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Approximation of Volterra and Fredholm Integral Equations by utilizing Non-dyadic Haar wavelets

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Abstract In this research article, the non-dyadic Haar wavelet collocation method is introduced for approximating the solution of integral equations of Volterra and Fredholm type. The unknown function involved in the integral equation is approximated by the truncated series of non-dyadic Haar wavelets. To explain the supremacy of the presented method, we compare the obtained results with those available in the literature. Different errors have been calculated at collocation points.

1 Introduction

Integral equations arise in various fields of science and engineering and find their applications in conformal mapping, Volterra's population growth model, water waves, diffraction problem, scattering in quantum mechanics [1], filtration theory, queuing theory, biomechanics, electrical engineering, approximation theory, heat and mass transfer, actuarial science, etc. [2]. These equations are usually difficult to solve analytically, so approximation methods are needed for finding their approximate solution. A variety of methods are available in the literature for solving integral equations including the Toeplitz matrix method [3], CAS wavelets [4], Chebyshev wavelet [5], Legendre wavelets [6], Gaussian radial basis function [7], Wavelet moment method [8], Triangular orthogonal functions [9], Two-dimensional Legendre wavelets method [10], Discrete Adomian decomposition [11], Iteration method [12], Walsh function [13], operational matrix of RH wavelet [14], Modified homotopy perturbation method [15], Finite difference method [16], Hermite cubic splines [17]. To determine the approximate solution of integral equations various researchers used different basis functions such as wavelets and orthogonal functions. Wavelet basis is used in a variety of fields of science and engineering to effectively approximate the solution of a large number of problems. A number of researchers used Gegenbauer and Bernoulli wavelets [18], Haar wavelets [19], Legendre wavelets [20], Hermite wavelets [21], and Jacobi wavelets [22] for solving differential equations. The Volterra integral equations and Fredholm integral equations have been solved in the literature using various wavelet bases in connection with a variety of collocation techniques. In [27] the author solved the Volterra integral equation by using the two-dimensional Haar wavelets. The mixed Volterra Fredholm integral equations involving the delay term has been solved by using Haar wavelets in [28]. The nondyadic haar wavelet is used in literature for solving Fisher Kolmogorov Petrovsky Equation [24], integrodifferential equations [31] and dispersive equation [32]. In this manuscript, we have approximated the solution of integral equations of both Volterra and Fredholm types by utilizing nondyadic Haar wavelets. The rest of this article is organized as: section 2 contains a brief idea about nondyadic Haar wavelets and their integrals. Section 3 contains the proposed nondyadic Haar wavelet collocation technique. In section 4 some examples from already published research articles are analyzed by the presented technique along with their errors. The conclusion is given in section 5 of the paper.

2 Non-Dyadic Haar Wavelet

In a dyadic Haar wavelet, the whole wavelet family is generated by only one mother wavelet but in the case of a non-dyadic Haar wavelet, the wavelet family is generated by two mother wavelets having different shapes and different characteristics. Standard representation of non-dyadic Haar wavelet is presented here [25]
Haar scaling Function

$$h_1(z) = \begin{cases} 1; & 0 \leq z < 1 \\ 0; & elsewhere \end{cases} \tag{2.1}$$

Symmetric Haar wavelet

$$h_i(z) = \varphi^1(3^j z - k) = \frac{1}{\sqrt{2}} \begin{cases} -1; & \alpha_1 \leq z < \alpha_2 \\ 2; & \alpha_2 \leq z < \alpha_3 \\ -1; & \alpha_3 \leq z < \alpha_4 \\ 0; & elsewhere \end{cases} \quad for\ \ even\ \ wavelet\ \ numbers\ \ i. \tag{2.2}$$

