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SOME RESULTS ON PURE PROJECTIVE MODULES

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Abstract We explore few novel results on direct sum and direct summand of pure projective (PP) modules and prove Schanuel's lemma in PP-modules over commutative ring. In this manuscript, we also discuss some interesting examples on PP-modules.

1 Introduction

A pure projective (PP) module was first introduced by Warfield [21] in 1969. A PP-module is defined to be direct summand of the direct sum of finitely presented modules. PP-modules can be observed as the generalization of projective modules. PP-modules play a vital role in module and ring theory. In particular, finitely generated projective modules appear naturally in many contexts such as Morita theory, homological algebra, tilting theory, and K-theory for example. The notion of pure projective resolution and dimension of modules were studied by Simson [20]. So far structure decompositions, sub-projectivity domain, orthogonal class, positive constructibility, dimension, global dimension, and applications of pure projective modules over various rings have been studied in [2], [3], [5], [6], [8], [9], [11], [12] and [17]. In [2], [5], and [17], we see various decompositions of a PP-module over several rings (one-sided perfect ring, Dedekind prime ring, Dedekind prime exchange ring, chain domain, etc) as the direct sum of finitely presented modules. Chelliah [3] studied coresolutions and dimensions of a right orthogonal class of PP-modules over a hereditary ring. Kucera [12] proved the result when a PP-module is +constructible and vice versa. Also, some applications are discussed in [5], and [9]. Dehkordi [6] investigated the rings where every ideal becomes pure projective and the answer was left pure hereditary ring. Jensen [11] determined the pure global dimension of rings which are regular and local, with uncountable residue class fields, and also the pure global dimension of domains with Krull dimension one.

Durgan[8] discussed properties of sub-projectivity domains of PP-modules and pure projective indigent and proved the existence of pp-indigent modules for numerous categories of modules. For example: cyclic, simple, singular, and finitely generated modules. And also explained the class of Noetherian ring over which every (cyclic, simple, finitely generated and singular) PP-module is projective or pure projective indigent and concluded that the set of flat modules is the smallest possible sub-projectivity domain of a PP-module. Alagoz [1] defined the pure sub-projectivity domain of module M and studied its properties. Clearly, all pure projective modules are included in the pure sub-projectivity domains. In 2012, Puninski[18] stated that over an exceptional chain ring, any PP-module is constructed inimitably using its dimension and classified these modules over an arbitrary exceptional chain non-coherent ring R.

Motivated by the above research work and literature review, in this article we set up some new interesting outcomes on PP-modules. In Section 3, we establish and prove some results on direct sum and direct summand of PP-modules. In Section 4, we prove Schanuel's lemma in PP-modules over a commutative ring. Section 2 is devoted to some preliminaries. We conclude this paper with some remarks which were observed during the literature survey. R represents a commutative ring with unity throughout the paper.

2 Preliminaries

Definition 2.1. [7] An exact sequence $0 \to M^{'} \to M \to M^{''} \to 0$ of R-modules is *pure exact* if, for every R-module M_1, sequence $0 \to M^{'} \otimes_R M_1 \to M \otimes_R M_1 \to M^{''} \otimes_R M_1 \to 0$ is also exact.

Definition 2.2. [15] A module M over ring R is said to be *finitely presented (f.p.)* if there exists an exact sequence $0 \to N \to F \to M \to 0$ with F (free) and N *finitely generated (f.g.)*.

Example 2.3. $\mathbb{Z}_n$ and $\mathbb{Z}$ are f.p. $\mathbb{Z}$-modules.

Definition 2.4. [19] A module is known to be a *PP-module* if it is projective with respect to pure exact sequence.

Example 2.5. (i) Abelian groups over ring of integers are PP-module.

(ii) All right Noetherian serial rings are PP-module.

(iii) Every *f.p.* module is PP-module.

Remark 2.6. An R-module M is called *stably free* if $M \oplus R^m \approx R^{n+m}$, for some $n, k \in \mathbb{Z}^+$ and M is *f.g.* R-module. Every stably free module is pure projective.

Example 2.7. Let $(x_1, x_2, ..., x_n)$ be a unimodular row in R^n, defining $f : R_n \to R$ by $f(e_i) = x_i$ and letting $P_1 = ker(f)$. Then $P_1 \oplus R \in R^n$, implying P_1 is stably free. Hence, pure projective.

Definition 2.8. [16] A submodule X of a module M over R is said to be *small* if for any submodule Y of M, $X + Y = M$ implies that $Y = M$.

Remark 2.9. [16] An R-module M_1 is *d-complement* of an R-module M_2 if and only if $M_1 \cap M_2$ is small in M_1.

Definition 2.10. [16] A module epimorphism $\phi : M_1 \to M_2$ is *direct* if there exists a homomorphism $\psi : M_2 \to M_1$ such that $\phi o \psi = Id_{M_2}$.

Corollary 2.11. *[13] If C is a f.p. right R-module, then an exact sequence*

$$0 \to A \to B \to C \to 0$$

of R-modules, is pure iff it splits.

