Palestine Journal of Mathematics

Vol. 12(Special Issue IV)(2023) , 147–158

Combination Difference Synchronization between Hyperchaotic Complex Lü Time-delay Systems via Adaptive Control

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MSC 2020 Classifications: 34D06, 34H10, 34D20, 65P20.

Keywords and phrases: Combination Difference Synchronization, Lyapunov Stabilty Theory, Time-delay System, Adaptive Control.

Abstract In this paper, we have investigated the hybrid projective combination difference synchronization scheme between hyper chaotic complex L$\ddot{u}$ time-delay systems through adaptive control. Using Lyapunov stability theory, we establish the stability of error states together with the controllers and parameters updated laws, that leads to achieve the required synchronization scheme among two identical hyperchaotic complex Lü time-delay systems and one slave hyperchaotic complex Lü time-delay system. Numerical simulations exhibit the validity of the theoretical work which have been done by using MATLAB.

1 Introduction

Chaos is deterministic and unpredictable phenomenon in nonlinear dynamics. Henri Poincare observed the three-body problem: earth, moon, and sun are connected under their mutual gravitational interactions and noticed that a small change in the initial phase of this problem can cause a large error in the final phase, which is the main feature of chaos. Poincare's [1] aided in the development of chaos theory. In 1963. E.N. Lorenz [2] introduced the ideas of chaos in weather model. The chaotic systems exhibit nonlinear and complex behavior that is affected by the initial conditions.

Due to practical significance and altering the dimensionality of dynamical systems, time delay dynamics has been a popular topic among researchers in the past few decades. According to Farmer [3], a nonlinear delay differential equation with a constant time delay is an infinite dimensional system. MacKey and Glass [4] discovered chaos in a time-delay system for the first time. Since, time delay systems show multi-stability, which has wide application in pattern recognition and memory storage devices. Because of these properties, time-delay chaotic systems have piqued the curiosity of many researchers. In a variety of disciplines, including physics, chemistry, biology and many more, the effects of time delay have been studied. Also, time-delay occurs in various physical systems such as artificial intelligence, secure communications, neural networks, automatic control systems, biological systems, population models, economic systems, and so on. Moreover, time-delay has two representations: delay differential equations and delay difference equations and also delay differential equations may describe models more precisely in real-world scenarios very often. Therefore time delay is an unavoidable element of real-world models; hence, additional research into this topic is required.

Chaos control and chaos synchronization of chaotic systems are very famous research problems in emerging literature. Therefore, the study of synchronization in chaotic systems has been an attractive research area for researchers. Pecora and Carroll [5] were the first to propose the idea of a synchronization problem using the master-slave system's endowment. Various types of synchronization schemes have been proposed for chaotic systems, such as anti-synchronization [6], compound synchronization [7], complete synchronization [8], hybrid projective synchronization [9], compound combination synchronization [10], difference synchronization [11], dual combination synchronization [12], combination-combination hybrid synchronization [13], modulas hybrid projective synchronization [14], double compound synchronization [15] and more. There are several types of useful and powerful methods have been applied to obtain synchronization and chaos control. Some of the methods are: adaptive sliding mode control [16], optimal control [17], active control [18], robust adaptive sliding mode control [19], sliding mode control

[20], time-delayed feedback control [21] etc. In 1998, Pyragus [22] was the first who studied the synchronisation of time delay systems. Furthermore, time delay system with lag synchronisation [23], phase synchronisation [24], and generalised synchronisation [25] were also developed.

We proposed an adaptive control technique based on the previous research work to investigate the problem of hybrid projective combination difference synchronization (HPCDS) in time-delay hyperchaotic complex L$\ddot{u}$ chaotic dynamical systems. Using the Lyapunov stability theory, synchronization has been achieved between two master time-delay hyperchaotic complex L$\ddot{u}$ system and one slave time-delay hyperchaotic complex L$\ddot{u}$ system along with unknown controllers.

This article has been drafted as follows: Section 2 deals with basic definitions and general principle of HPCDS. Section 3 contains the description of the time-delay hyperchaotic complex Lü chaotic system. Section 4 contains an example of an adaptive control technique HPCDS scheme of time-delay hyperchaotic complex L$\ddot{u}$ chaotic dynamical system. Section 5 provides the numerical simulations. In Section 6, we made a comparison between previous results and obtained result through the proposed method. Section 7 provides conclusions.

2 Synchronization Principle of Hybrid Projective Combination Difference Synchronization (HPCDS)

This section systematically describes the principle of hybrid projective combination difference synchronization among two identical master time-delay systems and one slave time-delay system through adaptive control technique. Two identical hyperchaotic time-delay master systems are described as,

$$\dot{x}_{m1} = \phi_1(x_{m1}, x_{m1}(t-\tau_1)) + \phi_2(x_{m1}, x_{m1}(t-\tau_1))\Theta, \tag{2.1}$$

$$\dot{x}_{m2} = \varphi_1(x_{m2}, x_{m2}(t-\tau_1)) + \varphi_2(x_{m2}, x_{m2}(t-\tau_1))\Theta, \tag{2.2}$$

The corresponding hyperchaotic time-delay slave system is:

$$\dot{x}_{s3} = \chi_1(x_{s3}, x_{s3}(t-\tau_1)) + \chi_2(x_{s3}, x_{s3}(t-\tau_1))\omega + Q(x_{m1}, x_{m2}, x_{s3}). \tag{2.3}$$

where x_{m1} and x_{m2} are state vectors of the master system which are given as $x_{m1} = (x_{m11}, x_{m12}, ..., x_{m1n}) \in \mathbb{R}^n$, $x_{m2} = (x_{m21}, x_{m22}, ..., x_{m2n}) \in \mathbb{R}^n$ and $x_{s3} = (x_{s31}, x_{s32}, ..., x_{s3n}) \in \mathbb{R}^n$ is the state vectors for the slave system and ϕ_1, ϕ_2, φ_1, φ_2, χ_1, and χ_2 represent $n \times n$ matrix function and Θ and ω are the real parameters with $\tau_1 > 0$, where Q is the controller to be constructed.