Anti-Symmetric Haar wavelet

$$h_i(z) = \varphi^2(3^j z - k) = \sqrt{\frac{3}{2}} \begin{cases} 1; & \alpha_1 \leq z < \alpha_2 \\ 0; & \alpha_2 \leq z < \alpha_3 \\ -1; & \alpha_3 \leq z < \alpha_4 \\ 0; & elsewhere \end{cases} \quad for\ \ odd\ \ wavelet\ \ numbers\ \ i. \tag{2.3}$$

where the values of parameters α_1, α_2, α_3, and α_4 are given as follows:

$$\alpha_1 = \frac{k}{p}\ ,\ \alpha_2 = \frac{3k+1}{3p}\ ,\ \alpha_3 = \frac{3k+2}{3p}\ ,\ \alpha_4 = \frac{k+1}{p}$$

j is called the dilation factor and the value of j varies from $j = 0, 1, 2, ...$ The translation parameter in the wavelet family is represented by k, and the value of $k = 0, 1, 2, ..., p-1$, and $p = 3^j$. i is the wavelet number and the values of the wavelet number i, is calculated from two different mathematical relations which are given as follows:

$$i - 1 = 3^j + 2k\ (for\ i = 2, 4, 6, ...3p - 1.)$$

$$i - 2 = 3^j + 2k\ (for\ i = 3, 5, 7, ...3p.)$$

Using the aforementioned formulae for varying j and k , the extended wavelet family would be produced, where $h_2(z)$ and $h_3(z)$ are referred to as mother wavelets and the other wavelets generated from these two mother wavelets are referred to as daughter wavelets.

With dyadic Haar wavelet, only a single mother wavelet is responsible for generating all sequential wavelets. However in nondyadic, two mothers with significantly different wavelet structures are responsible for creating the wavelet family, which helps to increase the rate at which solutions converge.

With the formula that is presented below, the integration of the equation $[(2.1) - (2.3)]$ can be performed effortlessly and rapidly throughout the interval $[c, d]$ the desired number of times.

$$\begin{aligned} q_{\delta,i}(z) &= \int_c^z \int_c^z \int_c^z ...\ \delta\ \ times\ ... \int_c^z h_i(x)(dx)^\delta = \frac{1}{(\delta-1)!} \int_c^z (z-x)^{(\delta-1)} h_i(x) dx \\ &= \frac{1}{\Gamma(\delta)} \int_c^z (z-x)^{(\delta-1)} h_i(x) dx \end{aligned} \tag{2.4}$$

$$where\ \ \delta = 1, 2, 3, ...\ and\ i = 1, 2, 3, ..., 3p.$$

After obtaining the abovementioned integrals,

$$q_{\delta,i}(z) = \frac{z^{\delta}}{\Gamma(\delta+1)}, for \;\; i = 1. \tag{2.5}$$

$$q_{\delta,i}(z) = \frac{1}{\sqrt{2}} \begin{cases} 0; \;\; 0 \le z < \alpha_1 \\ \frac{-1}{\Gamma(\delta+1)}(z-\alpha_1)^{\delta}; \;\; \alpha_1 \le z < \alpha_2 \\ \frac{1}{\Gamma(\delta+1)}[-(z-\alpha_1)^{\delta} + 3(z-\alpha_2)^{\delta}]; \;\; \alpha_2 \le z < \alpha_3 \\ \frac{1}{\Gamma(\delta+1)}[-(z-\alpha_1)^{\delta} + 3(z-\alpha_2)^{\delta} - 3(z-\alpha_3)^{\delta}]; \;\; \alpha_3 \le z < \alpha_4 \\ \frac{1}{\Gamma(\delta+1)}[-(z-\alpha_1)^{\delta} + 3(z-\alpha_2)^{\delta} - 3(z-\alpha_3)^{\delta} + (z-\alpha_4)^{\delta}]; \;\; \alpha_4 \le z < 1 \end{cases}$$

$$for \; i = 2, 4, 6, ..., 3p-1. \tag{2.6}$$

$$q_{\delta,i}(z) = \sqrt{\frac{3}{2}} \begin{cases} 0; \;\; 0 \le z < \alpha_1 \\ \frac{1}{\Gamma(\delta+1)}(z-\alpha_1)^{\delta}; \;\; \alpha_1 \le z < \alpha_2 \\ \frac{1}{\Gamma(\delta+1)}[(z-\alpha_1)^{\delta} - (z-\alpha_2)^{\delta}]; \;\; \alpha_2 \le z < \alpha_3 \\ \frac{1}{\Gamma(\delta+1)}[(z-\alpha_1)^{\delta} - (z-\alpha_2)^{\delta} - (z-\alpha_3)^{\delta}]; \;\; \alpha_3 \le z < \alpha_4 \\ \frac{1}{\Gamma(\delta+1)}[(z-\alpha_1)^{\delta} - (z-\alpha_2)^{\delta} - (z-\alpha_3(i))^{\delta} + (z-\alpha_4)^{\delta}]; \;\; \alpha_4 \le z < 1 \end{cases}$$

