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ON FUZZY PROJECTIVE MODULES - ITS MODIFIED CLASSES AND ESTIMATION OF PROJECTIVE DIMENSION

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Abstract Thee current research focuses on two key concepts: fuzzy quasi projectivity as a generalization of fuzzy projectivity and projective dimension estimation. We also touch on the concept of μ_A projective modules and utilize them to characterize fuzzy quasi projective modules while examining the fuzzy characteristics of quasi projective modules. In addition, we introduce and construct a methodology and algorithm for determining the projective dimension of a fuzzy module specified in terms of possible projective resolution lengths in this paper, which is supported by suitable examples. The projective dimension of a module indicates how far it is from being a projective module.

1 Introduction

Projective Modules were first unveiled by Cartan and Eilenberg[2]. Following this Banaschewski[1] and many other researchers expeditiously worked towards the generalization of projective and injective modules resulting in new concepts like quasi projective modules, pseudo projective-injective modules, pseudo semi-projective modules and small pseudo projective modules to mention a few. In 1972, Goldie[5] introduced the concept of finite Goldie dimension in modules. In the light of the foregoing, dimension theory became the center of interest and was discussed by numerous researchers like Yenumula [19], Satyanarayana [15], Prasad, and Nagaraju [17]. Numerous remarkable works on projective dimension over various fascinating rings, for example, Weyl algebra, polynomial ring, and Laurent polynomial ring have been studied and analyzed in [18], [6], [3] and [12]. In 1965 when the concept of fuzzy came into existence, extensive research was carried out to apply the concept to various algebraic structures, which resulted in fuzzy submodules, projective L - module, fuzzy projectivity and injectivity of modules [20], to name a few. A pivotal extension was made when the fuzzy dimension was examined by Satyanarayana [16] in terms of fuzzy pseudo basis. Here in this paper, our aim is to generalize a few prevalent consequences of homological algebra and to establish a theory for finding the projective dimension of a fuzzy module designated in the language of feasible lengths of projective resolutions and the corresponding work is affirmed through suitable examples. In addition, we looked into a key feature of fuzzy split short exact sequences and fuzzy quasi projective modules. The present work can further make room for the fuzzy version of the Auslander-Buchsbaum formula, which connects the projective dimension to the depth of a module by stating that they are complementary to each other. This allows one to solve the fuzzy version of the recognition theorem for Cohen- Macaulay ring [[4], corollary19.10]. The projective dimension calculated over the fuzzy polynomial ring here can give a new twist to one of the early results of homological algebra namely the Hilbert syzygy theorem. And, the fuzzy projective module worked on, can open windows to the concepts such as intuitionistic fuzzy modules, fuzzy factor rings, fuzzy hom, and tensor functors which play a vital role in module theory and fuzzy module categories.
Apart from the above, the projective dimension can be used to calculate the global dimension and Betti numbers of a fuzzy module.

2 Preliminaries

Terminology, definitions and results applied during the present study are discussed below.

(i) R is a ring with identity.

(ii) $_RM$ and M_R denotes the left and right R – module respectively for each module M.

(iii) Fuzzy module over the module M is denoted as μ_M.

(iv) $R[x^{[0,1]}]$ means fuzzy polynomial ring.

(v) For the ring of integers Z, $\mu: \ Z \to [0,1]$ written as $\mu_Z(x)$ means the fuzzy set over the ring of integers.

(vi) pd(M) is projective dimension of the module M.

(vii) $< a >$ is an ideal generated by the element "a".

Definition 2.1. [6] *Let R be a ring, μ be a fuzzy ideal of R and η be a fuzzy subring of R. We define a fuzzy set on R/μ called as fuzzy factor ring of η with respect to μ as follows :*
$\eta/\mu : R/\mu \to [0,1]$ and $\frac{\eta}{\mu}(a+\mu) = \underset{x+\mu=a+\mu}{sup}\eta(x)$, for all $x \in R$.

