (briefly, $awrs$) if $M = C + D$, where D contains a wrs of C in M, for every submodules C and D of M.

Proposition 2.5. *Let C and D be submodules of an $R-$module N with the property $C \cap D \subseteq_{cc} N$, where $D \subseteq_{cc} N$. If D is $ccwrs-$module, then N is $awrs-$module with $N = C + D$.*

Proof. Assume that, $C \cap D \subseteq_{cc} N = C + D$, so $C \cap D \subseteq_{cc} D$. By assumption D is $ccwrs$ then we get a wrs E of $C \cap D$ in D with $D = (C \cap D) + E$ and $(C \cap D) \cap E = C \cap E \subseteq RadD$, so $D \subseteq C+E$, hence $N = C+D \subseteq C+E$. Thus we have $N = C+E$ and $C \cap E \subseteq RadD \subseteq RadN$, so E is wrs of C in N lies inside D. Consequently, N is $awrs-$module. □

Corollary 2.6. *Let C and D be submodule of a semisimple $R-$module N, such that $N = C + D$, then N is $awrs-$module.*

Proposition 2.7. *Let N be $ccwrs-$module and D is a wrs of a $C \subseteq_{cc} N$. If $C \cap D$ has rs in D, then C has wrs in N.*

Proof. As N is $ccwrs-$module, then by definition, for any $C \subseteq_{cc} N$, there exists wrs D of C in N, with $N = C+D$ and $C \cap D \subseteq RadN$. Assume that $C \cap D$ has rs E in D, then $D = E+(C \cap D)$ and $E \cap (C \cap D) = E \cap C \subseteq RadE$. Thus $N = C + D = C + E + (C \cap D) = C + E$ and $C \cap [E \cap (C \cap D)] = E \cap C \subseteq RadE \subseteq RadN$. Therefore, C has wrs E in N. □

Lemma 2.8. *Every module $N = RadN$ is $ccwrs$.*

Proof. Let $C \subseteq_{cc} N = RadN$, then $C \subseteq RadN = N$. Clearly, we get a $D \subset N$ or N itself, such that $N = C + D$ and $C \cap D \subseteq N = RadN$. Hence N is $ccwrs-$module. □

Corollary 2.9. *The P(N) of a module N is $ccwrs$.*

Remark 2.10. It is observed that;
(1) A module M is $w-$local if and only if $RadM \subseteq_{max} M$.
(2) Every direct summand of a module M is either $w-$local or radical.

Lemma 2.11. *Every $w-$local module M is $ccwrs-$module.*

Proof. Let $C \subseteq_{cc} M$, M is $w-$local. So $RadM \subseteq_{max} M$, which is unique, as M is $w-$local, i.e., $M/RadM \subseteq M/C$ implies that $C \subseteq RadM$. Clearly, we get a $D \subset M$ or M itself, with $M = C + D$ and $C \cap D \subseteq RadM$. Hence M is $ccwrs-$module. □

Corollary 2.12. *Every $A \subseteq^{\oplus} M$ of a $w-$local module M is $ccwrs$.*

Proof. From above Remark 2.10(2), $A \subseteq^{\oplus} M$, M is $w-$local is either $Rad(A) = A$ or $w-$local. Applying Lemma 2.8 and Lemma 2.11, we get the desired result. □

Remark 2.13. It is observed by many authors that, the direct sum of $CS-$modules is not necessarily CS ([1], Example 2.4). Now consider the polynomial ring $R = \mathbb{Z}[x]$ is a commutative Noetherian domain with an indeterminate x, then R_R is $CS-$module while module $M = R \oplus R$ is not CS. Also, ([7], 7.6) for any prime p, $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^3\mathbb{Z}$ are $CS-$modules, however $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not CS. As every $CS-$module are $ccwrs$ so these examples work directly to conclude that direct sum of $ccwrs-$modules need not be $ccwrs$.

