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AP-INJECTIVE MODULES

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Abstract We introduce the notion of almost pseudo-injective modules (AP-injective modules), which generalizes the idea of pseudo-injective and almost injective modules. In support, we provide some examples. Here, we investigate properties of almost pseudo-injective modules related to uniform modules. For any two uniform right R-modules M and N, M is AP-N-injective if and only if for every monomorphism $f : E(N) \to E(M)$, either $f(N) \subseteq M$ or $f^{-1}(M) \subseteq N$.

1 Introduction

In [4], Y. Baba introduced the idea of almost injectivity. Recall, let M and N be two right R-modules, M is said to be *almost N-injective* if, for every submodule K of N and every homomorphism $f : K \to M$, either there exists $g : N \to M$ such that $f = g \circ i$ or there exist a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$ and a homomorphism $h : M \to N_1$ such that $h \circ f = \pi \circ i$, where $i : K \to N$ is an inclusion and $\pi : N \to N_1$ is a projection onto N_1. If M is almost M-injective, then M is called *almost self-injective*.

Recall from [5], a right R-module M is called *N-pseudo injective* if, every monomorphism from a submodule of N to M can be extended to a homomorphism from N to M. If M is M-pseudo injective, then M is called *pseudo-injective*.

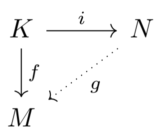
By the motivation, we introduce the idea of AP-injective and SAP-injective modules. We call a right R-module M *AP-injective* if, M is AP-M-injective. If M is AP-N-injective for all right R-modules N, M is called *SAP-injective*.

The class of AP-injective modules is bigger than the classes of almost self-injective and pseudo-injective modules, in support, we give Example 2.3 and 2.4. We observe that every semi-simple module is AP-injective but the converse need not be true. We find some new properties of AP-injective modules which are not analogous to pseudo-injective modules, for example the C_2 condition. We find a condition under which a submodule of an AP-injective module is AP-injective. In Theorem 3.2, we give a necessary and sufficient condition for a uniform module to be AP-N-injective. In Proposition 3.3, we prove that a uniform AP-injective module is co-Hopfian module. The endomorphism ring of a uniform AP-injective module is always a local ring Proposition 3.4.

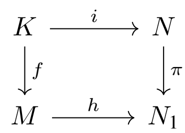
Throughout, we consider every ring R to be an associative ring with identity and every module a unitary right R-module. For a right R-module M, we denote $E(M)$ and $\subseteq^{\oplus}$ for the injective hull and direct summand, respectively. For all undefined facts and notions, we refer to [2].

2 Properties of AP-injective modules

Definition 2.1. For any two right R-modules M and N, we call M an *almost pseudo N-injective* if, for every submodule K of N and every monomorphism $f : K \to M$, either there exists $g : N \to M$ such that the diagram (1) commutes, or there exists $h : M \to N_1$ such that the diagram (2) commutes, where N_1 is nonzero direct summand of N with canonical projection $\pi : N \to N_1$. We denote it by *AP-N-injective*.



(1)



(2)

We call M *almost pseudo-injective* if, M is almost pseudo M-injective and denoted by *AP-injective*.

Definition 2.2. If M is AP-N-injective for all right R-modules N, then M is called *strongly almost pseudo-injective* and denoted by *SAP-injective*. M and N are called *mutually relative almost pseudo-injective* if, M is almost pseudo N-injective and N is almost pseudo M-injective.

Example 2.3. Every almost self-injective module is AP-injective. But the converse is not necessarily true. For example, suppose M is a right R-module whose only submodules are 0, N_1, N_2 and $N_1 \oplus N_2$, where $N_1 \ncong N_2$ and $End(N_i) \cong \mathbb{Z}_2$ such modules exist (see [7, Lemma 2]). By [7, Lemma 2], M is a pseudo-injective module but not quasi-injective. Therefore, M is AP-injective but not quasi-injective. If possible, assume that M is almost self-injective. We observe that M is indecomposable and a map $j_1 \circ \pi$ is not monomorphism, where $\pi : N_1 \oplus N_2 \to N_1$ is the projection and $j_1 : N_1 \to M$ is an injection map. It follows by the proof of [7, Lemma 2] that the map $j_1 \circ \pi$ can not be extended to M. But by [6], it should be extended to M. This is a contradiction to the fact that M is almost self-injective.

Example 2.4. Every pseudo-injective module is AP-injective but the converse need not be true. For example, let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. By [3, Remark 3.6], it follows that all R-modules are almost injective. Therefore, R as R-module is almost injective and hence it is AP-injective. By Example [9, Example 2.9], R_R does not satisfy C_3 condition. Since C_2 implies C_3 condition and a pseudo-injective module satisfies C_2 condition (see [5, Theorem 2.6]), it follows that R_R does not satisfy C_2 condition and hence R_R is not pseudo-injective.

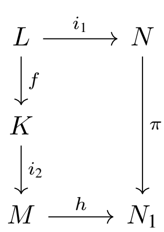
There are some properties of AP-injective modules which are not analogous to pseudo-injective modules, for example the C_2 condition.

