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A Hermite Wavelet Collocation Method for Solving Neutral Delay Differential Equations

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Abstract This paper looked into third order neutral differential equations by employing Hermite wavelet. In order to estimate the highest order derivative with regard to the Hermite wavelet, the integral operator technique is applied. Then, to estimate the lower order derivatives and unknown function, Hermite wavelet integrations are used. To develop an algebraic system of linear or nonlinear equations, the unknown function's and its derivative's estimated values are substituted in neutral differential equations. When the developed system is solved, we get undetermined wavelet coefficients and afterward the estimate solution. The error norm's upper bound is determined in order to assess the practicability of the technique from a theoretical standpoint. Moreover, the theoretical results are verified through few numerical experiments.

1 Introduction

In describing the problem of environment, medicine and other areas of pure and applied sciences, it is the differential equation which governs the behaviour of modeled system. For example, the physical behaviour of fluctuating environment is governed by the third order differential equations [1]. Other problems which are modelled by third order differential equation with boundary conditions are electromagnetic waves, study of aeroelasticity, theory of sandwich beams, fluid mechanics of incompressible flow, thin film flow and obstacle problem (see [2],[3],[4],[5],[6]).

Ordinary differential equation fails to capture the nuances of these models, due to the fact that there are often delays between the observation and control action, the word "delay" in differential equations is coined. So we can say that a differential equation in which delay exists in unknown variable and/or its derivatives, is known as delay differential equation. Moreover, if delay exist in the highest order derivative, then it is called neutral delay differential equation (NDDE). In contrast to ordinary differential equation, delay differential equation exhibit better picture of phenomenon whether it is natural or artificial, particularly in biological sciences and physical sciences.

We shall investigate the following type of NDDE:

$$\begin{aligned} \dddot{C}(\omega) =& F\big(\omega, C(\omega), C(\omega-\tau_1(\omega,C(\omega))), \dot{C}(\omega), \dot{C}(\omega-\tau_2(\omega,C(\omega))), \ddot{C}(\omega), \\ & \ddot{C}(\omega-\tau_3(\omega,C(\omega))), \dddot{C}(\omega), \dddot{C}(\omega-\tau_4(\omega,C(\omega)))\big), \qquad \omega\in[\alpha,\beta], \end{aligned} \tag{1.1}$$

with initial and delay conditions

$$C(\omega)=\xi, \quad \omega\le\alpha,$$

and boundary conditions (BCs):

$$C(\alpha)=\xi, \quad \dot{C}(\alpha)=\zeta, \quad C(\beta)=\eta,$$

where $F:[\alpha,\beta]\times\mathbb{R}^8\longrightarrow\mathbb{R}$ is a differentiable function, $\tau_1(\omega,C(\omega)), \tau_2(\omega,C(\omega)), \tau_3(\omega,C(\omega))$ and $\tau_4(\omega,C(\omega))$ are continuous functions on $[\alpha,\beta]\times\mathbb{R}$ such that $\omega-\tau_1(\omega,C(\omega)), \omega-\tau_2(\omega,C(\omega))$, $\omega-\tau_3(\omega,C(\omega)), \omega-\tau_4(\omega,C(\omega))<\beta$.
Also, $\phi(\omega)$ represents the initial function which is given in Vanani and Aminataei [7].
Many renowned researchers have discussed the boundedness, stability, and asymptotic behaviour of solutions of DDEs of third order. For example, Timothy and Olutola [8] and Ademola *et al.*

[9] discussed the uniform stability, asymptotic behaviour and boundedness of solution of third order nonlinear DDEs. Gui [10] has discussed the existence of positive periodic solutions of third order DDEs, and the existence of a periodic solution to a nonlinear differential equation with numerous diverging parameters and delay has been discussed by Tunc [11]. In [12], the authors have discussed about the criteria for the existence of periodic solution of third order and fourth order DDEs having constant delay. In recent time, various researchers have put their efforts in constructing uniformly convergent numerical algorithm to get the solution of third and higher order BVPs (see [13],[14]). In contrary, there are few researchers who have put efforts in developing numerical techniques for solving DDEs (see [15],[16]).

Wavelet theory is one of the emerging technique of the recent era, which is being applied to solve various sort of natural and artificial problems (see [17],[18],[19],[21],[22], [20]). This paper aims to propose integral operator technique wherein Hermite wavelets are employed to estimate the highest order derivative for solving third order NDDEs. The paper's outline is presented as follows. In the Section 2, we present definition of multiresolution analysis, basic definition of wavelet, Hermite wavelet and approximation of functions by Hermite wavelet. In Section 3, we discussed the method for solving NDDEs. We carried out the Hermite wavelet's convergence analysis in Section 4. Section 5, consist of two test problem to demonstrate the proposed method's validation using the maximum absolute errors. Moreover, we made a comparison between exact solution and the solution obtained through proposed method in this section.