Remark 2.1. Master systems (2.1), (2.2) and the slave system (2.3) achieved the combination difference synchronization, if for α_i and three constant matrices A, B and $C \in \mathbb{R}^{n\times n}$ with $C \neq 0$, we have

$$\lim_{t\to\infty} \|E(t)\| = \lim_{t\to\infty} \|Cx_{s3}(t) - \alpha_i(Bx_{m2}(t) - Ax_{m1}(t))\| = 0, \tag{2.4}$$

where $\alpha_i = diag(\alpha_{11}, \alpha_{22}, ..., \alpha_{nn})$ and $\|.\|$ describe the matrix norm.

Remark 2.2. If $A = B = 0$, then from (2.4), combination synchronization will reduce into general chaos control problem.

Remark 2.3. If $C = I$ and $A = B = \alpha_i I$, then for $\alpha_i = 1$, the equation (2.4) reduces to combination complete synchronization and if $\alpha_i = -1$, then it reduces to combination anti-synchronization.

Remark 2.4. The equation (2.4) represents that combination synchronization of two master systems and one slave system can be developed into many others, such as identical and non-identical systems.

Definition 2.5. Master systems (2.1), (2.2) and the slave system (2.3) are said to be in hybrid projective combination difference synchronization (HPCDS), if there exists a real number α_i such that,

$$\lim_{t\to\infty} \|E(t)\| = \lim_{t\to\infty} \|x_{s3}(t) - \alpha_i(x_{m2}(t) - x_{m1}(t))\| = 0, \tag{2.5}$$

where $\|.\|$ represents the matrix norm, and $\alpha_i \in \mathbb{R}$.

We describe the synchronization scheme through designed controllers by using adaptive control approach. The state error for hybrid projective combination difference synchronization is defined as,

$$E(t) = x_{s3}(t) - \alpha_i(x_{m2}(t) - x_{m1}(t)),$$

$$\dot{E}(t) = \dot{x}_{s3}(t) - \alpha_i(\dot{x}_{m2}(t) - \dot{x}_{m1}(t)). \tag{2.6}$$

Using the equations (2.1), (2.2) and (2.3), and we get,

$$\begin{aligned}\dot{E}(t) = \chi_1(x_{s3}, x_{s3}(t-\tau_1)) + \chi_2(x_{s3}, x_{s3}(t-\tau_1))\omega + Q - \alpha_i(\varphi_1(x_{m2}, x_{m2}(t-\tau_1))\\ +\varphi_2(x_{m2}, x_{m2}(t-\tau_1))\Theta - \phi_1(x_{m1}, x_{m1}(t-\tau_1)) - \phi_2(x_{m1}, x_{m1}(t-\tau_1))\Theta).\end{aligned} \tag{2.7}$$

Further, we design the suitable controller Q and the parameter update laws to obtain the synchronization among the two hyperchaotic master time-delay systems and one hyperchaotic slave time-delay system. In this regard, we will prove the following theorem:

Theorem 2.6. *The hybrid projective combination difference synchronization among the two master systems (2.1), (2.2) and the slave system (2.3) globally and asymptotically can be achieved if the controller Q is taken as,*

$$\begin{aligned}Q = -\chi_1(x_{s3}, x_{s3}(t-\tau_1)) - \chi_2(x_{s3}, x_{s3}(t-\tau_1))\hat{\omega} + \alpha_i(\varphi_1(x_{m2}, x_{m2}(t-\tau_1))\\ +\varphi_2(x_{m2}, x_{m2}(t-\tau_1))\hat{\Theta} - \phi_1(x_{m1}, x_{m1}(t-\tau_1)) - \phi_2(x_{m1}, x_{m1}(t-\tau_1))\hat{\Theta}) - d_Q E,\end{aligned}$$

and updated parameters are:

$$\begin{aligned}\dot{\hat{\Theta}} &= -\alpha_i(\varphi_2(x_{m2}, x_{m2}(t-\tau_1)) - \phi_2(x_{m1}, x_{m1}(t-\tau_1))E - d_\Theta\tilde{\Theta},\\ \dot{\hat{\omega}} &= -\chi_2(x_{s3}, x_{s3}(t-\tau_1))E - d_\omega\tilde{\omega},\end{aligned} \tag{2.8}$$

where $\hat{\omega}$ and $\hat{\Theta}$ are the estimated values of ω and Θ, and $d_Q > 0$ are chosen arbitrary numbers. Also $\tilde{\Theta} = \Theta - \hat{\Theta}$ and $\tilde{\omega} = \omega - \hat{\omega}$.

Proof. Since we have the error dynamics,

$$\begin{aligned}\dot{E}(t) = \chi_1(x_{s3}, x_{s3}(t-\tau_1)) + \chi_2(x_{s3}, x_{s3}(t-\tau_1))\omega + Q - \alpha_i(\varphi_1(x_{m2}, x_{m2}(t-\tau_1))\\ +\varphi_2(x_{m2}, x_{m2}(t-\tau_1))\Theta - \phi_1(x_{m1}, x_{m1}(t-\tau_1)) + \phi_2(x_{m1}, x_{m1}(t-\tau_1))\Theta).\end{aligned} \tag{2.9}$$

Using the equation (2.8) in equation (2.9), we obtain,

$$\begin{aligned}\dot{E}(t) =& \chi_2(x_{s3}, x_{s3}(t-\tau_1))\tilde{\omega} - \alpha_i(\varphi_2(x_{m2}, x_{m2}(t-\tau_1))\\ &-\phi_2(x_{m1}, x_{m1}(t-\tau_1)))\tilde{\Theta} - d_Q E.\end{aligned} \tag{2.10}$$