Definition 2.2. [10] *A fuzzy subset μ_M is called fuzzy submodule of module M if following conditions are satisfied :*

(i) $\mu(m+n) \geq min\{\mu(m), \mu(n)\}$

(ii) $\mu(xm) \geq \mu(m)$, for all $m, n \in M$ and $x \in R$

(iii) $\mu(-x) = \mu(x)$ for all $x \in M$

(iv) $\mu(0) = 1$

Definition 2.3. [11] *A fuzzy R-module μ_P is said to be projective subject to the condition that for every surjective fuzzy R-homomorphism $\bar{f}$: $\mu_A \to \mu_B$ and every fuzzy R-homomorphism $\bar{g}$: $\mu_P \to \mu_B$ there is always a fuzzy R-homomorphism $\bar{h}$: $\mu_P \to \mu_A$ such that figure 1 commutes.*

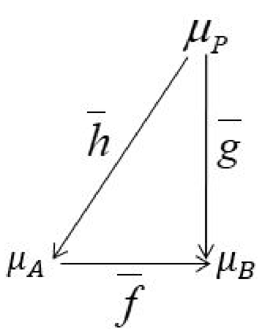

FIGURE 1. Fuzzy Projective Module

NOTE : In definition 2.3 the fuzzy R-module μ_P can also be called μ_A-projective or projective relative to μ_A. Also, the same can be termed as fuzzy quasi projective module if it is μ_P-projective.

Definition 2.4. [11] *A fuzzy R-homomorphism $\bar{\alpha} \in$ Hom(μ_A, ν_B) is termed as fuzzy split, if there is a fuzzy R-homomorphism $\bar{\beta} \in$ Hom(ν_B, μ_A)such that the composition $\bar{\alpha}\bar{\beta}$ = 1_{ν_B}.*

Definition 2.5. [20] *The sequence* "... $\to \mu_{n-1_{\lambda_{n-1}}} \xrightarrow{\bar{f_{n-1}}} \mu_{n_{\lambda_n}} \xrightarrow{\bar{f_n}} \mu_{n+1_{\lambda_{n+1}}} \to$ ..." is termed as fuzzy exact subject to Im$\bar{f_{n-1}}$= Ker$\bar{f_n}$ for every single n also here Im$\bar{f_{n-1}}$ along with Ker$\bar{f_n}$ stands for μ_n | Imf_{n-1} and μ_n | Kerf_n separately.

Definition 2.6. [20] *The exact sequence of the form* $0 \to \mu_A \xrightarrow{f} \eta_B \xrightarrow{g} \nu_B \to 0$ *is called as the fuzzy short exact sequence.*

Definition 2.7. [9] *Let* $[0, 1]$ *be a closed interval of the real line. Let R be a commutative ring with 1 or the field of reals. The fuzzy polynomial ring in the variable x with coefficients from R denoted by* $R[x^{[0,1]}]$ *consists of elements of the form* $a_0 + a_1x^{\gamma_1} + a_2x^{\gamma_2} + ... + a_nx^{\gamma_n}$ *where* a_0, a_1, a_2, ..., $a_n \in R$ *and* γ_1, γ_2, γ_3, ..., $\gamma_n \in [0, 1]$, $\gamma_1 < \gamma_2 < \gamma_3 < .... < \gamma_n$.

Lemma 2.8. *[12][Lemma 2.10] Let* A_1, A_2, A_3 *be R-modules and* $0 \to A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \to 0$ *be a split short exact sequence. Suppose* γ_1 *and* γ_2 *are the splittings corresponding to* α_1 *and* α_2 *respectively . Then the following sequence is an exact sequence* $0 \to A_3 \xrightarrow{\gamma_2} A_2 \xrightarrow{\gamma_1} A_1 \to 0$.

Example 2.9. [20][Example 2.6] *Let* $\bar{f}: \mu_M \to \eta_N$ *be a fuzzy homomorphism. Then the fuzzy sequence* $0 \to ker\bar{f} \xrightarrow{\bar{i}} \mu_M \xrightarrow{\bar{f}} \eta_N \xrightarrow{\bar{g}} coker\bar{f} \to 0$ *is exact where* $\bar{i}$ *is inclusion map and* $\bar{g}$ *is canonical map.*

Lemma 2.10. *[20][Lemma 2.11]. Let* $\bar{g}'$: $\eta_C \to \rho_B$ *be a fuzzy splitting for the fuzzy short exact sequence* $0 \to \mu_A \xrightarrow{\bar{f}} \rho_B \xrightarrow{\bar{g}} \eta_C \to 0$ *of a fuzzy R-modules. Then* $\rho_B \cong \mu_A \oplus \eta_C$.

