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Numerical study of sine-Gordon equation via differential quadrature based on modified quartic B-spline

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Abstract This manuscript studied the one dimensional sine Gordon equation (SGE) using differential quadratue (DQ) method with modified form of quartic B-spline (mQB spline) as base functions. The modification in quartic B-splines is done so as the diagonal dominance of the coefficient matrix is not effected and the method does not requires additional evaluations outside the computational domain as it is required with original form of quartic B-spline. At first, the SGE is transformed into a system of two ordinary differential equations and after the implementation of mQB-DQ method we get a system of ODEs which is solved by the SSP-RK43 method. The efficiency, accuracy and convergence of the method is demonstrated by implementing the method on three test problem of SGE, the solution profile of these test problem is also depicted graphically.

1 Introduction

To gain insight into the phenomenon of science and engineering, researchers are often presenting the relations between the involved variable in terms of differential equations. The study of modelled equations has always attracted mathematicians to get a deep understanding of the process. One of the well-known nonlinear partial differentials having applications in various fields of physics and whose solutions are in soliton form is a Sine-Gordon (SG) equation. It appears in the study of motion of pendulum, fluid motion, optics etc.

In the study of optics, the SG equation is administered as a solution to the classical Maxwell systems [1]. This equation is considered a prototype equation to describe the light bullet phenomenon. That plays an important role in communication systems and is an entrant to be opted in designing optical switches in optical devices. This equation presents a mathematical model to discuss the fault dynamics and to study the phenomena related to strain waves and earthquakes[2]. It plays a significant role in understanding the seismic distortion effects on the earth's crust. It has been applied to verify the theory behind the faulty medium in the analysis of natural substances. It is successfully implemented in the models due to the soliton solution of kinks form. SGE also appears in the literature in the geometrical study of the soliton in view of the canonical field [3]. The work is presented as a link in the evaporation of the black hole that was depicted in form of the one soliton solution phenomena. This study also describes a relationship between the soliton velocity and the black hole temperature.

The equation is given by:

$$\frac{\partial^2 \xi}{\partial \tau^2} + \alpha \frac{\partial^2 \xi}{\partial y^2} + \sin(\xi) = 0, \quad \tau \geq 0, \quad y \in [l, r] = {}_l\Omega_r \tag{1.1}$$

This equation has been solved for its soliton solutions by numerous researchers and scientist because of its applications in phenomena involving waves. Various analytical and numerical techniques have been used by the researchers and scientist to study SGE that includes but are not limited to the following: Kaya [4] and Ray [5] uses application of the modified decomposition method for evaluating the SGE in 1 and 2 dimensions, Wang [6] uses modified Adomian decomposition method to solve the SGE in higher dimensions, homotopy analysis method by Yucel [7]. Spline based approximation is implemented along with the finite difference scheme Rashidinia and Mohammadi [8]. Other than investigating the solution of the equation, researchers have put the effort in to studying the solution based on the properties shown by the equation. Such as the equation has been solved using finite difference based on the concept of conserved discrete energy Ben-yu et. al. [9]. The equation is discussed for the unstable nature using the nonlinear spectrum approach by Albowitz et. al. [10, 11]. Some other approaches that have been successfully implemented to study and solve the SG equation are Legendre spectral element method [12], virtual element method [13], MCB collocation technique [14], CFD6 scheme [15], boundary element and boundary integral approach [16, 17], localized method of approximate particular solutions [18].

Recently, the equation is solved using fourth order efficient collocation scheme [19]. The SG equation in fractional form has also been solved using homotopy perturbation method [20]. The equation in the fractional form in two-dimensional has been solved using finite difference meshfree methods in [21]. The equation in multidimensional form has been solved using radial basis functions [22], the concept of rational radial basis functions has been used to solve the equation in 1-dim [23].

In this work we proposed a modification in the quartic B-spline functions so that use of additional nodes outside the computation domain is not required while using these B-splines. We adopted DQ method with modified quartic B-splines to study the SGE. The DQ method was firstly introduced by [24], till then many variant form of DQ method has been introduced by various researchers to study the different type of phenomena arising in the field of engineering and sciences and thus this method is now one of the efficient and powerful technique to study the various phenomena modeled in the form of a PDEs. For more details about DQ method see [25, 26, 27, 28, 29, 30, 31, 32] and references therein. The rest of the manuscript is designed as: Section 2 presents the quartic and modified quartic B-splines and the procedure of the DQ method. Section 3 presents the implementation of the mQB-DQ method on the SGE. Three test problems are solved and their solutions are compared in the section 4 and in the last we concluded on our work in the section 5.