Now, we investigate the conditions which ensure the property of module being $ccwrs$ is inherited by direct sum.

Recall from [15], a module is said to be distributive (local distributive), if intersection distribute over addition or addition distribute over intersection for every submodules (closed submodules), D, E and F of M, i.e. $D \cap (E + F) = (D \cap E) + (D \cap F)$ or $D + (E \cap F) = (D + E) \cap (D + F)$. The class of distributive module includes all duo modules. The class of local distributive module includes all duo modules and distributive modules but the converse need not be true for example; consider the ring of integers $\mathbb{Z}$ as module over itself, then $M = \mathbb{Z} \oplus \mathbb{Z}$ is local distributive but not distributive module.

Proposition 2.14. *A distributive $R-$module $N = N_1 \oplus N_2$ is $ccwrs$ if and only if each components N_1 and N_2 are $ccwrs-$modules.*

Proof. Assume that $N = N_1 \oplus N_2$ is $ccwrs-$ module, then N_1 and N_2 are $ccwrs$ followed by Remark 2.3(2). Conversely, assume that $A \subseteq_{cc} N$ and N_1, N_2 be $ccwrs-$module. As distributive module includes all duo module, applying Lemma 1.1(1), we get, $A \cap N_1 \subseteq_c N_1$ and $A \cap N_2 \subseteq_c N_2$, clearly these are cofinite submodules by Lemma 1.4(1). By assumption there exists wrs, $B_i \subseteq N_i, i = 1, 2$ of $(A \cap N_i)$ such that $N_i = (A \cap N_i) + B_i$ and $(A \cap N_i) \cap B_i = A \cap B_i \subseteq RadN_i$. Take $C = B_1 \oplus B_2$, as N is distributive, we can write $N = N_1 \oplus N_2 = [(A \cap N_1) + B_1] \oplus [(A \cap N_2) + B_2] = (B_1 \oplus B_2) + [(A \cap N_1) \oplus (A \cap N_2)] = C + [A \cap (N_1 \oplus N_2)] = C + (A \cap N) = C + A$. Also, $A \cap C = A \cap (B_1 \oplus B_2) = (A \cap B_1) \oplus (A \cap B_2) \subseteq RadN_1 \oplus RadN_2 = RadN$ by Lemma 1.3(2). Thus $A \subseteq_{cc} N$ has wrs $C = B_1 \oplus B_2$ in N, and hence N is $ccwrs-$module. □

Corollary 2.15. *A local distributive $R-$module $M = M_1 \oplus M_2$ is $ccwrs$ if and only if each component M_1 and M_2 of M are $ccwrs-$modules.*

Proof. Proof is straightforward. □

Corollary 2.16. *Let every cofinite closed submodule of an $R-$module $M = M_1 \oplus M_2$ is fully invariant, then M is $ccwrs$ if and only if each components M_1 and M_2 of M are $ccwrs-$modules.*

Proof. Proof is straightforward. □

Theorem 2.17. *Let the ring $R = r_R(N_1) + r_R(N_2)$, then $R-$module $N = N_1 \oplus N_2$ is $ccwrs$ if and only if each components N_1 and N_2 are $ccwrs-$modules.*