2 Wavelets

2.1 Preliminaries and Notations

Definition 2.1. The term "multiresolution analysis" (MRA), popularly known as the "wavelet's heart," first emerged in 1989. It plays a vital role in writing the wavelet in a broad sense. It gives us the ability to write a function $f(\omega) \in \mathcal{L}^2(\mathbb{R})$ over the multiresolution approximation space. MRA's goal is to break down the entire function spaces into spaces, $\mathcal{W}^\jmath$ and $\mathcal{V}^\jmath$, namely wavelet subspace and scaling function subspace, respectively. Any function $f(\omega) \in \mathcal{L}^2(\mathbb{R})$ is projectable on $\mathcal{V}^\jmath$, if $\mathcal{V}^\jmath$ satisfies the following conditions:

(i) $\mathcal{V}^\jmath \subset \mathcal{V}^{\jmath+1}$,

(ii) The collection $\{\phi(\omega - k), k \in \mathbb{Z}\ \}$ serves as an orthonomal basis for scaling function subspace $\mathcal{V}^0$,

(iii) $\overline{\bigcup_{\jmath\in\mathbb{Z}} \mathcal{V}^\jmath} = \mathcal{L}^2(\mathbb{R}),\ i.e., \{\mathcal{V}^\jmath\}$'s are dense in $\mathcal{L}^2(\mathbb{R})$,

(iv) $f(.) \in \mathcal{V}^\jmath \Longleftrightarrow f(2.) \in \mathcal{V}^{\jmath+1},\ \forall\ \jmath \in \mathbb{N}$,

(v) $\bigcap_{\jmath\in\mathbb{Z}} \mathcal{V}^\jmath = \{0\}, i.e.,$ there is nothing common in the all the subspaces.

The wavelet subspace is defined in the following manner:

$$\mathcal{W}^\jmath = \{\psi_\jmath^k;\ \ k, \jmath \in \mathbb{Z}\},$$

where $\mathcal{W}^\jmath$ is perpendicular complement of $\mathcal{V}^\jmath$'s in $\mathcal{V}^{\jmath+1}$ such that

$$\mathcal{V}^{\jmath+1} = \mathcal{V}^\jmath \oplus \mathcal{W}^\jmath.$$

On repeating the above steps, we get

$$\mathcal{V}^J = \mathcal{V}^{J_0} \oplus \bigoplus_{\jmath=J_0}^{J-1} \mathcal{W}^\jmath, \quad J > J_0. \tag{2.1}$$

If P_{v^J} project any arbitrary function $f(\omega) \in \mathcal{L}^2(\mathbb{R})$ on $V^\jmath$, we can conclude from dense criteria of MRA that

$$P_{v^J} f(\omega) \longrightarrow f(\omega), \quad \text{as} \quad J \longrightarrow \infty. \tag{2.2}$$

From (2.1) and (2.2), we can define scaling function projection and wavelet projection in the following way

$$P_{v^J} f(\omega) \approx \sum_k \tilde{\mu}_J^k \phi_J^k(\omega),$$

$$P_{v^J} f(\omega) \approx \sum_k \tilde{\mu}_{J_0}^k \phi_{J_0}^k(\omega) + \sum_k \sum_{J=J_0}^{J-1} \mu_J^k \psi_J^k(\omega),$$

where the coefficients μ_J^k and $\tilde{\mu}_J^k$ can be evaluated by applying the orthogonal property of the wavelet $\psi(\omega)$ and scaling function $\phi(\omega)$ as

$$\tilde{\mu}_{J_0}^k = \int_{-\infty}^{\infty} f(\omega)\phi_J^k(\omega)d\omega, \quad \mu_J^k = \int_{-\infty}^{\infty} f(\omega)\psi_J^k(\omega)d\omega.$$