Theorem 2.11. *[20][Theorem 3.4]. For a is fuzzy R- module* θ_P *following are identical :*
(i) θ_P *is projective*
(ii) The induced sequence of homomorphisms
$\bar{0} \to hom_R(\theta_P, \mu_A) \xrightarrow{\bar{f}_*} hom_R(\theta_P, \rho_B) \xrightarrow{\bar{g}_*} hom_R(\theta_P, \eta_C) \to \bar{0}$ *for every short exact sequence of fuzzy modules*
$\bar{0} \to \mu_A \xrightarrow{\bar{f}} \rho_B \xrightarrow{\bar{g}} \eta_C \to \bar{0}$, *is exact.*
(iii) If $\bar{\alpha}: \rho_B \to \theta_p$ *is a fuzzy epimorphism then there exists a fuzzy homomorphism* $\bar{\phi}: \theta_p \to \rho_B$ *such that* $\bar{\alpha}\bar{\phi} = \bar{1}_{\theta_p}$
(iv) If θ_P *is the fuzzy homomorphic image of a fuzzy module* μ_A *then* θ_P *is the fuzzy direct summund of* μ_A
(v) θ_P *is fuzzy direct summund of a fuzzy free R-module.*

Theorem 2.12. *[7][Theorem 3.4]. Every free L-module is a projective L-module.*

3 Modified Class Of Fuzzy Projective Module

Definition 3.1. [8] μ_M *is said to be simple fuzzy left module if it has no proper sub modules*

Definition 3.2. [8]μ_M *is said to be semi-simple fuzzy left module if whenever for* ν_N, *a strictly proper fuzzy submodule of* μ_M *there exist a strictly proper fuzzy submodule* η_P *of* μ_M *such that* $\mu_M = \nu_N \oplus \eta_P$

NOTE : A ring is said to be semi-simple if, every left-module over it is semi-simple.

Definition 3.3. [20] *A fuzzy submodule* μ_T *of* ν_Q *is said to be fuzzy pseudo stable if whenever for* $\bar{g}, \bar{h}: \nu_Q \to \eta_A$ *epimorphisms such that* $\mu_T \subseteq Ker\bar{g} \bigcap Ker\bar{h}$. *There exists* $\bar{f} \in End(\nu_Q)$ *with* $\bar{g} = \bar{h}\ o\ \bar{f}$ *then* $\bar{f}(\mu_T) \subseteq \mu_T$.

Example 3.4. Fuzzy projective module is an example of μ_P-projective module.

Lemma 3.5. *Let* μ_1, η, μ_2 *be the fuzzy R- modules over* A_1, A_2, A_3 *respectively and* $\bar{0} \to \mu_1 \xrightarrow{\bar{\alpha_1}} \eta \xrightarrow{\bar{\alpha_2}} \mu_2 \to \bar{0}$ *be a fuzzy split short exact sequence. Where* $\bar{\gamma_1}$ *and* $\bar{\gamma_2}$ *are the fuzzy splittings of* $\bar{\alpha_1}$ *and* $\bar{\alpha_2}$, *then* $\bar{0} \to \mu_2 \xrightarrow{\bar{\gamma_2}} \eta \xrightarrow{\bar{\gamma_1}} \mu_1 \to \bar{0}$ *is exact.*

Proof. For the above we need to prove $Im\bar{\gamma_2} = ker\bar{\gamma_1}$. Now by definition we have $Im\bar{\gamma_2} = \eta \mid Im\gamma_2[x]$ which is equal to $\eta[x]$ for all $x \in Im\gamma_2$.
Also, $ker\bar{\gamma_1} = \eta \mid ker\gamma_1[y]$ which equals η[y] for all $y \in ker\gamma_1$.

Then from Lemma 2.8, we have *ker*γ_1 = *Im*γ_2. Therefore, *Im*$\bar{\gamma}_2$ = *ker*$\bar{\gamma}_1$ and hence the given sequence is a fuzzy exact.□

NOTE : The following lemma is already available in [20]. However, the proof of the same has been proved in a much interesting and unique way in [13].