Choosing the Lyapunov function,

$$V(t) = \frac{1}{2}(E^2 + \tilde{\Theta}^2 + \tilde{\omega}^2), \tag{2.11}$$

This implies that,

$$\dot{V} = E\dot{E} + \tilde{\Theta}\dot{\tilde{\Theta}} + \tilde{\omega}\dot{\tilde{\omega}}. \tag{2.12}$$

Thus,

$$\begin{aligned}\dot{V} =& E[\chi_2(x_{s3}, x_{s3}(t-\tau_1))\tilde{\omega} - \alpha_i(\varphi_2(x_{m2}, x_{m2}(t-\tau_1))\\ &-\phi_2(x_{m1}, x_{m1}(t-\tau_1)))\tilde{\Theta} - d_Q E] + \tilde{\Theta}(-\dot{\hat{\Theta}}) + \tilde{\omega}(-\dot{\hat{\omega}}).\end{aligned} \tag{2.13}$$

Using equation (2.8) in equation (2.13), we get

$$\begin{aligned}\dot{V} =& E[\chi_2(x_{s3}, x_{s3}(t-\tau_1))\tilde{\omega} - \alpha_i(\varphi_2(x_{m2}, x_{m2}(t-\tau_1)) - \phi_2(x_{m1}, x_{m1}(t-\tau_1)))\tilde{\Theta} \\ & -d_Q E] + \tilde{\Theta}(-\alpha_i(\varphi_2(x_{m2}, x_{m2}(t-\tau_1)) - \phi_2(x_{m1}, x_{m1}(t-\tau_1))E - d_\Theta \tilde{\Theta}) \\ & +\tilde{\omega}(-\chi_2(x_{s3}, x_{s3}(t-\tau_1))E - d_\omega \tilde{\omega}),\end{aligned}$$

$$\dot{V} = -d_Q E^2 - d_\Theta \tilde{\Theta}^2 - d_\omega \tilde{\omega}^2.$$

Selecting $d_Q > 0$, $d_\Theta > 0$ and $d_\omega > 0$ in such a manner so $\dot{V}$ is negative. Thus,

$$\dot{V} \leq 0.$$

Then by Lyapunov Stability Theory, we get

$$\lim_{t \to \infty} E(t) = 0.$$

Consequently, the error system is asymptotically globally stable which proves that required synchronization has been achieved among two master systems (2.1), (2.2) and the slave system (2.3). This ends the proof. □

3 System Description

In this section, we firstly describe about the time-delay hyperchaotic complex Lü system, proposed by Mahmood et. al. [26].

$$\begin{cases} \dot{x}_1 &= a_{11}(x_2 - x_1 + x_2 x_3) \\ \dot{x}_2 &= x_1 x_3 + b_{11} x_2 + x_4 \\ \dot{x}_3 &= \frac{1}{2}(\bar{x}_1 x_2 + x_1 \bar{x}_2) - c_{11} x_3 \\ \dot{x}_4 &= \dfrac{d_{11}}{2}(x_1(t-\tau_1) + \bar{x}_1(t-\tau_1)), \end{cases} \tag{3.1}$$

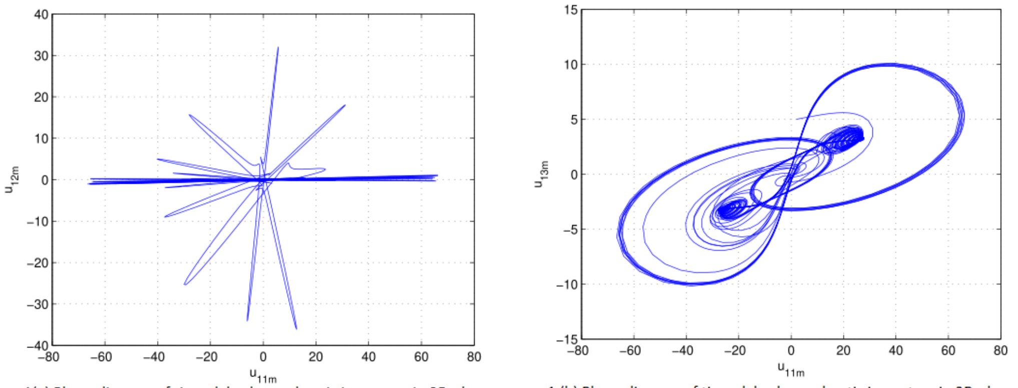
where $a_{11}, b_{11}, c_{11}, d_{11}$ denote the real parameters and $\tau_1 \geq 0$ is a constant time delay, $x_1 = x_{m11} + i x_{m12}$ and $x_2 = x_{m13} + i x_{m14}$ denote the complex variables and $x_3 = x_{m15}$ and $x_4 = x_{m16}$ are real variables and $i = \sqrt{-1}$.

$$\begin{cases} \dot{x}_{m11} + i\dot{x}_{m12} &= a_{11}(x_{m13} + i x_{m14} - x_{m11} - i x_{m12} + (x_{m13} + i x_{m14}) x_{m15}) \\ \dot{x}_{m13} + i\dot{x}_{m14} &= (x_{m11} + i x_{m12}) x_{m15} + b_{11}(x_{m13} + i x_{m14}) + x_{m16} \\ \dot{x}_{m15} &= \frac{1}{2}((x_{m11} - i x_{m12})(x_{m13} + i x_{m14}) \\ & +(x_{m11} + i x_{m12})(x_{m13} - i x_{m14})) - c_{11} x_{m15} \\ \dot{x}_{m16} &= \dfrac{d_{11}}{2}((x_{m11}(t-\tau_1) + i x_{m12}(t-\tau_1)) \\ & +(x_{m11}(t-\tau_1) - i x_{m12}(t-\tau_1))). \end{cases} \tag{3.2}$$