Proof. Assume that $N = N_1 \oplus N_2$ is $ccwrs-$ module, then by Remark 2.3(2), N_1 and N_2 are $ccwrs-$module. Conversely, assume that A be any cofinite closed submodule of N and N_1, N_2 are $ccwrs-$ modules. Since $R = r_R(N_1) + r_R(N_2)$, then there exists summands A_1 of N_1 and A_2 of N_2 such that $A = A_1 \oplus A_2$. As every direct summands are closed submodule, so $A_1 \subseteq_c N_1$ and $A_2 \subseteq_c N_2$, also by Lemma 1.4(1) $A_1 \subseteq_{cf} N_1$ and $A_2 \subseteq_{cf} N_2$, thus $A_1 \subseteq_{cc} N_1$ and $A_2 \subseteq_{cc} N_2$. By assumption there exists wrs B_i of A_i in N_i, $i = 1, 2$ such that $N_i = A_i + B_i$ and $A_i \cap B_i \subseteq RadN_i$. Take $C = B_1 \oplus B_2$, then N can be written as $N = N_1 \oplus N_2 = (A_1 + B_1) \oplus (A_2 + B_2) = (A_1 \oplus A_2) + (B_1 \oplus B_2) = A + C$ and $A \cap C = (A_1 \oplus A_2) \cap (B_1 \oplus B_2) = (A_1 \cap B_1) \oplus (A_2 \cap B_2) \subseteq RadN_1 \oplus RadN_2 = RadN$ by Lemma 1.3(2). Thus any cofinite closed submodule $A \subseteq_{cc} N$ has a wrs $C = B_1 \oplus B_2$ in N, and hence N is $ccwrs-$module. □

Remark 2.18. The homomorphic image of $CS-$modules is not necessarily be CS ([1], Example 2.3). Thus we conclude that homomorphic image of $ccwrs-$module need not be $ccwrs$. Now we, investigate the conditions which ensure that the property of module being $ccwrs$ is inherited by homomorphic image.

A right $R-$module M is relatively $c-$Rickart to a right $R-$module N, if $Kerg \subseteq_c M$ for every homomorphism $g : M \longrightarrow N$. Every simple and semisimple modules are relatively $c-$Rickart.

Corollary 2.19. *If a fg module N is $ccwrs$ and relatively $c-$Rickart to a module P, then homomorphic image of g is $ccwrs$ for any module homomorphism $g : N \longrightarrow P$.*

Proof. Let $g : N \longrightarrow P$ be module homomorphism, then by assumptions $Kerg \subseteq_c N$. Applying Remark 2.3(1), we get $N/Kerg \cong Img$ is $ccwrs-$module. □

A right $R-$module N is small cover of a right $R-$module Q if there exists a small epimorphism $g : N \longrightarrow Q$, i.e., $Kerg \ll N$.

Proposition 2.20. *Let P be a $ccwrs-$module and $g : N \longrightarrow P$ be a small epimorphism. If each $0 \neq A \subseteq_{cc} N$ contains $Kerg$, then N is $ccwrs-$module.*

Proof. Let $0 \neq A \subseteq_{cc} N$ and suppose that $g(A) \subseteq_{cc} B \subseteq P$, as $g : N \longrightarrow P$ is a small epimorphism, so $A = A + Kerg = g^{-1}(g(A)) \subseteq_c g^{-1}(B)$. Hence $A = g^{-1}(B)$ and consequently $g(A) = B \subseteq_{cc} P$. By assumption P is $ccwrs$, so $g(A)$ has a wrs in P. Using Lemma 2.6 [11], A has a wrs in N i.e., N is $ccwrs-$module. □

Theorem 2.21. *Every non-singular homomorphic image of a $ccwrs-$module N is again a $ccwrs$.*

Proof. Assume that $g : N \longrightarrow P$ be a module epimorphism and $P = Img$ is a non-singular module. Let, $A \subseteq_{cc} P$, then by Lemma 1.1(3), $B = g^{-1}(A) \subseteq_{cc} N$. Since N is $ccwrs$ then there exists a wrs C of B in N such that $N = B + C$ and $B \cap C \subseteq RadN$. Hence we can write $P = g(N) = g(B) + g(C) = A + g(C)$ and $A \cap g(C) = g(B) \cap g(C) = g(B \cap C)$. Since $Kerg = g^{-1}(0) \subseteq B$, then by Lemma 1.2(3) and Lemma 1.3(3) we get $A \cap g(C) = g(B) \cap g(C) = g(B \cap C) \subseteq g(RadN) \subseteq RadP$. Thus $g(C)$ is wrs of A in P and hence module $P = Img$ is $ccwrs-$module. □

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