2 Modified Quartic B-spline Differential Quadrature Method (mQB-DQ Method)

The quartic B-spline $\mathbb{Q}_k(y)$ at different nodes y_j of the uniform partition $\chi = \{l = y_0 < y_1 < y_2 < \ldots < y_{n-1} < y_n = r\}$, of the domain of computation ${}_l\Omega_r = [l, r]$ is defined as [33]:

$$24\ \mathbb{Q}_k(y) = \begin{cases} \mathbb{A}_{k-2}^4, & [y_{k-2}, y_{k-1}] \\ \mathbb{A}_{k-2}^4 - 5\mathbb{A}_{k-1}^4, & [y_{k-1}, y_k] \\ \mathbb{A}_{k-2}^4 - 5\mathbb{A}_{k-1}^4 + 10\mathbb{A}_k^4, & [y_k, y_{k+1}] \\ \mathbb{A}_{k+3}^4 - 5\mathbb{A}_{k+2}^4, & [y_{k+1}, y_{k+2}] \\ \mathbb{A}_{k+3}^4, & [y_{k+2}, y_{k+3}] \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\mathbb{A}_k = (y_k - y)/h$. The set of these splines $\{\mathbb{Q}_{-2}, \ldots, \mathbb{Q}_n, \mathbb{Q}_{n+1}\}$ make a basis for the function defined over the considered computation domain ${}_l\Omega_r$.

The values of quartic splines $\mathbb{Q}_k$ and their derivatives at various nodes ($y_j \in \chi$) can be computed from equation (2.1), which is given as

$$\begin{aligned} &24\mathbb{Q}_k(y_{k-1}) = 24\mathbb{Q}_k(y_{k+2}) = 1, \quad 24\mathbb{Q}_k(k_i) = 24\mathbb{Q}_k(y_{k+1}) = 11, \\ &2h\mathbb{Q}_k'(y_{k-1}) = -2h\mathbb{Q}_k'(y_{k+2}) = 1, \quad 2h\mathbb{Q}_k'(y_k) = -2h\mathbb{Q}_k'(y_{k+1}) = 1, \\ &2h^2\mathbb{Q}_k''(y_{k-1}) = 2h^2\mathbb{Q}_k''(y_{k+2}) = 1, \quad 2h^2\mathbb{Q}_k''(y_k) = 2h^2\mathbb{Q}_k''(y_{k+1}) = -1, \\ &h^3\mathbb{Q}_k'''(y_{k-1}) = -h^3\mathbb{Q}_k'''(y_{k+2}) = 1, \quad h^3\mathbb{Q}_k'''(y_k) = -h^3\mathbb{Q}_k'''(y_{k+1}) = -1, \end{aligned} \tag{2.2}$$

also values of $\mathbb{Q}_k$, $\mathbb{Q}_k'$, $\mathbb{Q}_k''$ and $\mathbb{Q}_k'''$ vanishes at other nodes .

One can easily observe that the some of these splines $\mathbb{Q}_k,\ k \in \{-2,-1,0,1,n-2,n-1,n,n+1\}$ needs additional nodes outside the computation domain ${}_l\Omega_r$ so these splines aren't supported completely inside the domain of computation. So during the use of these splines the additional nodes outside the computation domain increase the complexity of the computation. To reduce the complexity of the computation wee have redefined the quartic B-splines at the boundary of the domain as follows

$$\begin{aligned}
\mathcal{Q}_0(y) &= Q_0(y) + 2\,Q_{-1}(y) + 3\,Q_{-2}(y),\\
\mathcal{Q}_1(y) &= Q_1(y) - Q_{-1}(y) - 2\,Q_{-2}(y),\\
\mathcal{Q}_k(y) &= Q_k(y), \qquad k \in \Delta_{n-2} \setminus \{0,1\}\\
\mathcal{Q}_{n-1}(y) &= Q_{n-1}(y) - Q_{n+1}(y),\\
\mathcal{Q}_n(y) &= Q_n(y) + 2\,Q_{n+1}(y).
\end{aligned} \tag{2.3}$$

2.1 Procedure of Differential Quadrature (DQ) Method

DQ method is a well known and powerful technique to compute the approximations of high accuracy for a unknown of the partial differential equations (PDEs), in DQ method partial derivatives of unknown are transformed into the weighted linear sum of the functional values at the considered nodes of computation domain. As a result, if the PDE is time-independent it transform into a set of algebraic equations whereas if the PDE is time dependent it transform into a set of ordinary differential equations (ODEs).