Proposition 2.12. *[15] If $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ is a split exact sequence with $t : M_3 \to M_2$ a splitting, then*

$$M_2 = f(M_1) \oplus t(M_3) \simeq M_1 \oplus M_3.$$

Proposition 2.13. *[10] Let P be a projective module over R, then P/PI is cyclic PP-module for every ideal I of R.*

Proposition 2.14. *[10] Let R be a right Noetherian ring then every cyclic pure projective R-module is PP-module.*

3 Direct Sum of Pure Projective Modules

Theorem 3.1. *Direct sum of PP-modules $\oplus_{i=1}^{n} \Sigma_i P_i$ is pure projective if and only if every P_i is pure projective for $1 \leq i \leq n, n \in \mathbb{N}$.*

Proof. Let P_1 and P_2 be two PP-modules over R. Then there exist R-modules Q_1, Q_2 and finitely presented modules M_1, M_2, M_3 and M_4 such that,

$$P_1 \oplus Q_1 = M_1 \oplus M_2$$

and

$$P_2 \oplus Q_2 = M_3 \oplus M_4.$$

From above we can write

$$P_1 \oplus Q_1 \oplus P_2 \oplus Q_2 = M_1 \oplus M_2 \oplus M_3 \oplus M_4,$$

$$(P_1 \oplus P_2) \oplus Q = \oplus \Sigma_{i=1}^{4} M_i,$$

where $Q = Q_1 \oplus Q_2$.

This shows that $P_1 \oplus P_2$ is a direct summand of direct sum of *f.p.* modules. Hence, direct sum of PP-modules is also pure projective.

Conversely, let $P_1 \oplus P_2$ be PP-module over R. Then there exist an R-module Q and finitely presented modules M_1 and M_2 such that,

$$P_1 \oplus P_2 \oplus Q = M_1 \oplus M_2$$

$$P_1 \oplus (P_2 \oplus Q) = M_1 \oplus M_2$$

$$P_2 \oplus (P_1 \oplus Q) = M_1 \oplus M_2$$

From above, it's clear that P_1 and P_2 both are direct summand of direct sum of *f.p.* modules. Hence P_1 and P_2 both are PP-modules. In similar manner by principle of mathematical induction, we can prove that $\oplus_{i=1}^{n} \Sigma_i P_i$ is pure projective if and only if every P_i is PP-module for $1 \leq i \leq n, n \in \mathbb{N}$. □

Remark 3.2. Theorem 3.1 shows that direct summand of a pure projective module is also pure projective.

Corollary 3.3. *Let A and B be two R-modules and B is finitely presented. Then a sufficient condition for an epimorphism $\alpha : A \to B$ to be direct is that $A \oplus B$ is pure projective.*

Proof. Let $A \oplus B$ be PP-module. Then A and B both are pure projective by Theorem 3.1 and given $\alpha : A \to B$ is an epimorphism,
therefore we can construct a pure exact sequence

$$0 \to Ker(\alpha) \to A \xrightarrow{\alpha} B \to 0 \tag{3.1}$$

Since sequence (3.1) is pure and B is finitely presented, exact sequence (3.1) splits.[13]
So we get a homomorphism $\beta : B \to A$ such that $\alpha o \beta = Id_B$.
Hence, α is a direct epimorphism. □

Corollary 3.4. *Let A be a PP-module and $\alpha : A \to B$ is an epimorphism, then B is PP-module iff $A \oplus B$ is pure projective.*

Proof. It can be easily seen that this proof follows from Theorem 3.1. □

Theorem 3.5. *Let $M = A + B$ be a pure projective module, where A and B are d-compliments of each other. Then $M = A \oplus B$.*

Proof. Let $M = A + B$, we can construct an exact sequence with an epimorphism $\beta : M \to B$ and a monomorphism $\alpha : A \to M$ as follows

$$0 \to A \xrightarrow{\alpha} M \xrightarrow{\beta} B \to 0 \tag{3.2}$$

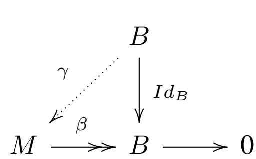
Now, $Id_B : B \to B$ can be lifted to a map γ,

$$\gamma : B \to M$$

with

$$\beta o \gamma = Id_B.$$

This can be seen by the following diagram



This implies that above exact sequence (3.2) splits. Thus, we can write

$$M = \alpha(A) \oplus \gamma(B)$$

Since $\alpha(A) \subseteq A$ and A is d-compliment. Also $\gamma(B) \subseteq B$ and B is a d-compliment. We can write

$$M = A \oplus B.$$

□

Remark 3.6. We can easily check that Theorem 3.5 is true for any module $M = A + B$, where A and B are d-compliments of each other.

Corollary 3.7. *In a pure projective module, d-compliments are also pure projective.*

Proof. Let M be a PP-module and A, B be its d-compliment submodules. Then, by Theorem 3.5, $M = A \oplus B$ and by Theorem 3.1, we can say A and B both are pure projective as well. It follows the proof of corollary. □

4 Schanuel's Lemma in Pure Projective Modules

In the study of projective modules, we see a lemma related to the equivalence of two modules N_1 and N_2, provided two projective modules P_1 and P_2. Then $N_1 \oplus P_2$ is isomorphic to $P_1 \oplus N_2$. This is known as Schanuel's lemma in projective modules. As we discussed earlier, pure projective modules are the general case of projective modules. So here we will generalize this lemma for pure projective modules.