Lemma 3.6. *For a fuzzy module θ_P following are indistinguishable :*
(i) θ_P is fuzzy projective
(ii) Every short exact sequence $\bar{0} \to \mu_A \xrightarrow{\bar{f}} \mu_B \xrightarrow{\bar{g}} \theta_p \to \bar{0}$ splits.
(iii) θ_P is a direct summund of a free fuzzy R-module.□

Lemma 3.7. *Let η_P be projective and ν_Q quasiprojective. Then a sufficient condition for an exact sequence $\bar{0} \to \mu_K \to \eta_P \xrightarrow{\bar{g}} \nu_Q \to \bar{0}$ to split is that $\eta_P \oplus \nu_Q$ be fuzzy quasi projective.*

Proof. Let $\eta_P \oplus \nu_Q$ is quasiprojective and $\bar{\alpha}$, $\bar{\beta}$: $\eta_P \oplus \nu_Q \to \nu_Q$ by $\bar{\alpha}\{(\eta(p), \nu(q)\} = \nu(q)$, $\bar{\beta}\{(\eta(p), \nu(q)\}$ = $\eta(p)\lambda$. By quasiprojectivity there exists an endomorphism $\bar{\theta}$ of $\eta_P \oplus \nu_Q$ such that figure 2 commute.

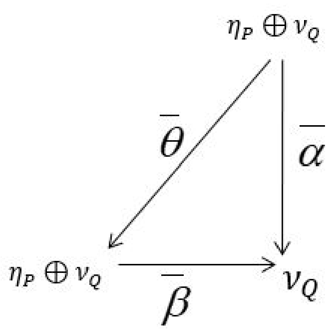

FIGURE 2. Quasiprojectivity in $\eta_P \oplus \nu_Q$

Define $\bar{\gamma}$: $\nu_Q \to \eta_P$ by $\nu(q)\bar{\gamma} = (0, \nu(q))\bar{\theta}\bar{\pi}$ where $\bar{\pi}$: $\eta_P \oplus \nu_Q \to \eta_P$ is the canonical map. Then for all $\nu(q) \in \nu_Q$, $\nu(q)\bar{\gamma}\bar{\lambda} = (0, \nu(q))\bar{\theta}\bar{\pi}\bar{\lambda}$ = (0, $\nu(q))\bar{\theta}\bar{\beta}$ = (0, $\nu(q))\bar{\alpha})$ = $\nu(q)$. Thus, $\bar{\gamma}\bar{\lambda}$ is identity on ν_Q. Hence, the equation splits.□

Corollary 3.8. *A sufficient condition for R to be semisimple is that R $\oplus$ μ_M be quasiprojective for every simple module μ_M.*

Proof. If μ_M is simple then there exists an exact sequence $\bar{0} \to \mu_K \to \eta_P \to \nu_Q \to \bar{0}$ which splits by lemma 3.6. As a result, every fuzzy simple module is fuzzy projective, implying that R is semi simple. □

Lemma 3.9. *Quotient of a fuzzy pseudo stable submodule of a fuzzy quasi projective module is a fuzzy pseudo stable submodule. Particularly, if μ_T is fuzzy pseudo stable submodule of a fuzzy quasi projective module η_Q and $\nu_A \nsubseteq \mu_T$ then μ_T/ν_A is a fuzzy pseudo stable submodule of η_Q /ν_A.*

Proof. Let $\bar{\lambda}, \bar{\phi}$: $\eta_Q/\nu_A \to \pi_B$ be epimorphism with $\mu_T/\nu_A \subseteq Ker\bar{\lambda} \bigcap Ker\bar{\phi}$ such that there exists $\bar{\theta} \in$ End (η_Q/ν_A) satisfying $\bar{\theta}$ o $\bar{\phi}$ = $\bar{\lambda}$. Let $\bar{\psi}$: $\eta_Q \to \eta_Q/\nu_A$ be the natural epimorphism, then since η_Q is fuzzy quasi projective there exists a homomorphism $\bar{\tau} \in$ End(η_Q) such that

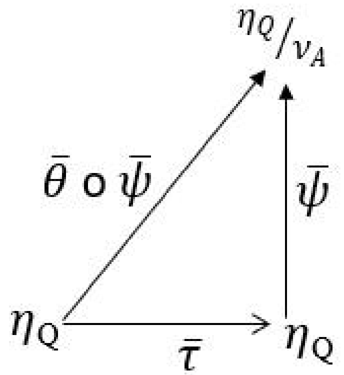

FIGURE 3. Commutes as $\bar{\theta}$ o $\bar{\psi}$ = $\bar{\psi}$ o $\bar{\tau}$