$$for \; i = 3, 5, 7, ..., 3p. \tag{2.7}$$

The collocation point for the interval $[c, d]$ in the nondyadic Haar wavelet collocation technique is determined by the following relation:

$$z_m = c + (d-c)\frac{m - \frac{1}{2}}{3p}; \quad m = 1, 2, 3, ..., 3p. \tag{2.8}$$

3 Non-Dyadic Haar wavelet Collocation Method

In this section, a numerical method has been designed by utilizing non-dyadic Haar wavelets for approximating the solutions of variety of integral equations.
1. Nonhomogenous linear Volterra integral equations of second kind

$$u(z) = g(z) + \int_0^z w(z,t)u(t)dt \tag{3.1}$$

2. Nonhomogenous linear Fredholm integral equations of second kind

$$u(z) = g(z) + \int_0^1 w(z,t)u(t)dt \tag{3.2}$$

3. Nonhomogenous nonlinear mixed Volterra Fredholm hammerstein integral equations of second kind

$$u(z) = g(z) + + \int_0^1 w(z,t)u(t)^n dt + \int_0^z w(z,t)u(t)^n dt \quad where \; n \ge 2 \tag{3.3}$$

4. Nonhomogenous nonlinear Fredholm integral equations of second kind

$$u(z) = g(z) + \int_0^1 w(z,t)u(t)^n dt \quad where \; n \ge 2 \tag{3.4}$$

here $g(z)$ and the kernel function $w(z,t)$ are known function, we have to calculate the unknown function $u(z)$. non-dyadic Haar wavelet collocation approach has been introduced for the interval $[0, 1)$.The unknown function is approximated by the truncated series of non-dyadic Haar functions and then integrals are calculated by the process of integration.

3.1 Approximation of solution

Using the characteristics of non dyadic Haar wavelets, any member of $l_2(R)$ can be expressed as follows:

$$u(z) = \sum_{i=0}^{\infty} c_i h_i(z) = c_1 h_1(z) + \sum_{eveni} c_i \varphi^1(3^j z - k) + \sum_{oddi} c_i \varphi^2(3^j z - k) \tag{3.5}$$

here c_i is the unknown wavelet coefficients, that will be calculated by the proposed method. By considering only finite $3p$ terms for computation.

$$u(z) = u_{3p}(z) = \sum_{i=0}^{3p} c_i h_i(z) \tag{3.6}$$

Now, for approximating the solution of equation (3.1)-(3.4) consider

$$u(z) = \sum_{i=1}^{3p} c_i h_i(z) \tag{3.7}$$

Integrating (3.7) both sides from 0 to z

$$\int_0^z u(z)dz = \sum_{i=1}^{3p} c_i P_{i,1}(z); \quad where\ P_{i,1}(z) = \int_0^z h_i(z)dz \tag{3.8}$$

Again integrating (3.8) from 0 to z

$$\int_0^z \int_0^z u(z)dzdz = \sum_{i=1}^{3p} c_i P_{i,2}(z); \quad where\ P_{i,2}(z) = \int_0^z P_{i,1}(z)dz \tag{3.9}$$

and so on. By repeating this process and making the required substitutions to the given integral equations, as well as substituting the collocation points from equation (2.8), one obtains a $N \times N$ system of algebraic equations that can be solved by any sequential iterative technique. In order to solve this set of linear equations, we used the Gauss - Jordan technique. As a result of solving this system, we obtain the unknown Haar coefficients. The solution at the collocation points can be determined by substituting the corresponding Haar coefficients c_i's into Eq.(3.7). The method is explained in detail for example 2 of the paper.