Real part and imaginary part of system (3.2) are:

$$\begin{cases} \dot{x}_{m11} &= a_{11}(x_{m13} - x_{m11} + x_{m13} x_{m15}) \\ \dot{x}_{m12} &= a_{11}(x_{m14} - x_{m12} + x_{m14} x_{m15}) \\ \dot{x}_{m13} &= -x_{m11} x_{m15} + b_{11} x_{m13} + x_{m16} \\ \dot{x}_{m14} &= -x_{m12} x_{m15} + b_{11} x_{m14} \\ \dot{x}_{m15} &= x_{m11} x_{m13} + x_{m12} x_{m14} - c_{11} x_{m15} \\ \dot{x}_{m16} &= -d_{11}(x_{m11}(t-\tau_1)). \end{cases} \tag{3.3}$$

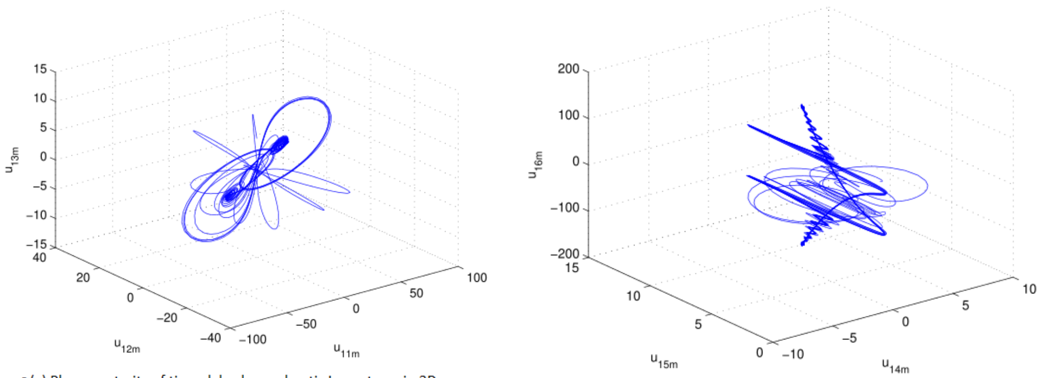
Choosing the parameter values as $a_{11} = 70$, $b_{11} = 15$, $c_{11} = 12$ and $d_{11} = 5$ along with time deal $\tau_1 = 0.5$ which confirms that the systems have the chaotic behavior.



1(a) Phase diagram of time-delay hyperchaotic Lu system in 2D-plane

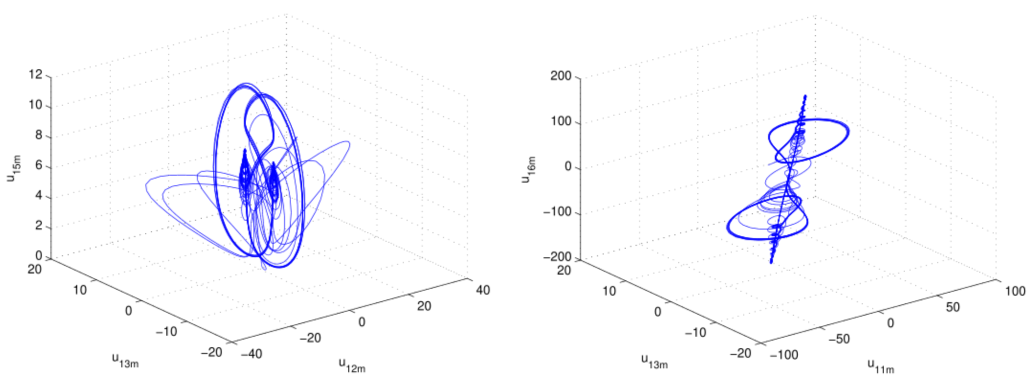
1.(b) Phase diagram of time-delay hyperchaotic Lu system in 2D-plane

Figure 1.



2(a) Phase portrait of time-delay hyperchaotic Lu systems in 3D-space

2(b) Phase portrait of time-delay hyperchaotic Lu system in 3D-space



2(c) Phase portrait of time-delay hyprchaotic Lu system in 3D-space

2(d) Phase portrait of time-delay hyperchaotic Lu system in 3D-space

Figure 2.

4 Example

To demonstrate the methodology of HPCDS among the two identical hyperchaotic master time-delay systems and one hyperchaotic slave time-delay system. Let three hyperchaotic time-delay systems be as follows:

$$\begin{cases} \dot{x}_{m11} &= a_{11}(x_{m13} - x_{m11} + x_{m13}x_{m15}) \\ \dot{x}_{m12} &= a_{11}(x_{m14} - x_{m12} + x_{m14}x_{m15}) \\ \dot{x}_{m13} &= -x_{m11}x_{m15} + b_{11}x_{m13} + x_{m16} \\ \dot{x}_{m14} &= -x_{m12}x_{m15} + b_{11}x_{m14} \\ \dot{x}_{m15} &= x_{m11}x_{m13} + x_{m12}x_{m14} - c_{11}x_{m15} \\ \dot{x}_{m16} &= -d_{11}(x_{m11}(t - \tau_1)), \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dot{x}_{m21} &= a_{11}(x_{m23} - x_{m21} + x_{m23}v_{m15}) \\ \dot{x}_{m22} &= a_{11}(x_{m24} - x_{m22} + x_{m24}x_{m25}) \\ \dot{x}_{m23} &= -x_{m21}x_{m25} + b_{11}x_{m23} + x_{m26} \\ \dot{x}_{m24} &= -x_{m22}x_{m25} + b_{11}x_{m24} \\ \dot{x}_{m25} &= x_{m21}x_{m23} + x_{m22}x_{m24} - c_{11}x_{m25} \\ \dot{x}_{m26} &= -d_{11}(x_{m21}(t - \tau_1)), \end{cases} \tag{4.2}$$