$\bar{\theta}$ o $\bar{\psi}$ = $\bar{\psi}$ o $\bar{\tau}$ which implies $\bar{\lambda}$ o $\bar{\psi}$ = $\bar{\theta}$ o $\bar{\phi}$ o $\bar{\psi}$ = $\bar{\phi}$ o $\bar{\psi}$ o $\bar{\tau}$ which proves that the fig 3 commutes.
Also, $\bar{\lambda}$ o $\bar{\psi}$ (μ_T) = $\bar{\lambda}(\mu_T/\nu_A)$ = 0........(1)
and
$\bar{\phi}$ o $\bar{\psi}$ (μ_T) = $\bar{\phi}(\mu_T/\nu_A)$ = 0..........(2)

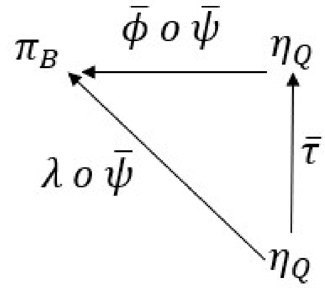

FIGURE 4. Commutativity using $\bar{\lambda}$ o $\bar{\psi}$

(1) and (2) implies $\mu_T \subseteq Ker(\bar{\lambda}$ o $\bar{\psi}) \bigcap Ker(\bar{\phi}$ o $\bar{\psi})$. Hence, $\bar{\tau}(\mu_T)$ = μ_T. It follows that $\bar{\theta}(\mu_T/\nu_A)$ = $\bar{\theta}$ o $\bar{\psi}(\mu_T)$ = $\bar{\psi}$ o $\bar{\tau}(\mu_T) \subseteq \bar{\psi}(\mu_T)$ = $\mu_T/\nu_A)$. Thus, $\mu_T/\nu_A \subseteq \eta_Q/\nu_A$ is fuzzy pseudo stable submodule.□

Proposition 3.10. *Any fuzzy R module μ_P can also be called μ_A projective if and only if for any given diagram of fuzzy R-modules and R-homomorphism, where the row is fuzzy exact and $\bar{g}$ $\bar{h}$ = 0 then there exists $\bar{\phi}$: $\mu_P \to \mu_A$ such that $\bar{f}$ $\bar{\phi}$ = $\bar{h}$.*

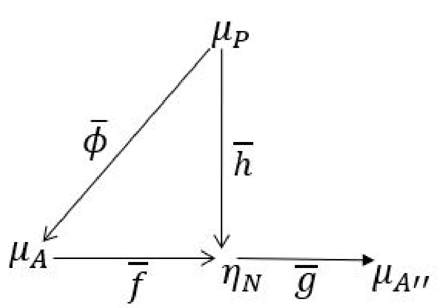

FIGURE 5. Fuzzy exact row of fuzzy R-modules and R-homomorphisms

Proof. Let μ_P can also be called μ_A projective. Since $\bar{g}$ $\bar{h}$ = 0 we have *Im* $\bar{h} \subseteq$ *Ker* $\bar{g}$ = *Im* $\bar{f}$. Let $\bar{f}'$: $\mu_A \to Im\bar{f}$ and $\bar{h}'$: $\mu_P \to Im\bar{f}$ be the R-homomorphisms induced by $\bar{f}$ and $\bar{h}$ respectively. Then $\bar{f}^{'}$ is an epimorphism and we can have figure 6 below;

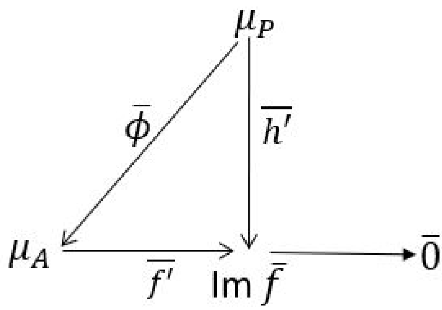

FIGURE 6. μ_P is μ_A projective

in which the row is exact. We find $\bar{\phi}$: $\mu_P \to \mu_A$ such that $\bar{f}^{'}\ \bar{\phi} = \bar{h}^{'}$ it is clear that $\bar{i}\bar{f}^{'}\ \bar{\phi} = \bar{i}\bar{h}^{'}$ where $\bar{i}^{'}$: $Im\bar{f} \to \eta_N$ be the canonical inclusion map. Thus, $\bar{f}\ \bar{\phi} = \bar{h}$. Conversely if we put $\mu_{A''}$ = 0, in figure 6 we can easily say the conditions are satisfied. Hence, μ_P is μ_A projective.□