The r-th order derivative of unknown $\xi(y,\tau)$ i.e $\frac{\partial^r \xi}{\partial y^r},\ r=1,\ 2$ at the certain node $y=y_j,\ j=0,1,\ldots,n$ is read as

$$\frac{\partial^r \xi}{\partial y^r}(y_j) = \sum_{k=0}^{n} \eta_{jk}^{(r)} \xi(y_k), \qquad j=0,1,\ldots,n, \tag{2.4}$$

where $\eta_{jk}^{(r)}$ denote rth derivative's weighting coefficients at jth node $y=y_j$. We will compute these coefficients by adopting the mQB splines as trial functions in the DQ method, the procedure of computation of these coefficients is described in following section.

Computing procedure for weighting coefficient $\eta_{jk}^{(r)},\ r=1,2$

For the computation of weighting coefficients $\eta_{jk}^{(r)},\ r=1,2$ we will utilize the modified quartic B-splines (2.3) as trial functions in the DQ (2.4), which leads to the equation

$$\frac{\partial^r \mathcal{Q}_i}{\partial y^r}(y_j) = \sum_{k=0}^{n} \eta_{jk}^{(r)} \mathcal{Q}_i(y_k), \qquad j=0,1,\ldots,n; \quad r=1,2. \tag{2.5}$$

For a fix node point y_j, the equation (2.5) can be written as

$$\begin{bmatrix}
\mathcal{Q}_{00} & \mathcal{Q}_{01} & \mathcal{Q}_{02} & \cdots & \mathcal{Q}_{0n}\\
\mathcal{Q}_{10} & \mathcal{Q}_{11} & \mathcal{Q}_{12} & \cdots & \mathcal{Q}_{1n}\\
\mathcal{Q}_{20} & \mathcal{Q}_{21} & \mathcal{Q}_{22} & \cdots & \mathcal{Q}_{2n}\\
\vdots & \vdots & \vdots & & \vdots\\
\mathcal{Q}_{(n-1)0} & \mathcal{Q}_{(n-1)1} & \mathcal{Q}_{(n-1)2} & \cdots & \mathcal{Q}_{(n-1)n}\\
\mathcal{Q}_{n0} & \mathcal{Q}_{n1} & \mathcal{Q}_{n2} & \cdots & \mathcal{Q}_{nn}
\end{bmatrix}
\begin{bmatrix}
\eta_{j0}^{(r)}\\ \eta_{j1}^{(r)}\\ \eta_{j2}^{(r)}\\ \vdots\\ \eta_{j(n-1)}^{(r)}\\ \eta_{jn}^{(r)}
\end{bmatrix} = \Psi_j, \tag{2.6}$$

where $\mathcal{Q}_{ij} = \mathcal{Q}_i(y_j)$ and $\Psi_j = \left[\frac{\partial^r \mathcal{Q}_0}{\partial y^r}(y_j), \frac{\partial^r \mathcal{Q}_1}{\partial y^r}(y_j), \ldots, \frac{\partial^r \mathcal{Q}_n}{\partial y^r}(y_j)\right]^T$. Values of $\mathcal{Q}_{ij}$ and $\frac{\partial^r \mathcal{Q}_i}{\partial x^r}(y_j)$ can be computed directly from (2.3) with the help of equation (2.2). As the matrix arises in Equation (2.6) is nonsingular, therefore equation (2.6) can be easily solve by Gauss elimination method, and hence we will get the weighting coefficient for the $r^{th}(=1,2)$ order derivative.