4 Numerical Examples

For checking the accuracy of the method, we implemented the method to different examples and the results obtained by this method are compared with previous results. The maximum Absolute error, $l_2 - error$, $E_{max} - error$, and $l_\infty - error$ has been calculated for checking the accuracy of the presented algorithm by using the MATLAB software. Where u_{ap} is the approximate and u_{ex} is the exact solution at different collocation points z_m.

$$l_2 - error = \frac{\sqrt{\sum_{i=1}^{3p} |u_{ex}(z_m) - u_{ap}(z_m)|^2}}{\sum_{i=1}^{3p} |u_{ex}(z_m)|^2}, \quad E_{max} - error = \sqrt{\sum_{i=1}^{3p} |u_{ex}(z_m) - u_{ap}(z_m)|^2}$$

$$l_\infty - error = max|u_{ex}(z_m) - u_{ap}(z_m)|, \quad Absolute\ \ error = |u_{ex}(z_m) - u_{ap}(z_m)|$$

Example 1

Consider the second kind of Fredholm integral equation [23]

$$u(z) = 0.9z^2 + \int_0^z z^2 t^2 u(t)\ dt \tag{4.1}$$

z	Analytical solution	Approximate solution	Absolute Error
0.055555556	0.003086420	0.003083488	2.93E-03
0.166666667	0.027777778	0.027751394	2.64E-02
0.277777778	0.077160494	0.077087206	7.33E-02
0.388888889	0.151234568	0.151090923	1.44E-01
0.500000000	0.250000000	0.249762546	2.37E-01
0.611111111	0.373456790	0.373102076	3.55E-01
0.722222222	0.521604938	0.521109511	4.95E-01
0.833333333	0.694444444	0.693784851	6.60E-01
0.944444444	0.891975309	0.891128098	8.47E-01

Table 1. Computation of exact and approximated solution for example 1

J	$l_2 - error$	$l_\infty - error$	$E_{max} - error$	$E_{max} - error$ [26]
0	8.35E-03	5.80E-03	6.17E-03	———
1	9.50E-04	8.47E-04	1.27E-03	———
2	1.06E-04	1.02E-04	2.46E-04	———
3	1.18E-05	1.16E-05	4.73E-05	8.23E-04
4	1.31E-06	1.30E-06	9.11E-06	1.92E-04
5	1.45E-07	1.45E-07	1.75E-06	4.66E-05
6	1.61E-08	1.61E-08	3.37E-07	1.14E-05

Table 2. Computations of different errors for example 1

The exact solution for example 1 is $u(z) = z^2$ The Fredholm integral equation presented in example 1 has been solved by using nondyadic Haar wavelet collocation method. The results obtained by using the presented method is tabulated in table 1 which clearly explains the comparability among the exact and approximated solution for level of resolution 1. Table 1 clearly depicts that NHWCA provides more accurate results for small number of collocation points. $l_2 - error, l_\infty - error$ and $E_{max} - error$ for fredholm integral equation 1 (for level of resolution 1) are $9.50E-04, 8.47E-04$ and $1.27E-03$ respectively. From table 2, figure 1 and figure 2, it can be observed that approximated solution converges to the exact solution. From table 2 we can observe that by increasing the number of collocation points, accuracy of the solution gets better. Also, it can be observed from table 2 that NHWCM gives more accurate results than previous methods.

Example 2

Next consider Volterra integral equation of second kind [23]

$$u(z) = z + \int_0^z (t - z)u(t)\, dt \tag{4.2}$$

Let

$$u(z) = \sum_{i=1}^{3p} c_i \psi_i(z) \tag{4.3}$$

Integrating (4.3) both sides from 0 to z

$$\int_0^z u(z)dz = \sum_{i=1}^{3p} c_i L_{i,1}(z); \quad where\ L_{i,1}(z) = \int_0^z \psi_i(z)dz \tag{4.4}$$