Corollary 3.11. *Any fuzzy R module μ_P will be fuzzy quasi projective if and only if for any given diagram of fuzzy R-modules and R-homomorphism, where the row is fuzzy exact and $\bar{g}$. $\bar{h}$ = 0 then there exists $\bar{\phi}$: $\mu_P \to \mu_P$ such that $\bar{f}\ \bar{\phi} = \bar{h}$.*

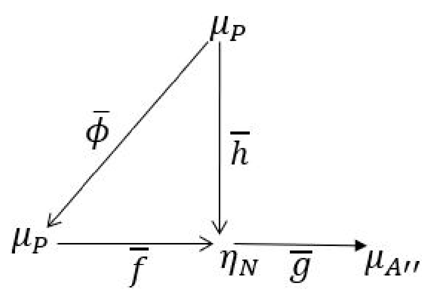

FIGURE 7. Fuzzy exact row of fuzzy R-modules and R-homomorphisms

4 Projective Dimension Of A Fuzzy Module

Let R be a ring and μ_M be a fuzzy finitely generated R - module. An fuzzy exact sequence .... $\to$ $\mu_{n-1}\xrightarrow{\bar{f_{n-1}}}\mu_n\xrightarrow{\bar{f_n}}\mu_{n+1}$.... $\to$ with only fuzzy free(resp. projective) modules μ_i [i = 0,1,2,...] is termed as free(resp. projective) resolution of μ_M. The minimum length of which, is called as projective dimension of μ_M.

4.1 Procedure for Projective Dimension

STEP 1 - Let μ be a fuzzy subset on a ring R. Define a fuzzy ring and a fuzzy factor ring quotient by the fuzzy ideal.
STEP 2- For a given fuzzy module η_p choose a free fuzzy module μ_A and define a fuzzy epimorphism $\bar{\phi}$: $\mu_A \to \eta_p$, which results in the following fuzzy exact sequence $\bar{0} \to ker\bar{\phi} \xrightarrow{\bar{\psi}} \mu_A \xrightarrow{\bar{\phi}} \eta_p \to \bar{0}$ [by Example 2.9]. Now if $ker\bar{\phi} \neq 0$, then define a fuzzy epimorphism say $\bar{\phi}_1$: $\mu_B \to ker\bar{\phi}$ with $ker\bar{\phi}_1 \neq 0$ where μ_B be a free fuzzy module generated by $ker\bar{\phi}$. Next in sequence choose μ_C such that $\bar{\phi}_2$:$\mu_C \to ker\bar{\phi}_1$ with $ker\bar{\phi}_2 \neq 0$ resulting in an exact sequence of the form : $\bar{0} \to ker\bar{\phi}_2 \xrightarrow{\bar{\psi}_2} \mu_C \xrightarrow{\bar{\phi}_2} ker\bar{\phi}_1 \xrightarrow{\bar{\psi}_1} \mu_B \xrightarrow{\bar{\phi}_1} ker\bar{\phi} \xrightarrow{\bar{\psi}} \mu_A \xrightarrow{\bar{\phi}} \eta_p \to \bar{0}$ proceeding in the same way we have the following fuzzy exact sequence :
$\bar{0} \to \mu_K \xrightarrow{\bar{\alpha}_k} \mu_{K-1} \xrightarrow{\bar{\alpha}_k - 1}$ ....$\mu_B \xrightarrow{\bar{\alpha}_1} \mu_A \xrightarrow{\bar{\alpha}_0} \eta_p \to \bar{0}$...........(1)
STEP 3- For K = 1 we have $\bar{0} \to \mu_B \xrightarrow{\bar{\alpha}_1} \mu_A \xrightarrow{\bar{\alpha}_0} \eta_p \to \bar{0}$..........(2)

as a free resolution of a fuzzy module η_p. By Lemma 3.6, equation (2) splits as η_p is fuzzy projective, also there exists $\bar{\gamma}_1 : \mu_A \rightarrow \mu_B$ such that $\bar{\gamma}_1\bar{\alpha}_1 = I_{\mu_B}$.