the corresponding slave system;

$$\begin{cases} \dot{y}_{s11} &= a_{11}(y_{s13} - y_{s11} + y_{s13}y_{s15}) + Q_{11} \\ \dot{y}_{s12} &= a_{11}(y_{s14} - y_{s12} + y_{s14}y_{s15}) + Q_{12} \\ \dot{y}_{s13} &= -y_{s11}y_{s15} + b_{11}y_{s13} + y_{s16} + Q_{13} \\ \dot{y}_{s14} &= -y_{s12}y_{s15} + b_{11}y_{s14} + Q_{14} \\ \dot{y}_{s15} &= y_{s11}y_{s13} + y_{s12}y_{s14} - c_{11}y_{s15} + Q_{15} \\ \dot{y}_{s16} &= -d_{11}(y_{s11}(t - \tau_1)) + Q_{16}, \end{cases} \tag{4.3}$$

where $Q_{11}(t)$, $Q_{12}(t)$, $Q_{13}(t)$, $Q_{14}(t)$,$Q_{15}(t)$, $Q_{16}(t)$ are the adaptive controllers to be designed. Now, the error states of the given systems defined as:

$$\begin{cases} E_{11}(t) &= y_{s11} - \alpha_1(x_{m21} - x_{m11}) \\ E_{12}(t) &= y_{s12} - \alpha_2(x_{m22} - x_{m12}) \\ E_{13}(t) &= y_{s13} - \alpha_3(x_{m23} - x_{m13}) \\ E_{14}(t) &= y_{s14} - \alpha_4(x_{m24} - x_{m14}) \\ E_{15}(t) &= y_{s15} - \alpha_5(x_{m25} - x_{m15}) \\ E_{16}(t) &= y_{s16} - \alpha_6(x_{m26} - x_{m16}). \end{cases} \tag{4.4}$$

From (4.4) we obtain error dynamics as,

$$\begin{cases} \dot{E}_{11}(t) &= \dot{y}_{s11} - \alpha_1(\dot{x}_{m21} - \dot{x}_{m11}) \\ \dot{E}_{12}(t) &= \dot{y}_{s12} - \alpha_2(\dot{x}_{m22} - \dot{x}_{m12}) \\ \dot{E}_{13}(t) &= \dot{y}_{s13} - \alpha_3(\dot{x}_{m23} - \dot{x}_{m13}) \\ \dot{E}_{14}(t) &= \dot{y}_{s14} - \alpha_4(\dot{x}_{m24} - \dot{x}_{m14}) \\ \dot{E}_{15}(t) &= \dot{y}_{s15} - \alpha_3(\dot{x}_{m25} - \dot{x}_{m15}) \\ \dot{E}_{16}(t) &= \dot{y}_{s16} - \alpha_4(\dot{x}_{m26} - \dot{x}_{m16}). \end{cases} \tag{4.5}$$

Using equations (4.1), (4.2), (4.3), and (4.5) we obtain,

$$\begin{aligned}
\dot{E}_{11}(t) &= a_{11}(y_{s13} - y_{s11} + y_{s13}y_{s15}) + Q_{11}\\
&\quad - \alpha_1(a_{11}(x_{m23} - x_{m21} + x_{m23}x_{m25})) + \alpha_1(a_{11}(x_{m13} - x_{m11} + x_{m13}x_{m15}))\\
\dot{E}_{12}(t) &= a_{11}(y_{s14} - y_{s12} + y_{s14}y_{s15}) + Q_{12}\\
&\quad - \alpha_2(a_{11}(x_{m24} - x_{m22} + x_{m24}x_{m25})) + \alpha_2(a_{11}(x_{m14} - x_{m12} + x_{m14}x_{m15}))\\
\dot{E}_{13}(t) &= -y_{s13}y_{s15} + b_{11}y_{s13} + y_{s16} + Q_{13}\\
&\quad - \alpha_3(-x_{m23}x_{m25} + b_{11}x_{m23} + x_{m26} + x_{m13}x_{m15} - b_{11}x_{m13} - x_{m16})\\
\dot{E}_{14}(t) &= -y_{s12}y_{s15} + b_{11}y_{s14} + Q_{14}\\
&\quad - \alpha_4(-x_{m22}x_{m25} + b_{11}x_{m24} + x_{m12}x_{m15} - b_{11}x_{m14})\\
\dot{E}_{15}(t) &= y_{s11}y_{s13} + y_{s12}y_{s14} - c_{11}y_{s15} + Q_{15}\\
&\quad - \alpha_5(x_{m21}x_{m23} + x_{m22}x_{m24} - c_{11}x_{m25} - x_{m11}x_{m13} - x_{m12}x_{m14} + c_{11}x_{m15})\\
\dot{E}_{16}(t) &= -d_{11}y_{s11}(t-\tau_1) + Q_{16}\\
&\quad - \alpha_6(-d_{11}x_{m21}(t-\tau_1) + d_{11}x_{m11}(t-\tau_1)). \qquad (4.6)
\end{aligned}$$

To design the controllers, we prove the following theorem:

Theorem 4.1. *If the controllers and parameter updated laws are chosen as,*

$$\begin{aligned}
Q_{11}(t) &= -k_1E_{11}(t) - \hat{a}_{11}(y_{s13} - y_{s11} + y_{s13}y_{s15})\\
&\quad + \alpha_1(\hat{a}_{11}(x_{m23} - x_{m21} + x_{m23}x_{m25})) + \alpha_1(\hat{a}_{11}(x_{m13} - x_{m11} + x_{m13}x_{m15}))\\
Q_{12}(t) &= -k_2E_{12}(t) - \hat{a}_{11}(y_{s14} - y_{s12} + y_{s14}y_{s15})\\
&\quad + \alpha_2(\hat{a}_{11}(x_{m24} - x_{m22} + x_{m24}x_{m25})) + \alpha_2(\hat{a}_{11}(x_{m14} - x_{m12} + x_{m14}x_{m15}))\\
Q_{13}(t) &= -k_3E_{13}(t) + y_{s13}y_{s15} - \hat{b}_{11}y_{s13} - y_{s16}\\
&\quad + \alpha_3(-x_{m23}x_{m25} + \hat{b}_{11}x_{m23} + x_{m26} - x_{m13}x_{m15} - \hat{b}_{11}x_{m13} - x_{m16})\\
Q_{14}(t) &= -k_4E_{14}(t) + y_{s12}y_{s15} - \hat{b}_{11}y_{s14}\\
&\quad + \alpha_4(-x_{m22}x_{m25} + \hat{b}_{11}x_{m24} + x_{m12}x_{m15} - \hat{b}_{11}x_{m14})\\
Q_{15}(t) &= -k_4E_{14}(t) - y_{s11}y_{s13} - y_{s12}y_{s14} + \hat{c}_{11}y_{s15}\\
&\quad + \alpha_5(x_{m21}x_{m23} + x_{m22}x_{m24} - \hat{c}_{11}x_{m25} - x_{m11}x_{m13} - x_{m12}x_{m14} + \hat{c}_{11}x_{m15})\\
Q_{16}(t) &= -k_4E_{14}(t) + \hat{d}_{11}y_{s11}(t-\tau_1) + \alpha_6(-\hat{d}_{11}x_{m21}(t-\tau_1) + \hat{d}_{11}x_{m11}(t-\tau_1)), \qquad (4.7)
\end{aligned}$$

and updated parameters laws,

$$\begin{aligned}
\dot{\hat{a}}_{11} &= E_{11}(y_{s13} - y_{s11} + y_{s13}y_{s15})\\
&\quad - \alpha_1((x_{m23} - x_{m21} + x_{m23}x_{m25}) - (x_{m13} - x_{m11} + x_{m13}x_{m15}))\\
&\quad + E_{12}(y_{s14} - y_{s12} + y_{s14}y_{s15}) - \alpha_2((x_{m24} - x_{m22} + x_{m24}x_{m25})\\
&\quad - (x_{m14} - x_{m12} + x_{m14}x_{m15})) + K_7(a_{11} - \hat{a}_{11}),\\
\dot{\hat{b}}_{11} &= E_{13}(y_{s13} - \alpha_3(x_{m23} - x_{m13})) + E_{14}(y_{s14}\\
&\quad - \alpha_4(x_{m24} - x_{m14})) + K_8(b_{11} - \hat{b}_{11}),\\
\dot{\hat{c}}_{11} &= E_{15}(-y_{s15} - \alpha_5(-x_{m25} + x_{m15})) + K_9(c_{11} - \hat{c}_{11})\\
\dot{\hat{d}}_{11} &= E_{16}(-y_{s11}(t-\tau_1) - \alpha_6(-x_{m21}(t-\tau_1) + x_{m11}(t-\tau_1))) + K_{10}(d_{11} - \hat{d}_{11}), \qquad (4.8)
\end{aligned}$$

note that $K_j > 0$, $\forall\, j = 1, 2, ..., 10$ *are real numbers.*
Then the two identical delay systems and one slave delay system are in the hybrid projective combination difference synchronization.

Proof. Consider the Lyapunov function as,

$$V = \frac{1}{2}[E_{11}^2 + E_{12}^2 + E_{13}^2 + E_{14}^2 + E_{15}^2 + E_{16}^2 + \tilde{a}_{11}^2 + \tilde{b}_{11}^2 + \tilde{c}_{11}^2 + \tilde{d}_{11}^2], \tag{4.9}$$

$$\begin{aligned}\dot{V} = {} & E_{11}\dot{E}_{11} + E_{12}\dot{E}_{12} + E_{13}\dot{E}_{13} + E_{14}\dot{E}_{14} + E_{15}\dot{E}_{15} + E_{16}\dot{E}_{16} + \tilde{a}_{11}\dot{\tilde{a}}_{11} \\ & + \tilde{b}_{11}\dot{\tilde{b}}_{11} + \tilde{c}_{11}\dot{\tilde{c}}_{11} + \tilde{d}_{11}\dot{\tilde{d}}_{11},\end{aligned} \tag{4.10}$$

where $\tilde{a}_{11} = a_{11} - \hat{a}_{11}$, $\dot{\tilde{a}}_{11} = -\dot{\hat{a}}_{11}$ and $\tilde{b}_{11} = b_{11} - \hat{b}_{11}$, $\dot{\tilde{b}}_{11} = -\dot{\hat{b}}_{11}$ and $\tilde{c}_{11} = c_{11} - \hat{c}_{11}$, $\dot{\tilde{c}}_{11} = -\dot{\hat{c}}_{11}$ and $\tilde{d}_{11} = d_{11} - \hat{d}_{11}$, $\dot{\tilde{d}}_{11} = -\dot{\hat{d}}_{11}$,

$$\begin{aligned}\dot{V} = {} & E_{11}\dot{E}_{11} + E_{12}\dot{E}_{12} + E_{13}\dot{E}_{13} + E_{14}\dot{E}_{14} + E_{15}\dot{E}_{15} + E_{16}\dot{E}_{16} - \tilde{a}_{11}\dot{\hat{a}}_{11} \\ & - \tilde{b}_{11}\dot{\hat{b}}_{11} - \tilde{c}_{11}\dot{\hat{c}}_{11} - \tilde{d}_{11}\dot{\hat{d}}_{11}.\end{aligned} \tag{4.11}$$