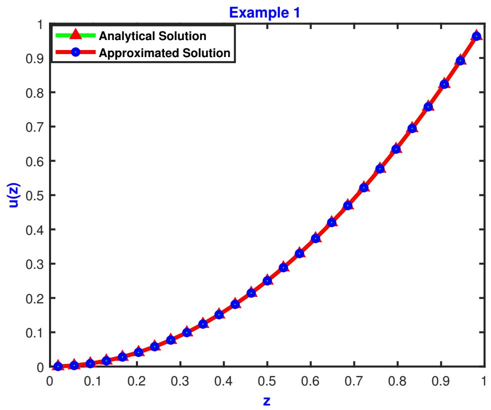

Figure 1. Comparability of exact and approximated solution for example 1

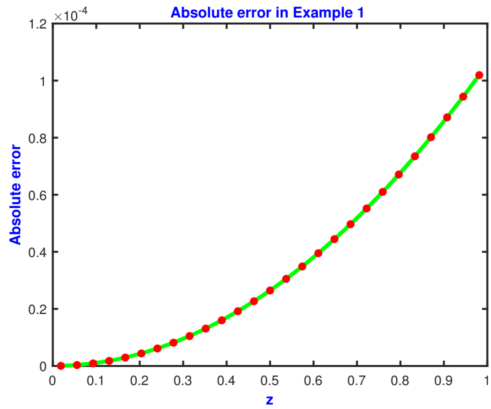

Figure 2. Graph of absolute error for example 1

Again integrating (4.4) from 0 to z

$$\int_0^z \int_0^z u(z)dzdz = \sum_{i=1}^{3p} c_i L_{i,2}(z); \quad where\ L_{i,2}(z) = \int_0^z L_{i,1}(z)dz \tag{4.5}$$

Substitute equation (4.3), (4.4), and (4.5) in (4.2) and simplifying we get

$$\sum_{i=1}^{3p} c_i \psi_i(z) = z - \sum_{i=1}^{3p} c_i L_{i,2}(z)$$

$$\sum_{i=1}^{3p} c_i(\psi_i(z) + L_{i,2}(z)) = z \tag{4.6}$$

Which is the required $AX = B$ form. The exact solution found from literature for example 2 is $u(z) = sin(z)$ The Volterra integral equation presented in example 2 has been solved by using nondyadic Haar wavelet collocation method. The results obtained by using the presented method is tabulated in table 3 which clearly explains the comparability among the exact and approximated solution for level of resolution 1. Table 3 clearly depicts that NHWCA provides more accurate results for small number of collocation points. $l_2 - error, l_\infty - error$ and $E_{max} - error$ for Volterra integral equation 2 (for level of resolution 1) are $9.76E-04, 7.66E-04$ and $1.53E-03$ respectively. From table 4, figure 3 and figure 4, it can be observed that approximated solution converges to the exact solution. From table 4, we can observe that by increasing the number of collocation points, accuracy of the solution gets better.

Example 3

Consider the Volterra integral equations having separable kernel [23]

$$u(z) = 1 - z - \frac{z^2}{2} + \int_0^z (z-t)u(t)\, dt \tag{4.7}$$

The exact solution found from the literature for example 3 is $u(z) = 1 - sinh(z)$ The Volterra integral equation presented in example 3 has been solved by using nondyadic Haar wavelet collocation method. The results obtained by using the presented method is tabulated in table 5 which clearly explains the comparability among the exact and approximated solution for level of resolution 1. Table 5 clearly depicts that NHWCA provides more accurate results for small number of collocation points. $l_2 - error, l_\infty - error$ and $E_{max} - error$ for Volterra integral

z	Analytical solution	Approximate solution	Absolute Error
0.055555556	0.055526982	0.055469954	5.70E-02
0.166666667	0.165896133	0.165726102	1.70E-01
0.277777778	0.274219289	0.273939402	2.80E-01
0.388888889	0.379160504	0.378775945	3.85E-01
0.500000000	0.479425539	0.478943447	4.82E-01
0.611111111	0.573777826	0.573207178	5.71E-01
0.722222222	0.661053722	0.660405180	6.49E-01
0.833333333	0.740176853	0.739462595	7.14E-01
0.944444444	0.810171396	0.809404909	7.66E-01

Table 3. Computation of exact and approximated solution for example 2

J	$l_2 - error$	$l_\infty - error$	$E_{max} - error$
0	8.71E-03	6.36E-03	7.82E-03
1	9.76E-04	7.66E-04	1.53E-03
2	1.09E-04	8.68E-05	2.95E-04
3	1.21E-05	9.71E-06	5.67E-05
4	1.34E-06	1.08E-06	1.09E-05
5	1.49E-07	1.20E-07	2.10E-06
6	1.65E-08	1.33E-08	4.04E-07