3 Implementation of mQB-DQ Method to sine-Gordon equation

Before implementing the mQB-DQ Method to sine-Gordon equation, we consider the transformation $\vartheta = \xi_\tau$, and this transformation reduces the SGE (1.1) into a system of first order ODEs

$$\frac{\partial \xi}{\partial \tau} = \vartheta; \qquad \frac{\partial \vartheta}{\partial \tau} = -\alpha \frac{\partial^2 \xi}{\partial y^2} - \sin(\xi). \tag{3.1}$$

Now implementing the mQB-DQ Method with boundary conditions: $\xi(l,\tau) = \psi_1(\tau), \quad \xi(r,\tau) = \psi_2(t)$ the system (3.1) is reduces to following system of ODEs

$$\begin{cases} \dfrac{\partial \xi_i}{\partial \tau} = \vartheta_i, \\ \dfrac{\partial \vartheta_i}{\partial \tau} = -\alpha \displaystyle\sum_{j=1}^{n-1} \eta_{ij}^{(2)} \xi_j - \sin(\xi_i) + F_i, \end{cases} \tag{3.2}$$

where $F_i = \eta_{i0}^{(2)} \psi_1 + \eta_{in}^{(2)} \psi_2$. This system of ODEs (3.2) can be easily solved by implementing a number of admissible procedures of integration. Among them, we preferably chose SSP-RK43 technique [34] due to its accuracy and stability. The procedure of SSP-RK43 to solve $\dfrac{d\bar{\xi}}{d\tau} = L(\bar{\xi})$ with time-step $\Delta\tau$ is given as

$$\gamma^{(1)} = \bar{\xi}^m + \frac{\Delta\tau}{2} L(\bar{\xi}^m); \qquad \gamma^{(2)} = \gamma^{(1)} + \frac{\Delta\tau}{2} L(\gamma^{(1)}); \qquad \gamma^{(3)} = \frac{2}{3}\bar{\xi}^m + \frac{\gamma^{(2)}}{3} + \frac{\Delta\tau}{6} L(\gamma^{(2)});$$
$$\bar{\xi}^{m+1} = \gamma^{(3)} + \frac{\Delta\tau}{2} L(\gamma^{(3)})$$

The initial solution is required before the implementation of SSP-RK43 on equation (3.2) and this initial solution can be easily get from the initial conditions of SGE as $\xi(y_j, 0) = \phi_1(y_j), \vartheta(y_j, 0) = \phi_2(y_j), j = 0, 1, \ldots, n$. With these initial solution we can find the solution of SGE at desired level of τ.

4 Numerical Illustrations

In this section we will assess the efficiency, convergence and accuracy of the proposed scheme by implementing it on some test problems of SGE and evaluating the L_2, and L_∞ errors given as

$$L_2 := \sqrt{h \sum_{j=0}^{n} |\xi_j - \xi_j^*|^2}, \quad L_\infty := \max\left\{ \left|\xi_j - \xi_j^*\right|_{j=0}^{n} \right\}, \tag{4.1}$$

where ξ_j^* is the exact solution and ξ_j is approximate solution at the node y_j. The order of convergence of the proposed scheme is assessed numerically by the following formula

$$\frac{\ln(E_{n_1}/E_{n_2})}{\ln(n_2/n_1)},$$

where E_{n_k} stands for the L_2/L_∞ error in the evaluated approximate solution while taking $n_k \quad (k = 1, 2)$ nodes.

Example 4.1. Consider the one dimensional SGE (1.1) in the computational domain $_{-\ell}\Omega_\ell, \ell = 1, 2$ with $\alpha = -1$ together with the initial conditions $\xi(y, 0) = 0, \quad \xi_\tau(y, 0) = 4\text{sech}(y)$, and boundary conditions extracted from the analytical solution (4.2) as given in [14]

$$\xi(y, t) = 4\arctan(\tau \text{sech}(y)) \tag{4.2}$$

For $\ell = 1$, we calculated the approximate solution at various values of $\tau(= 0.25, 0.50, 0.75, 1.0)$ by taking the parametric values as $h \in \{0.02, 0.04\}, \Delta\tau = 0.001$. The L_2 and L_∞ errors in these approximations are reported in the table 1 and compared with that of obtained in [19, 14, 16]. Comparison in table 1 shows that we are getting the improved solution than [14, 16] and comparable to [19]. In addition, the absolute errors at different time level and various nodes of the computation domain is reported and compared in the table 2, which shows the efficiency and accuracy of proposed technique.