Example 2.5. Every semi-simple module is AP-injective. The converse is not necessarily true. In support, consider one example of $\mathbb{Q}$ as $\mathbb{Z}$-module.

In the following, we find a sufficient condition for a submodule of an AP-N-injective module to be AP-N-injective.

Proposition 2.6. *Let M be an AP-N-injective module. Let L be a submodule of N such that any monomorphism from L to M can not be extended to N. Then every submodule of M is AP-N-injective.*

Proof. Let K be a submodule of M, $L \leq N$ and $f : L \to K$ be a monomorphism. Let $i_2 : K \to M$ be an injection. Clearly, $i_2 \circ f$ is a monomorphism from L to M. By assumption, it has no extension from N to M. But M is AP-N-injective so there exists a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$ and an R-homomorphism $h : M \to N_1$ such that $(h \circ i_2) \circ f = \pi \circ i_1$, where π is a natural projection from N to N_1. Therefore K is AP-N-injective.



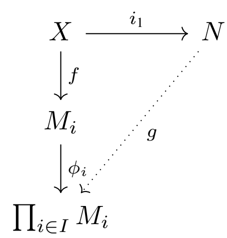
□

Theorem 2.7. *Let M, N and K be any right R-modules.*

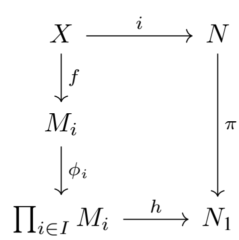
(i) Let $\{M_i : i \in I\}$ be a family of right R-modules. Let $\prod_{i\in I} M_i$ be the direct product of AP-N-injective right R-modules. Then M_i is AP-N-injective for all $i \in I$.

(ii) Let $M \cong N$. Then M is AP-K-injective if and only if N is AP-K-injective.

Proof. (i) Let $f : X \to M_i$ be a monomorphism where X is a submodule of N and $\phi_i : M_i \to \prod_{i\in I} M_i$ be the natural injection. Consider the following diagram:



Clearly, $\phi_i \circ f$ is a monomorphism. Since $\prod_{i\in I} M_i$ is AP-N-injective, if diagram (1) holds, then $\phi_i \circ f$ extends to N i.e. there exists $g : N \to \prod_{i\in I} M_i$ such that $\phi_i \circ f = g$ on X. Now consider the homomorphism $\pi_i \circ g : N \to M_i$, where $\pi_i : \prod_{i\in I} M_i \to M_i$ is the natural projection. For $x \in X$, $(\pi_i \circ g)(i(x)) = (\pi_i \circ \phi_i)(f(x)) = Id_{M_i}(f(x)) = f(x)$. Thus each M_i is AP-N-injective. If diagram (2) holds, then there exists a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$, and an R-homomorphism $h : \prod_{i\in I} M_i \to N_1$ such that $(h \circ \phi_i)(f(x)) = \pi(i(x))$, where $\pi : N \to N_1$ is projection with kernel N_2. Therefore each M_i is AP-N-injective.



(ii). Clear.

□

Corollary 2.8. *Any direct summand of an AP-N-injective module M is AP-N-injective.*

3 Uniform AP-injective modules

We know that every almost self-injective module is AP-injective, but the converse need not be true (see Example 2.3). In the following, we give a sufficient condition for an AP-injective module to be almost self-injective.

Proposition 3.1. *Every uniform nonsingular AP-injective right R-module is almost self-injective.*

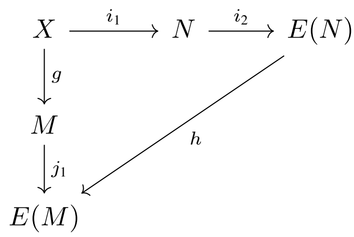
Proof. Let M be an AP-injective right R-module. If M satisfies diagram (1), then M is pseudo-injective. By [5, Theorem 3.1(a)], M is quasi-injective and hence almost self-injective. Now suppose that M satisfies diagram (2). Let L be a submodule of M and f be a homomorphism from L to M. Then by [5, Theorem 3.1(a)], f is a trivial homomorphism or a monomorphism. If f is a monomorphism, by assumption, there exists a homomorphism $g : M \to M$ such that $(g \circ f)(x) = x$, $\forall x \in L$. It follows that M is almost self-injective. □

Theorem 3.2. *Let M and N be any two uniform right R-modules. Then the following are equivalent:*

(i) M is AP-N-injective.

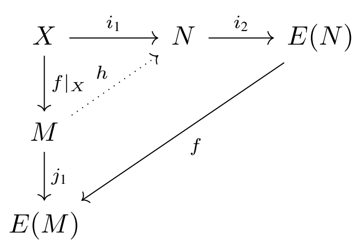
(ii) For every monomorphism $f : E(N) \to E(M)$, either $f(N) \subseteq M$ or $f^{-1}(M) \subseteq N$.