Table 4. Computations of different errors for example 2

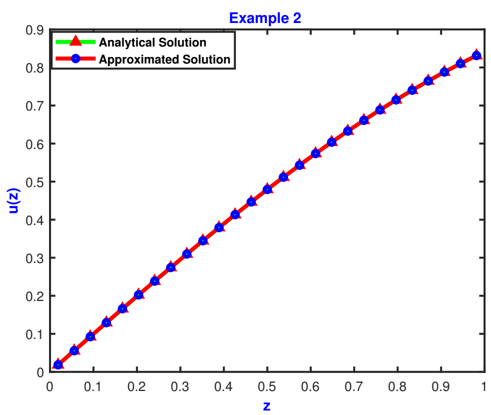

Figure 3. Comparability of exact and approximated solution for example 2

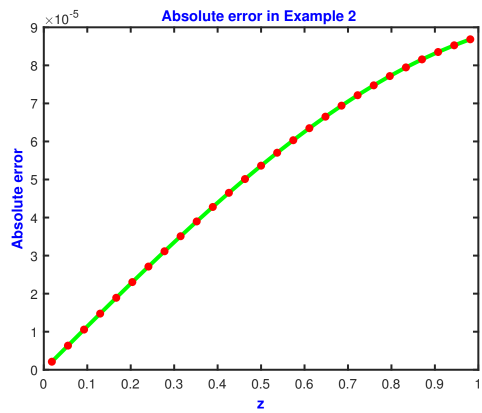

Figure 4. Graph of absolute error for example 2

z	Analytical solution	Approximate solution	Absolute Error
0.055555556	0.944415862	0.944358578	5.73E-05
0.166666667	0.832560656	0.832387741	1.73E-04
0.277777778	0.718636170	0.718344419	2.92E-04
0.388888889	0.601234481	0.600818493	4.16E-04
0.500000000	0.478904695	0.478356784	5.48E-04
0.611111111	0.350135011	0.349445080	6.90E-04
0.722222222	0.213334046	0.212489422	8.45E-04
0.833333333	0.066811159	0.065796383	1.01E-03
0.944444444	-0.091244435	-0.092447859	1.20E-03

Table 5. Computation of exact and approximated solution for example 3

J	$l_2 - error$	$l_\infty - error$	$E_{max} - error$
0	1.10E-02	9.25E-03	1.06E-02
1	1.22E-03	1.20E-03	2.06E-03
2	1.35E-04	1.41E-04	3.98E-04
3	1.50E-05	1.60E-05	7.65E-05
4	1.67E-06	1.78E-06	1.47E-05
5	1.85E-07	1.99E-07	2.83E-06
6	2.06E-08	2.21E-08	5.45E-07

Table 6. Computations of different errors for example 3

equation 3 (for level of resolution 1) are $1.22E-03, 1.20E-03$ and $2.06E-03$ respectively. From table 6, figure 5 and figure 6, it can be observed that approximated solution converges to the exact solution. From table 6 we can observe that by increasing the number of collocation points, accuracy of the solution gets better.

Example 4

Consider the nonlinear Hammerstein integral equations of mixed Volterra Fredholm type [29]

$$u(z) = \frac{z}{2} - \frac{z^4}{12} - \frac{1}{3} + \int_0^1 (z+t)\, u(t)\, dt + \int_0^z (z-t)[u(t)]^2\, dt \; and \;\; 0 \leq z, t, \leq 1 \qquad (4.8)$$

The exact solution for example 4 is $u(z) = z$. The Volterra Fredholm hammerstein integral equation presented in example 4 has been solved by using nondyadic Haar wavelet collocation method. The results obtained by using the presented method is tabulated in table 7 which clearly explains the comparability among the exact and approximated solution for level of resolution 1. Table 7 clearly depicts that NHWCA provides more accurate results for small number of collocation points. $l_2 - error, l_\infty - error$ and $E_{max} - error$ for integral equation 4 (for level of resolution 1) are $1.08E-02, 9.59E-03$ and $1.87E-02$ respectively. From table 8, figure 7 and figure 8, it can be observed that approximated solution converges to the exact solution. From table 8 we can observe that by increasing the number of collocation points, accuracy of the solution gets better.