Using equations (4.6), (4.7), (4.8), and (4.11), it reduces to,

$$\begin{aligned}\dot{V} = & - K_1E_{11}^2 - K_2E_{12}^2 - K_3E_{13}^2 - K_4E_{14}^2 - K_5E_{15}^2 - K_6E_{16}^2 - K_7\tilde{a}_{11}^2 \\ & - K_8\tilde{b}_{11}^2 - K_9\tilde{c}_{11}^2 - K_{10}\tilde{d}_{11}^2,\end{aligned} \tag{4.12}$$

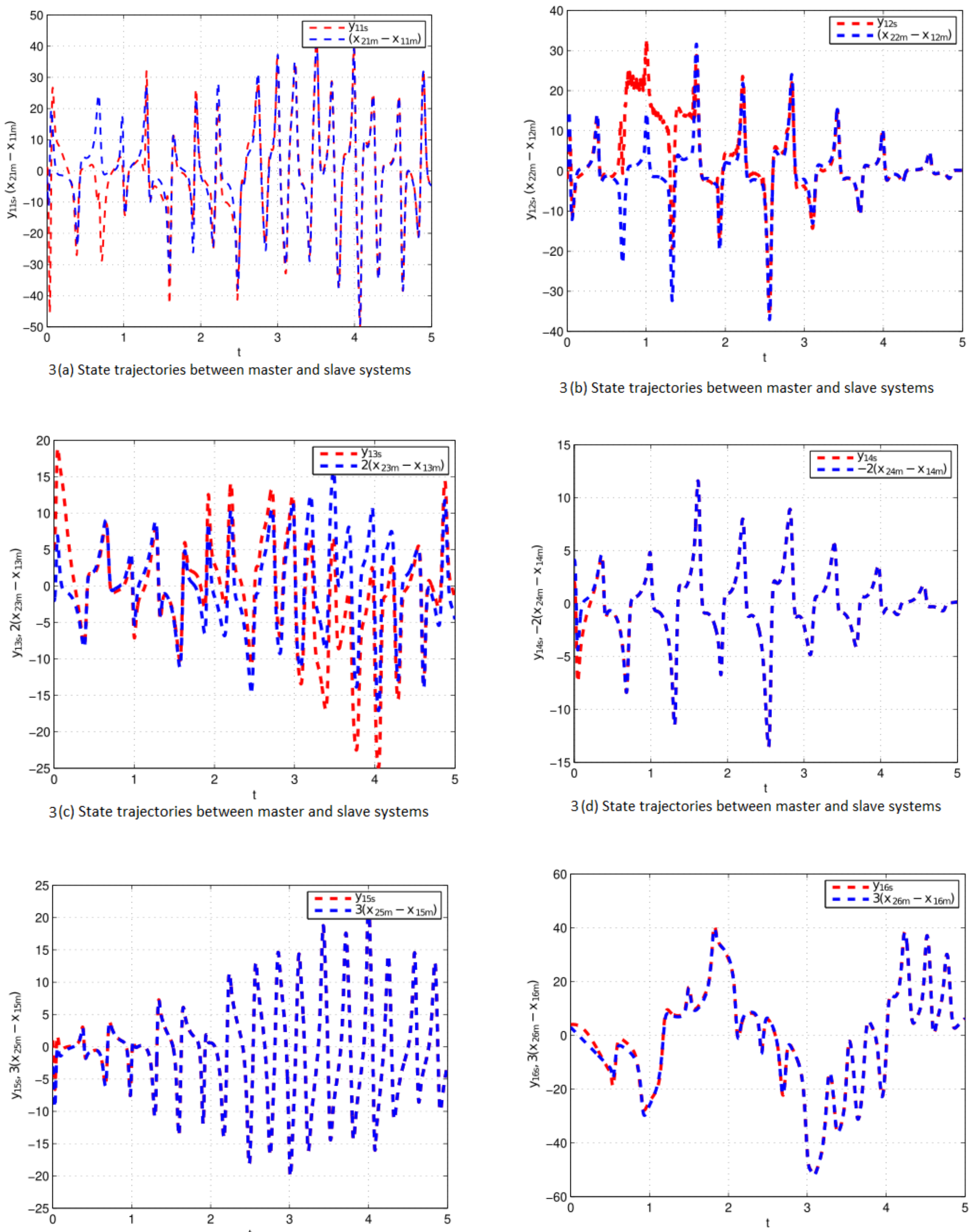
$$\dot{V} \leq 0,$$

where $K_i > 0$ for $i = 1, 2, ..., 10$.
Clearly, $\dot{V}$ is a negative definite function. Applying the Lyapunov stability theory, for every initial condition $E_{1j}(0)$, the error states $E_{ij}(t)$ approach to zero whenever $t \to \infty$, $\forall\ i = 1, j = 1, 2, 3, 4, 5, 6$, which means that error states are asymptotically globally stable. It proves that two hyperchaotic master time-delay systems (4.1), (4.2) and one hyperchaotic slave time-delay system (4.3) have achieved the required HPCDS. □

5 Numerical Simulation

Mainly, we talk about the numerical simulations to demonstrate the impact of the investigated HPCDS among two identical master time-delay system and one slave time-delay system through adaptive control technique. To carry out the simulation, we apply the Runge–Kutta formula to delay-differential equations.