STEP 4 - Using Lemma 3.5, we can construct $\bar{0} \rightarrow \eta_p \xrightarrow{\bar{\gamma}_0} \mu_A \xrightarrow{\bar{\gamma}_1} \mu_B \rightarrow \bar{0}$ where $\bar{\gamma}_0$ and $\bar{\gamma}_1$ are the splittings of $\bar{\alpha}_0$ and $\bar{\alpha}_1$ respectively, it implies $\mu_B \oplus \eta_p = \mu_A$ and is $\cong ker\bar{\gamma}_1 \oplus \mu_B$ by Lemma 2.10. Since $\bar{\alpha}_0 : \mu_A \rightarrow \eta_p$ is a fuzzy epimorphism, sequence $\bar{0} \rightarrow ker\bar{\alpha}_0 \xrightarrow{\bar{\alpha}_1} \mu_A \xrightarrow{\bar{\alpha}_0} \eta_p \rightarrow \bar{0}$........(3) is fuzzy short exact. Since η_p is projective equation (3) splits, implying $ker\bar{\alpha}_0 = Im\bar{\alpha}_1$ is fuzzy projective and hence pd(η_p) = 0.

STEP 5- Suppose η_p is not fuzzy projective then equation (3), does not split therefore $ker\bar{\alpha}_0 = Im\bar{\alpha}_1$ is not fuzzy projective. For K = 2, $ker\bar{\alpha}_1 = Im\bar{\alpha}_2$ is not fuzzy projective since the sequence (1) does not split. Taking this further $Im\bar{\alpha}_1$, $Im\bar{\alpha}_2$ ....$Im\bar{\alpha}_k$ are not fuzzy projective, but $Im\bar{\alpha}_{k+1}$ is fuzzy projective. Similarly, after definite number of steps the minimal length of free resolution of η_p is obtained. And the same is termed as the "PROJECTIVE DIMENSION" of η_p with pd(η_p) = K.□

Example 4.1. Let $\eta_z(x)$ be the fuzzy subset over the ring of integers Z, then it is defined as

$$\eta_z(x) = \begin{cases} 0.8, \text{if } x = 0 \\ 0.1, \text{if } x \neq 0 \end{cases}$$

whenever we want to emphasis on the role of a ring, we write $\eta_z(x)$ is a fuzzy subring of Z. Now let us define a fuzzy ideal $\mu_I : \eta_z(x) \rightarrow [0, 1]$ on it as

$$\mu_I = \begin{cases} 0.6, \text{if } x = 0 \\ 0.2, \text{if } x \neq 0 \end{cases}$$

then the fuzzy factor ring $\frac{\eta_z}{\mu_I} : \frac{Z}{\mu_I} \rightarrow [0, 1]$ will be $\frac{Z}{\mu_I}(x + \mu_I) = \underset{x+\mu_I=a+\mu_I}{sup} \eta_Z(x)$ Let us now define a mapping $\phi : \eta_z \rightarrow \frac{\eta_z}{\mu_I}$ as $\phi(x) = \mu_I + x$ for all
$x \in \eta_z$ with kerϕ = $x \in \eta_z : \phi(x) = \mu_I$ then by associating a probability mapping with this kerϕ, the fuzzy kernel will be

$$ker\phi_\psi = \begin{cases} 0.6, \text{if } \phi(x) = (\mu_I) \\ 0, \text{otherwise} \end{cases}$$

where $\psi : ker\phi \rightarrow [0, 1]$ defined as $\psi(x) = x$ for all $x \in$ kerϕ, resulting in the number of elements in kerϕ_ψ = 2.
Now choose $\phi_1 : \eta_{Z_2} \rightarrow \eta_Z$ with

$$ker\phi_{1_{\psi_1}} = \begin{cases} 0.8, \text{if } \phi(x) = (\mu_I) \\ 0, \text{otherwise} \end{cases}$$

where $\psi_1 : ker\phi_1 \rightarrow [0, 1]$ defined as $\psi_1(x_1) = x_1$ with $\phi_1(x_1) = \{\mu_I\}$. Thus the number of elements in $ker\phi_{1_{\psi_1}} = 2$. Similarly define $\phi_2 : \eta_{Z_2} \rightarrow \eta_{Z_2}$ with $ker\phi_{2_{\psi_2}}$ where $\psi_2 : ker\phi_2 \rightarrow [0, 1]$ is $\psi_2(x_2) = x_2$ implying the number of elements in the fuzzy kernel are 0. As a result of which the chain stops and the projective dimension is 2. The above can be represented using the following diagram:□