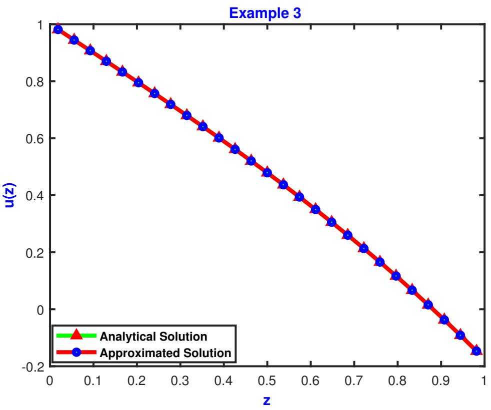

Figure 5. Comparability of exact and approximated solution for example 3

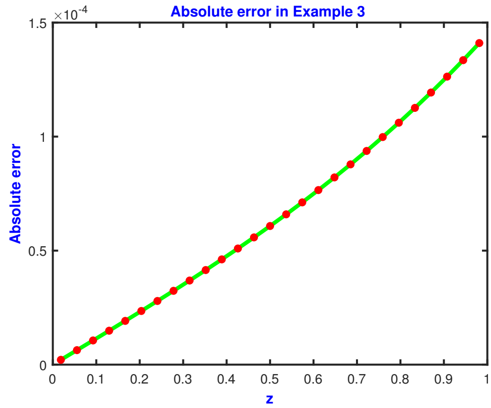

Figure 6. Graph of absolute error for example 3

z	Analytical solution	Approximate solution	Absolute Error
0.055555556	0.055555556	0.058392521	2.84E-03
0.166666667	0.166666667	0.170158473	3.49E-03
0.277777778	0.277777778	0.281939105	4.16E-03
0.388888889	0.388888889	0.393748617	4.86E-03
0.500000000	0.500000000	0.505605179	5.61E-03
0.611111111	0.611111111	0.617531404	6.42E-03
0.722222222	0.722222222	0.729555086	7.33E-03
0.833333333	0.833333333	0.841710272	8.38E-03
0.944444444	0.944444444	0.954038787	9.59E-03

Table 7. Computation of exact and approximated solution for example 4

J	$l_2 - error$	$l_\infty - error$	$E_{max} - error$
1	1.08E-02	9.59E-03	1.87E-02
2	1.19E-03	1.11E-03	3.57E-03
3	1.32E-04	1.25E-04	6.87E-04
4	1.47E-05	1.39E-05	1.32E-04
5	1.63E-06	1.55E-06	2.54E-05
6	1.81E-07	1.72E-07	4.89E-06

Table 8. Computations of different errors for example 4

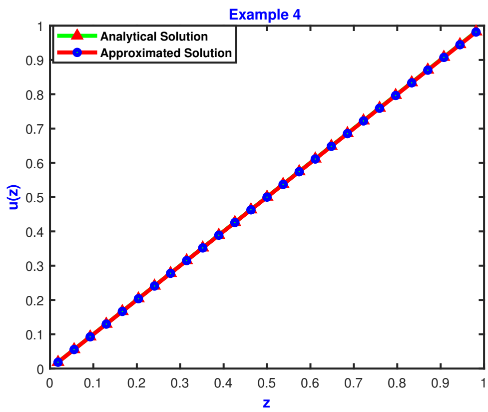

Figure 7. Comparability of exact and approximated solution for example 4

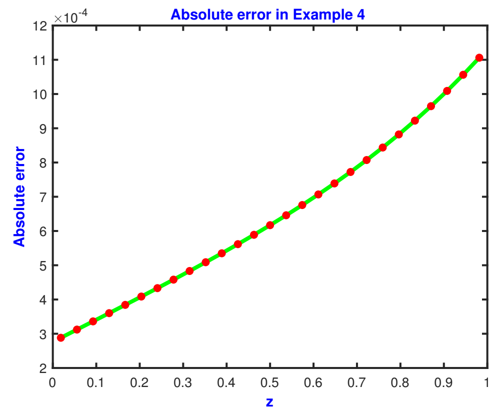

Figure 8. Graph of absolute error for example 4

z	Analytical solution	Approximate solution	Absolute Error
0.055555556	0.055555556	0.057953754	2.40E-03
0.166666667	0.166666667	0.168817106	2.15E-03
0.277777778	0.277777778	0.279680457	1.90E-03
0.388888889	0.388888889	0.390543809	1.65E-03
0.500000000	0.500000000	0.501407160	1.41E-03
0.611111111	0.611111111	0.612270512	1.16E-03
0.722222222	0.722222222	0.723133863	9.12E-04
0.833333333	0.833333333	0.833997215	6.64E-04
0.944444444	0.944444444	0.944860567	4.16E-04