Lemma 4.1. *Let*

$$0 \to N_1 \xrightarrow{\alpha_1} P_1 \xrightarrow{\beta_1} M \to 0, \tag{4.1}$$

$$0 \to N_2 \xrightarrow{\alpha_2} P_2 \xrightarrow{\beta_2} M \to 0, \tag{4.2}$$

be two exact sequences with P_1, P_2 pure projective. Then $N_1 \oplus P_2 \simeq P_1 \oplus N_2$.

Proof. From P_1 and P_2 being pure projective, we have $P_1 \oplus P_2$ also pure projective by Theorem 3.1. Let's form

$$A = \{(p_1, p_2) \in P_1 \oplus P_2;\ p_1 \in P_1 \ and\ p_2 \in P_2 | \beta_1(p_1) = \beta_2(p_2)\}.$$

Since, $(0,0) \in A$, $A \neq \emptyset$. Also, let $(a_1, a_2), (b_1, b_2) \in A$ and $r \in R$, we can see that

$$\beta_1(a_1 + b_1) = \beta_2(a_2 + b_2),$$

$$\Rightarrow (a_1 + b_1, a_2 + b_2) \in A,$$
$$\Rightarrow (a_1, a_2) + (b_1, b_2) \in A,$$

and

$$\beta_1(a_1.r) = \beta_1(a_1).r = \beta_2(a_2).r = \beta_2(a_2.r)$$
$$\Rightarrow (a_1 r, a_2 r) \in A$$
$$\Rightarrow (a_1, a_2).r \in A$$

This shows that A is a submodule of $P_1 \oplus P_2$. Pure projective modules are closed under submodules. Hence A is also pure projective.
Next, we have β_1 is surjective (epimorphism) (by exactness of (4.2)) $\Rightarrow M = \beta_1(P_1)$.
Since β_2 is also surjective, there exists $p_2 \in P_2$ for each $\beta_1(p_1) \in M$ such that

$$\beta_1(p_1) = \beta_2(p_2)$$

.

Let's define a homomorphism (natural projection map)

$$\phi_1 : A \to P_1$$

with

$$\phi_1(p_1, p_2) = p_1.$$

Then

$$\begin{aligned} ker(\phi_1) &= \{(p_1, p_2) \in A | \phi_1(p_1, p_2) = 0\} \\ &= \{(p_1, p_2) \in A | p_1 = 0\} \\ &= \{(0, p_2) \in A | \beta_2(p_2) = 0\} \\ &\simeq ker(\beta_2) \\ &= Im(\alpha_2) \\ &\simeq N_2 \end{aligned}$$

As α_2 is monomorphism (from (4.2)).

Now, a short exact sequence can be constructed as follows

$$0 \to N_2 \to A \xrightarrow{\phi_1} P_1 \to 0 \tag{4.3}$$

Since, P_1 and A both are pure projective, $Id_{P_1} : P_1 \to P_1$ can be lifted to a map ψ_1,

$$\psi_1 : P_1 \to A$$

with

$$\phi_1 \psi_1 = Id_{P_1}$$

thus (4.3) splits.

$$\Rightarrow A \simeq N_2 \oplus P_1$$

In analogous way, we can form another short exact sequence

$$0 \to N_1 \to A \xrightarrow{\phi_2} P_2 \to 0 \tag{4.4}$$

and get

$$A \simeq N_1 \oplus P_2$$

therefore,

$$N_2 \oplus P_1 \simeq N_1 \oplus P_2.$$

Hence Schanuel's lemma is proved for pure projective modules. □

We conclude this paper with some enthralling facts about projectivity and pure projectivity of modules:

Remark 4.2. Let R be a Noetherian ring and $A = R[x]$. Let P be a projective A-module, then P/mP is pure projective for any ideal m in A.

Proof. Let P be a projective module over A. P/mP will be a projective $\frac{R}{m}[x]$-module. Since R is commutative, R/m is commutative. Hence $\frac{R}{m}[x]$ is commutative. This gives P/mP is cyclic pure projective. Also, R is Noetherian. Then polynomial ring $A = R[x]$ is Noethirian, resulting P/mP is pure projective for any ideal m in A. □

Remark 4.3. By the definition of projectivity and pure projectivity we can observe that every projective module is PP-module. But interestingly, in general its converse is not true.

Example 4.4. Abelian groups over ring of integers are pure projective but not projective since they are not free. Infact, only free abelian group over ring of integers is $\mathbb{Z}$ itself.

Remark 4.5. Let R be a principal ideal domain (PID) and M be a torsion free *f.g.* module over R. Then M is free. Particulary, f.g. projective modules over R are free. But, if M is a pure projective R-module then it need not to be free.

Example 4.6. Abelian groups over ring of integers are pure projective but not torsion free, hence not free.

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