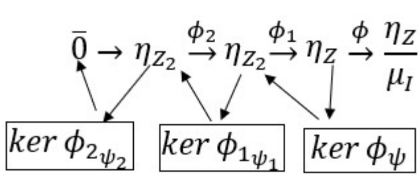

FIGURE 8. Chain for estimating projective dimension

Example 4.2. Let Z$[x^{[0,1]}]$ be the fuzzy polynomial ring and
$\mu(x) : Z[x^{[0,1]}] \to [0, 1]$ be the fuzzy set on this fuzzy polynomial ring defined as

$$\mu(x) = \begin{cases} 0.8, if < x^{\gamma_1} > \\ 1, if < x^{\gamma_2} > \\ 0.5 elsewhere \end{cases}$$

where γ_1 is equal to digit after the decimal in γ_1 is even and γ_2 is equal to digit after the decimal in γ_2 is odd. Then this μ will be fuzzy ideal of the above mentioned ring. Define a mapping say $\phi : Z[x^{[0,1]}] \to \frac{Z[x^{[0,1]}]}{\mu(x)}$, fuzzy kernel of which is $\mu(x)$ so the chain extends to
$Z_3[x^{[0,1]}] \xrightarrow{\phi_1} Z[x^{[0,1]}] \xrightarrow{\phi} \frac{Z[x^{[0,1]}]}{\mu(x)}$ where $\phi_1 : Z_3[x^{[0,1]}] \to Z[x^{[0,1]}]$ is defined as ϕ_1(p(x)) = p(x), which implies the number of elements in the kernel of this fuzzy polynomial ring is one. Thus, the projective dimension of the above will be 1.□

Now we present an algorithm to find out projective dimension of a fuzzy module over a fuzzy ring.

4.2 ALGORITHM

Let $\eta(x) \cong \frac{\eta}{\mu}$
Objective : To find out the projective dimension of a fuzzy module.
Input: A fuzzy Module η_p for a positive integer p.
Output: Projective dimension of η_p such that $\bar{0} \to \mu_K \xrightarrow{\bar{\alpha}_k} \mu_{K-1} \xrightarrow{\bar{\alpha}_k - 1}$ ....$\mu_B \xrightarrow{\bar{\alpha}_1} \mu_A \xrightarrow{\bar{\alpha}_0}$ $\eta_p \to \bar{0}$ is a fuzzy free resolution of η_p and $\bar{\alpha}_i$ are the fuzzy homomorphisms between fuzzy modules.
If pd(η_p)= 0 then η_p is fuzzy projective.
The algorithm returns $\mu_B \oplus \eta_p \cong \mu_A$.
START
initialize for $i = 1$
if ($\bar{\alpha}_1$ does not split)
$\bar{\alpha}_1$ = map ($\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$,......, $\bar{\alpha}_m$)
= pd(η_p) = 1
else
Let $\bar{\beta}_1$ be the fuzzy split of $\bar{\alpha}_1$ then pd(η_p) = 0
and $\mu_B \oplus \eta_p \cong \mu_A \cong ker\bar{\beta}_1 \oplus \mu_B$
end if
for $i = m$
if $\bar{\alpha}_1$ does not split
pd$(\eta_p) = m$
else
let $\bar{\beta}_l$ be the fuzzy split of $\bar{\alpha}_l$
then pd$(\eta_p) = m - 1$
end if
return pd(η_p).
STOP□

5 CONCLUSION AND FUTURE SCOPE

The concept of a fuzzy module's projective dimension is presented simply in this work along with a study of the features of fuzzy quasi projective modules, which can be used to generalize fuzzy pseudo projective modules. Furthermore, the research offered here is of high quality and can steer one in the right direction for constructing a long exact sequence of fuzzy modules using fuzzy connecting homomorphisms. It is a useful tool in homological algebra and has important applications in other fields also like algebraic topology. Aside from the aforementioned, this

research has paved the path for future research in the following areas:
(i) Characterize global and Goldie's dimensions in terms of fuzzy pseudo projective-injective and small pseudo projective modules. Further more, across Gorenstein rings, the global dimension can be extended to the Gorenstein dimension.
(ii) Spanning dimension of fuzzy modules
(iii) Corank of fuzzy modules.
(iv) Fuzzy lifting modules with chain condition on fuzzy small submod ules
(v) The exact sequences and fuzzy splitness studied here can be com bined to produce complexes of long exact sequences and fuzzy proper exact sequences
(vi) [14] and the current study can be integrated to create an algo rithm and a technique for evaluating dimension with fuzzy projective semimodules.

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