For the parameter values, $a_{11} = 70$, $b_{11} = 15$, $c_{11} = 12$, $d_{11} = 5$ and initial state vectors of master systems (4.1), (4.2) as, $(x_{m11}, x_{m12}, x_{m13}, x_{m14}, x_{m15}, x_{m16}) = (2, 1, 5, 3, 4, 7)$, $(x_{m21}, x_{m22}, x_{m23}, x_{m24}, x_{m25}, x_{m26}) = (-1, -2, 5, 2, 1, 4)$ with initial state vector of slave system (4.3) are taken as $(y_{s11}, y_{s12}, y_{s13}, y_{s14}, y_{s15}, y_{s16}) = (-1, -2, 5, 2, 1, 4)$ exhibit the chaotic behavior.



3(a) State trajectories between master and slave systems

3(b) State trajectories between master and slave systems

3(c) State trajectories between master and slave systems

3(d) State trajectories between master and slave systems

3(e) State trajectories between master and slave systems

3(f) State trajectories between master and slave systems

Figure 3.

Fig.(1) displays the phase portraits of hyper chaotic complex Lü time-delay system in 2D-plane. While Fig.(2) displays the phase portraits of hyper chaotic complex Lü time-delay system in 3D-space, where $u_{1jm} = x_{m1j}\ \forall\ j = 1,2,3,4,5,6$. For required formulation, control gains are selected as $K_i = 4$ for $i = 1,2,3,4,5,6$ and time delay $\tau_1 = 0.5$, whereas the scaling functions are selected as $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) = (1,-1,2,-2,3,-3)$. Fig.(3) represents the state trajectories of two master systems (4.1), (4.2) and one slave system(4.3) which are synchronized as t tend to infinity.

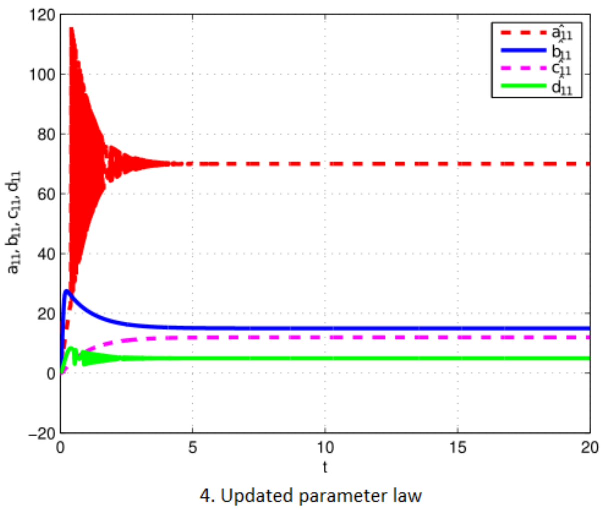


Figure 4. Updated parameter law

Fig.(4) exhibit the estimated values $(\hat{a}_{11}, \hat{b}_{11}, \hat{c}_{11}, \hat{d}_{11})$ of unknown parameters which approach asymptotically and globally to required values as t tends to infinity. We can seen that Fig.(5) displays the synchronization error states $(E_{11}, E_{12}, E_{13}, E_{14}, E_{15}, E_{16}) = (1, -3, 7, -2, 4, 1)$ goes to zero as $t \to \infty$. Hence, it proves that the proposed synchronization scheme for two hyperchaotic time-delay master systems (4.1), (4.2) and one hyperchaotic time-delay slave system (4.3) computationally justified.

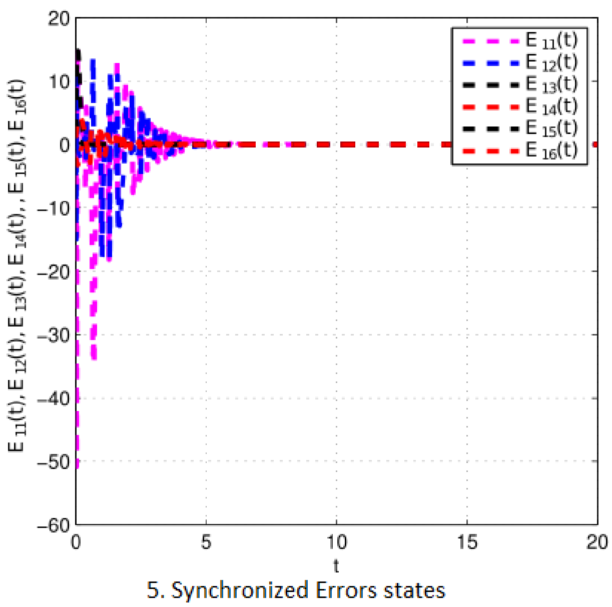


Figure 5. Synchronized error states

6 A comparative analysis

In this section, a thorough comparative study between the work we've just provided and previously published work is conducted.

Table 1. Comparison between various results

Methods	Synchronization time(approx.)
Combination synchronizing method [27]	$t = 20$
Synchronizing method using adaptive control [28]	$t = 10.5$
Combination synchronization using scaling matrix [29]	$t = 30$
Combination synchronization in Caputo–Hadamard [30]	$t = 14$
Combination synchronization using adaptive SMC [31]	$t = 6$
Present Method	$t = 5$

As a result, shown in the Table 1, the synchronization time attained in our research combination difference synchronization in the end of table, strategy is the shortest in comparison to all the aforementioned techniques.

7 Conclusion

We have designed the principle of HPCDS for hyperchaotic complex time-delay systems using an adaptive control technique. The synchronization scheme has been achieved among two identical hyperchaotic complex Lü master time-delay systems and hyperchaotic complex Lü slave time-delay system. Required controllers have been developed, and according to the Lyapunov stability theory, we have stabilized the error states and delay-differential equations. Due to time-delay, our systems exhibits more complexity in their behavior which may help to secure the messages, which can be treated as exceptional application in the field of secure communication and image encryption. In the future, we can exercise to time-delay systems which are interrupted by model uncertainty and disturbance.

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