Now for $\ell = 2$, we have computed the L_2, and L_∞ errors in the approximate solution at $\tau = 0.1 \times k$, $k = 1, 2, \ldots, 10$ by taking the parametric values $h = 0.01$, $\Delta\tau = 0.001$ and reported and compared them with [36, 19, 14] in the table 3. Further, in table 4 we reported the L_2, L_∞ errors at $t = 1, 2$ using different number of nodes and $\Delta t = 0.001$. The order of convergence of the scheme evaluated in this table indicates it to be two. The solution profile of the problem is depicted in the Fig. 3.

Table 1. Comparison of mQB-DQ solution of Ex. 4.1

τ	[19]		[14]		[16]		Present	
$h = 0.02$	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2
0.25	1.34E-06	7.40E-07	8.20E-06	3.71E-06			2.60E-06	8.27E-07
0.5	3.96E-06	1.42E-06	1.62E-05	1.34E-05			2.93E-06	1.60E-06
0.75	1.61E-05	7.07E-06	2.54E-05	2.40E-05			2.94E-06	1.79E-06
1	3.22E-05	1.78E-05	4.14E-05	3.00E-05			1.04E-05	4.00E-06
$h = 0.04$								
0.25	9.67E-06	4.20E-06	2.32E-05	1.18E-05	5.89E-06	3.91E-05	9.84E-06	3.17E-06
0.5	9.36E-06	6.92E-06	4.11E-05	4.19E-05	2.01E-05	1.30E-04	1.17E-05	6.50E-06
0.75	2.25E-05	1.09E-05	1.02E-04	7.78E-05	3.63E-05	2.35E-04	1.18E-05	7.48E-06
1	6.38E-05	3.27E-05	1.64E-04	1.30E-04	5.07E-05	3.27E-04	3.91E-05	1.59E-05

Table 2. Comparison absolute errors at various nodes Ex. 4.1

	[14]			[19]			Present		
y	$\tau = 1$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 1$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 1$	$\tau = 0.1$	$\tau = 0.01$
-0.8	1.11E-05	4.24E-08	4.19E-11	1.86E-05	7.25E-10	5.57E-11	4.63E-06	1.23E-11	2.40E-14
-0.6	6.17E-07	1.94E-08	1.72E-11	8.23E-06	8.03E-10	8.00E-11	4.25E-07	1.03E-11	2.04E-14
-0.4	1.47E-05	3.01E-08	3.32E-11	1.24E-06	1.05E-09	1.06E-10	2.87E-06	1.27E-12	2.01E-14
0	4.13E-05	1.09E-07	1.15E-10	1.01E-07	1.29E-09	1.33E-10	7.83E-08	2.65E-11	3.02E-14
0.4	1.47E-05	3.01E-08	3.32E-11	1.24E-06	1.05E-09	1.06E-10	4.12E-07	1.27E-12	2.01E-14
0.6	6.17E-07	1.94E-08	1.72E-11	8.23E-06	8.03E-10	8.00E-11	6.03E-07	1.03E-11	2.04E-14
0.8	1.11E-05	4.24E-08	4.19E-11	1.86E-05	7.25E-10	5.57E-11	7.43E-07	8.13E-12	2.40E-14

Table 3. Comparison of mQB-DQ solution of Ex. 4.1 at various τ

	[19]		[14]	[36]	Present	
τ	L_2	L_∞	L_∞	L_∞	L_2	L_∞
0.1	2.05E-07	5.89E-07	7.20E-06	1.54E-06	1.26E-07	6.70E-07
0.2	4.35E-07	8.33E-07	2.26E-05	9.25E-05	3.68E-07	1.40E-06
0.3	5.99E-07	7.59E-07	4.54E-05	9.02E-05	6.79E-07	2.12E-06
0.4	6.81E-07	7.64E-07	7.52E-05	1.62E-04	1.04E-06	2.81E-06
0.5	7.76E-07	1.01E-06	1.12E-04	2.58E-04	1.44E-06	3.46E-06
0.6	1.11E-06	2.47E-06	1.55E-04	3.73E-04	1.86E-06	4.07E-06
0.7	1.83E-06	4.35E-06	2.04E-04	4.98E-04	2.30E-06	4.62E-06
0.8	2.93E-06	6.62E-06	2.59E-04	6.24E-04	2.76E-06	5.12E-06
0.9	4.40E-06	9.28E-06	3.19E-04	7.44E-04	3.21E-06	5.55E-06
1	6.23E-06	1.23E-05	3.84E-04	8.49E-04	3.67E-06	5.93E-06