Proof. Assume condition (ii). Let g be a monomorphism from X to M, where X is any submodule of N. Consider the following diagram:



Here, $j_1 : M \to E(M)$ is a natural injection. Since $E(M)$ is injective, there exists a homomorphism $h : E(N) \to E(M)$, which extends $j_1 \circ g$. We claim that h is a monomorphism. Let $x \in X \cap kerh$. Then $x \in X$ and $h(x) = 0$. Since $j_1 \circ g = h$ on X, for any $x \in X$, $(j_1 \circ g)(x) = 0$. So, $g(x) = 0$. This implies that $x = 0$ because g is a monomorphism. Thus $X \cap kerh = 0$. Since $E(N)$ is uniform, $kerh = 0$. Thus h is a monomorphism. Since $E(N)$ is injective and $E(M)$ indecomposable, h is an isomorphism. Also, by assumption, either $h(N) \subseteq M$ or $h^{-1}(M) \subseteq N$. If $h(N) \subseteq M$, then diagram (1) holds. If $h^{-1}(M) \subseteq N$, $h^{-1} \circ j_1 \circ g = i_1$ on X and so diagram (2) holds. It follows that M is AP-N-injective.

Conversely, assume condition (i) that M is AP-N-injective. Let $f : E(N) \to E(M)$ be any monomorphism. Since $E(N)$ is injective and $E(M)$ indecomposable, f is an isomorphism. Let $X = \{n \in N \mid f(n) \in M\}$. Then X is a submodule of N and $f|_X : X \to M$ is a monomorphism as $f : E(N) \to E(M)$ is an isomorphism. If diagram (1) holds, M is N-pseudo injective. By [5, Proposition 2.1(5)], it follows that $f(N) \subseteq M$. If diagram (2) holds, consider the following:



Now, by above diagram, we have $f^{-1}(M) \subseteq N$ whose proof is similar to [1, Proposition 2]. □

Recall [11], a module M is said to be Hopfian (resp. co-Hopfian) if for every surjective (resp. injective) homomorphism $f : M \to M$ is an isomorphism.

Proposition 3.3. *Let M be a uniform AP-injective module. Then M is a co-Hopfian module.*

Proof. Let $f : M \to M$ be any injective homomorphism and $I_M : M \to M$ be the identity map. We have to show that f is an isomorphism.

Case (i). Since M is AP-injective, assume that it satisfies diagram (1). Then, there exists a homomorphism $g : M \to M$ such that $I_M = g \circ f$. Clearly, g is a surjective homomorphism. We show that g is injective also. Let $x \in kerg \cap Imf$. Then, $x \in kerg$ and $x \in Imf$. So $g(x) = 0$ and $x = f(y)$ for some $y \in M$. Hence $y = I_M(y) = (g \circ f)(y) = g(f(y)) = g(x) = 0$. This implies that $x = f(y) = 0$. Thus, $kerg \cap Imf = 0$. Since M is uniform and Imf is nonzero as f is injective, we have $kerg = 0$. Therefore, g is injective and hence an isomorphism. It follows that $f = g^{-1} \circ I_M$ is an isomorphism.

Case (ii). If M satisfies diagram (2), there exists a homomorphism $g : M \to M$ such that $g \circ f = I_M$. This implies that g is surjective but g is also injective by assumption. So, g is an isomorphism and therefore f is an isomorphism. □

Recall [8], a nonzero ring R is said to be *local ring* if R has a unique maximal left ideal or, equivalently, if R has unique maximal right ideal.

Proposition 3.4. *Let M be a uniform AP-injective module. Then the endomorphism ring $End(M)$ is a local ring.*

Proof. Let $f \in End(M)$. If $ker f = 0$, then by Proposition 3.3, f is an isomorphism and so, f is invertible. Now, suppose that $ker f \neq 0$. Then, we show that $ker(I_M - f) = 0$ and so, $I_M - f$ is invertible by Proposition 3.3. Let $x \in ker f \cap ker(I_M - f)$. Then $f(x) = 0$ and $(I_M - f)(x) = 0$, which implies that $x = f(x) = 0$. So, $ker f \cap ker(I_M - f) = 0$. Since M is uniform and $ker f \neq 0$, we have $ker(I_M - f) = 0$. Thus, we have shown that either f or $I_M - f$ is invertible. So by [2, Proposition 15.15], $End(M)$ is a local ring. □

In the following, we generalize [4, lemma A] and the proof is analogous.

Proposition 3.5. *Let M be a uniform module and N be an indecomposable module. Let M be AP-N-injective and composition length of M is greater than or equal to the composition length of N. Then M is pseudo N-injective.*

Recall [10], let M and N be two modules, M is called *essentially pseudo N-injective* if for any essential submodule A of N, any monomorphism $f : A \to M$ can be extended to some $g \in Hom(N, M)$. Also module M is called *essentially pseudo-injective* if, M is essentially pseudo M-injective.

Example 3.6. Let N be a uniform right R-module. Then every essentially pseudo N-injective module is AP-N-injective.

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