Table 9. Computation of exact and approximated solution for example 5

Example 5

Consider the nonlinear integral equations of Fredholm type [30]

$$u(z) = \frac{3z}{4} + \frac{1}{5} + \int_0^1 (z-t)[u(t)]^3\, dt \ and \ \ 0 \le t \le 1 \tag{4.9}$$

The exact solution for example 5 is $u(z) = z$. The Fredholm integral equation presented in example 5 has been solved by using nondyadic Haar wavelet collocation method. The results obtained by using the presented method is tabulated in table 9 which clearly explains the comparability among the exact and approximated solution for level of resolution 1. Table 9 clearly depicts that NHWCA provides more accurate results for small number of collocation points. $l_2 - error, l_\infty - error$ and $E_{max} - error$ for integral equation 5 (for level of resolution 1) are $2.68E-03, 2.40E-03$ and $4.64E-03$ respectively. From table 10, figure 9 and figure 10, it can be observed that approximated solution converges to the exact solution. From table 10 we can observe that by increasing the number of collocation points, accuracy of the solution gets better.

5 Conclusions

In this paper, a numerical method based on non-dyadic Haar wavelets has been introduced for finding the approximate solution of Fredholm integral equations and Volterra integral equations. The integral equations are converted to the corresponding linear algebraic system of equations

J	$l_2 - error$	$l_\infty - error$	$E_{max} - error$
1	2.68E-03	2.40E-03	4.64E-03
2	2.99E-04	2.76E-04	8.96E-04
3	3.32E-05	3.10E-05	1.73E-04
4	3.69E-06	3.46E-06	3.32E-05
5	4.10E-07	3.85E-07	6.39E-06
6	4.56E-08	4.28E-08	1.23E-06

Table 10. Computations of different errors for example 5

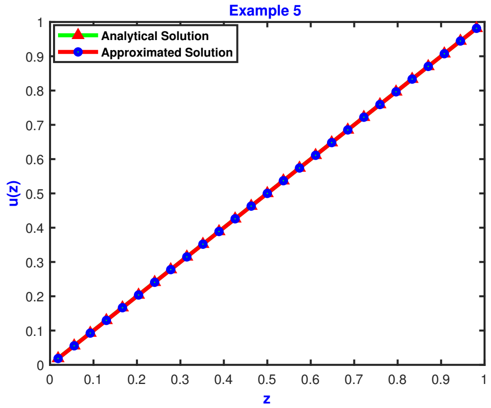

Figure 9. Comparability of exact and approximated solution for example 5

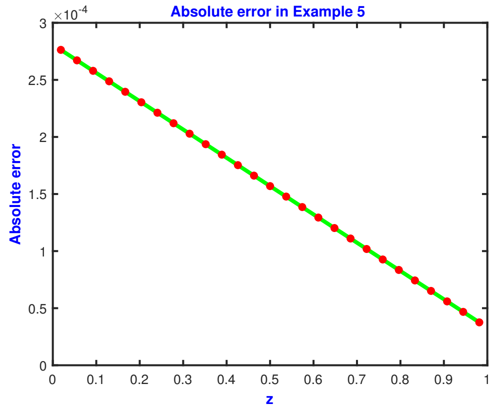

Figure 10. Graph of absolute error for example 5

which are then solved by the gauss elimination method. For the nonlinear equation, Quasilinearization technique is used. The proposed method is applied to some examples found in the literature for which the exact solutions are known. The results obtained by using this method are compared with the exact solution. From the table and graphs, it is observed that by increasing the level of resolution, approximated solutions converge to the exact solutions. MATLAB software is used for all the computational purposes.

Conflict of interest

The author(s) have no conflict of interest.

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