Table 4. Order of convergence(OC) of mQB-DQ method for 4.1

	$\tau = 1$				$\tau = 2$			
n	L_∞	OC	L_2	OC	L_∞	OC	L_2	OC
11	5.64E-03	1.6908	5.17E-03	1.7389	1.02E-02	1.9617	1.51E-02	1.9517
21	1.89E-03	1.8563	1.68E-03	1.8715	2.87E-03	1.9962	4.29E-03	1.9744
31	9.16E-04	1.9051	8.10E-04	1.9145	1.32E-03	1.9902	1.99E-03	1.9829
41	5.38E-04	1.9293	4.74E-04	1.9360	7.56E-04	1.9946	1.14E-03	1.9874
51	3.53E-04		3.11E-04		4.89E-04		7.39E-04	

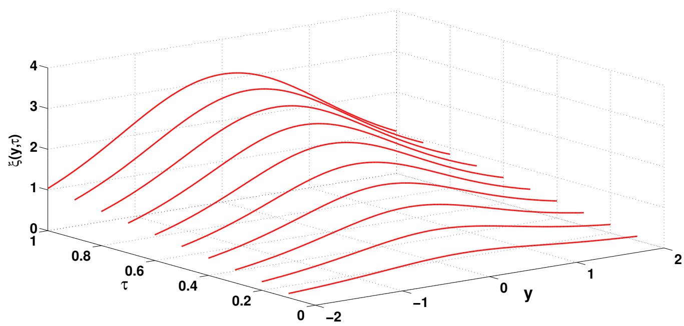


Figure 1. solution profile of Ex. 4.1 in ${}_{-2}\Omega_2$ and $0 \le \tau \le 1$

Example 4.2. Consider the one dimensional SGE (1.1) in the computational domain ${}_{-3}\Omega_3$ with $\alpha = -1$ together with the initial conditions $\xi(y,0) = 4\arctan\left(\exp\left(\frac{y}{\sqrt{1-\kappa^2}}\right)\right)$, $\xi_\tau(y,0) = \dfrac{-\frac{4\kappa}{\sqrt{1-\kappa^2}}\exp\left(\frac{y}{\sqrt{1-\kappa^2}}\right)}{1+\exp\left(\frac{2y}{\sqrt{1-\kappa^2}}\right)}$, and boundary conditions extracted from the analytical solution (4.3) as given in [14]

$$\xi(y,\tau) = 4\arctan\left(\exp\left(\frac{y-\kappa\,\tau}{\sqrt{1-\kappa^2}}\right)\right), \tag{4.3}$$

where κ denotes the velocity of solitary-wave. For the comparison purpose we have chosen the two values of κ $(0.05,\ \&\ 0.5)$.

For $\kappa = 0.5$, we calculated the approximate solution at various values of $\tau(= 0.25, 0.50, 0.75, 1.0)$ by taking the parametric values as $h \in \{0.02, 0.04\}, \Delta\tau = 0.0001$. The L_2 and L_∞ errors in these approximations are reported in the table 5 and compared with that of obtained in [19, 14, 16]. Comparison in table 5 shows that we are getting the improved solution than [14] for both values of h and comparable to [19, 16].

For $\kappa = 0.05$, the absolute errors at different time level and various nodes of the computation domain is reported and compared with [19, 14] in the table 6 by taking the parametric values as $\Delta\tau = 0.0001,\ h = 0.02,$, which shows the accuracy and efficiency the proposed method. In addition, to evaluate the order of convergence of the proposed scheme we have evaluated the $L_2,\ \&\ L_\infty$ errors, at $\tau = 1, 2$ by considering the various number of node values, and reported them in table 7. Table 7 again indicates that the order of convergence of the proposed scheme is two. The solution profile of the problem is depicted in the Fig. 2

Example 4.3. Consider the one dimensional SGE (1.1) in the computational domain ${}_{-3}\Omega_3$ with $\alpha = -1$ together with the initial conditions $\xi(y,0) = 0, \quad \xi_\tau(y,0) = 4\frac{1}{\sqrt{1+\kappa^2}}\operatorname{sech}\left(\frac{y}{\sqrt{1+\kappa^2}}\right)$ and boundary conditions extracted from the analytical solution (4.4) as given in [14]

$$\xi(y,t) = 4\arctan\left(\frac{1}{\kappa}\sin\left(\frac{\kappa\,\tau}{\sqrt{1+\kappa^2}}\right)\operatorname{sech}\left(\frac{y}{\sqrt{1+\kappa^2}}\right)\right), \tag{4.4}$$

where κ denotes the velocity of solitary-wave.

Table 5. Comparison of mQB-DQ solution of Ex. 4.2

τ	[19]		[14]		[16]		Present	
$h = 0.02$	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.25	7.10E-06	6.62E-06	9.26E-06	1.21E-06			3.66E-06	6.15E-06
0.5	1.23E-05	7.54E-06	2.24E-05	1.89E-05			5.05E-06	6.29E-06
0.75	1.60E-05	1.01E-05	3.98E-05	3.57E-05			5.96E-06	6.14E-06
1	1.79E-05	1.18E-05	5.66E-05	5.25E-05			6.61E-06	6.34E-06
$h = 0.04$								
0.25	1.60E-05	2.73E-05	3.66E-05	4.90E-05	1.76E-05	4.95E-06	1.42E-05	2.38E-05
0.5	2.37E-05	3.09E-05	9.00E-05	7.55E-05	4.31E-05	8.42E-06	1.99E-05	2.44E-05
0.75	2.91E-05	3.52E-05	1.60E-04	1.43E-04	8.25E-05	1.65E-05	2.35E-05	2.47E-05
1	3.25E-05	4.01E-05	2.27E-04	2.10E-04	1.27E-04	2.51E-05	2.62E-05	2.49E-05

Table 6. Comparison absolute errors at various nodes Ex. 4.2

	Present			[19]			[14]		
y	$\tau = 0.01$	$\tau = 0.1$	$\tau = 1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 1$
-2.5	4.27E-15	5.29E-13	5.66E-06	4.02E-10	4.02E-09	5.66E-06	6.05E-10	5.96E-08	5.28E-06
-2	4.21E-14	4.48E-12	2.74E-06	6.41E-10	6.41E-09	4.07E-06	8.76E-10	8.69E-08	1.21E-06
-1.5	1.47E-13	1.51E-11	8.28E-10	9.63E-10	9.63E-09	8.26E-08	5.64E-10	5.89E-08	9.16E-08
-1	9.53E-14	9.86E-12	1.02E-09	1.24E-09	1.23E-08	8.87E-08	2.68E-09	2.53E-07	2.02E-05
0	2.58E-14	8.87E-13	2.94E-10	2.87E-13	3.35E-12	3.97E-10	5.80E-11	5.64E-08	2.51E-05
1	1.30E-13	9.87E-12	1.29E-09	1.24E-09	1.23E-08	8.97E-08	2.72E-09	2.92E-07	4.82E-05
1.5	1.82E-13	1.55E-11	8.42E-10	9.64E-10	9.63E-09	8.32E-08	5.57E-10	5.08E-08	1.27E-05
2	6.57E-14	4.81E-12	2.74E-06	6.41E-10	6.41E-09	4.07E-06	8.78E-10	8.81E-08	2.21E-06
2.5	3.38E-14	8.11E-13	5.82E-06	4.03E-10	4.03E-09	5.97E-06	6.07E-10	6.15E-08	3.41E-06

Table 7. Order of convergence(OC) of mQB-DQ method for 4.2

	$\tau = 1$				$\tau = 2$			
n	L_∞	OC	L_2	OC	L_∞	OC	L_2	OC
11	5.26E-03	1.9926	6.78E-03	2.1540	5.07E-03	2.2084	7.07E-03	2.0186
21	1.45E-03	2.1515	1.68E-03	1.9766	1.21E-03	1.9258	1.92E-03	1.9635
31	6.27E-04	1.8153	7.80E-04	1.9884	5.74E-04	1.8144	8.92E-04	1.9837
41	3.78E-04	1.9873	4.47E-04	1.9860	3.45E-04	1.7946	5.12E-04	1.9894
51	2.45E-04		2.90E-04		2.34E-04		3.32E-04	

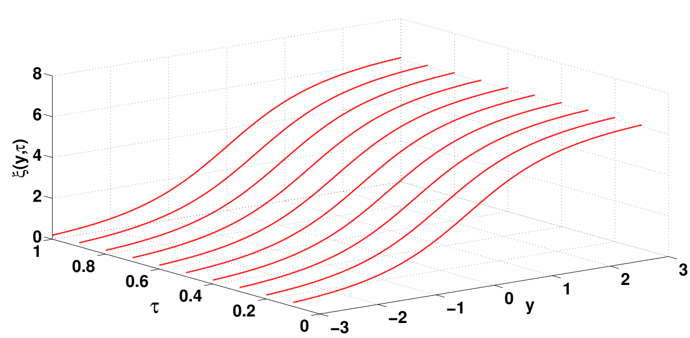
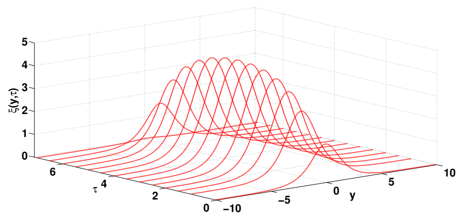
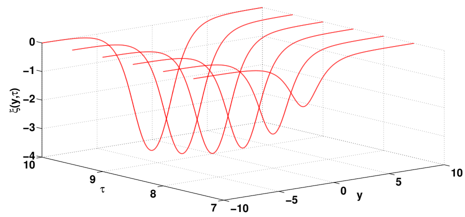


Figure 2. solution profile of Ex. 4.2 in ${}_{-3}\Omega_3$ and $0 \leq \tau \leq 1$ with $\kappa = 0.05$



(a) $0 \leq \tau \leq 7$



(b) $7 \leq \tau \leq 10$

Figure 3. solution profile of Ex. 4.3 in ${}_{-10}\Omega_{10}$ and $0 \leq \tau \leq 10$ with $\kappa = 0.5$

For $\kappa = 0.5$, we calculated the approximate solution at various values of $\tau(= 1.0,\ 10.0,\ 20.0)$ by taking the parametric values as $h = 0.01,\ \&\ \Delta\tau = 0.001$. The L_2 and L_∞ errors in these approximations are reported in the table 8 and compared with that of obtained in [19, 14, 37, 35]. Comparison in table 8 shows that we are getting the improved solution than [19, 14, 37, 35]. The solution profile of the problem is depicted in the Fig. 3.

Table 8. Comparison of mQB-DQ solution of Ex. 4.3

	[19]		[35]	[14]		[37]	Present	
τ	L_∞	L_2	L_∞	L_∞	L_2	L_∞	L_∞	L_2
1	4.77E-07	6.89E-07	9.88E-04	7.03E-06	7.45E-06	1.47E-03	5.83E-09	3.31E-09
10	3.49E-07	7.37E-07	1.63E-03	2.23E-05	4.00E-05	9.22E-03	1.31E-08	9.71E-09
20	3.13E-05	5.74E-05	1.04E-03	3.57E-04	6.47E-04	3.04E-01	2.71E-08	1.52E-08

5 Conclusion

The one dimensional sine Gordon equation (SGE) is studied using differential quadratue (DQ) method with modified form of quartic B-spline (mQB spline) as base functions. We have done the modification in quartic B-splines so as the diagonal dominance of the coefficient matrix is not effected and the requirement of additional nodes outside the computation domain does not exists. At first, the SGE is transformed into a system of two ordinary differential equations and after the implementation of mQB-DQ method we get a system of ODEs which is solved by the SSP-RK43 method. We implemented the mQB-DQ method on three test problem of SGE and the L_2, L_∞ error in the computed approximations are compared with that of existed in the some literature. We found that the method is easy to implement, efficient and accurate. The solution profile of considered test problem is also depicted graphically. The proposed mQB-DQ Method seems straightforward, easy and powerful method to obtain accurate and efficient solutions for various kinds of linear and nonlinear problems arising in various fields.

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