2.2 Wavelet and Hermite wavelet

Definition 2.2. Mother wavelets are any orthogonal systems that result from MRA whose total integration is essentially zero., i.e,

$$\int_{-\infty}^{\infty} \psi(\omega)d\omega = 0.$$

The dilation and translation of mother wavelet produces a function's group which is referred as wavelet, and defined in the following manner:

$$\psi_{\mathcal{D}}^{\mathcal{T}}(\omega) = \mathcal{D}^{-\frac{1}{2}}\psi(\frac{\omega - \mathcal{T}}{\mathcal{D}}), \quad \mathcal{D} \neq 0, \mathcal{T} \in \mathbb{R},$$

where $\mathcal{D}$ is dilation and $\mathcal{T}$ is translation parameter ([23],[24]). On restricting the parameters $\mathcal{D} = \mathcal{D}_0^{-J}$, $\mathcal{T} = k\mathcal{T}_0\mathcal{D}_0^{-J}$, where $\mathcal{D}_0 > 1$ and $\mathcal{T}_0 > 1$, we get the following family of discrete wavelet:

$$\psi_J^k = (\sqrt{\mathcal{D}_0})^J(\mathcal{D}_0^J\omega - k\mathcal{T}_0).$$

Definition 2.3. The n^{th}-order Hermite polynomials denoted by $Hp_n(\omega)$, are defined as an orthogonal system over the domain $(-\infty, \infty)$ with weight function $e^{-\omega^2}$. These Hermite polynomials are given by:

$$\begin{aligned} Hp_0(\omega) &= 1, \quad Hp_1(\omega) = 2\omega, \\ Hp_{n+1}(\omega) &= 2\omega Hp_n(\omega) - 2nHp_{n-1}(\omega), \qquad n = 1, 2, 3.... \\ Hp_n'(\omega) &= 2nHp_{n-1}(\omega). \end{aligned}$$

Now, we define Hermite wavelet $\mathcal{H}p_{n,m}(\omega) = \mathcal{H}p(k, m, n, \omega)$ over the interval $[0, 1)$ as [25],

$$\mathcal{H}p_{n,m}(\omega) = \begin{cases} \sqrt{\frac{1}{2^n\sqrt{\pi}n!}}2^{\frac{J}{2}} Hp_n(2^J\omega - \lambda), & \omega \in [\kappa_1, \kappa_2) \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda = 1, 2, ...2^J - 1, \kappa_1 = \frac{\lambda-1}{2^J}, \quad \kappa_2 = \frac{\lambda+1}{2^J}$ and 'm' denotes the order of the Hermite polynomial vary from 0 to a fixed positive value $M - 1$. The collection of Hermite wavelets generates an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$.
We can rewrite the Hermite wavelet for each pair of m and n in the the following way :

$$\mathcal{H}p_\imath(\omega) = \mathcal{H}p_{n,m}(\omega),$$

where $\imath$ satisfies, $\imath = n + 2^{J-1}m$.

Approximation of function by Hermite wavelet

Any arbitrary function $C(\omega) \in \mathcal{L}^2[0,1)$ is capable of being expanded into Hermite wavelet's series as [26],

$$C(\omega) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \mu_{n,m}\mathcal{H}p_{n,m}(\omega) = \sum_{\imath=1}^{\infty} \mu_\imath \mathcal{H}p_\imath(\omega). \tag{2.3}$$

To get the best approximation, we can truncate the series given in equation (2.3) for a fixed natural number N in the manner described below:

$$C(\omega) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} \mu_{n,m}\mathcal{H}p_{n,m}(\omega) = \sum_{\imath=1}^{N} \mu_\imath \mathcal{H}p_\imath(\omega) = \mu^T \mathcal{H}p(\omega),$$

where

$$\begin{aligned}
\mu^T &= [\mu_{1,0}, \mu_{1,1}, \ldots\mu_{1,M-1}, \mu_{2,0}, \mu_{2,1}\ldots\mu_{2,M-1}, \ldots\ldots, \mu_{2^{k-1},0}, \ldots., \mu_{2^{k-1},M-1}],\\
\mu^T &= [\mu_1, \mu_2, \ldots.\mu_N],\\
\mathcal{H}p(\omega) &= [\mathcal{H}p_{1,0}(\omega), \ldots., \mathcal{H}p_{1,M-1}(\omega), \mathcal{H}p_{2,0}(\omega), \ldots., \mathcal{H}p_{2,M-1}(\omega), \mathcal{H}p_{2^{k-1},0}(\omega), \ldots,\\
&\qquad \mathcal{H}p_{2^{k-1},M-1}(\omega)]^T,\\
\mathcal{H}p(\omega) &= [\mathcal{H}p_1(\omega), \mathcal{H}p_2(\omega), \ldots., \mathcal{H}p_N(\omega)]^T,
\end{aligned}$$

where $N = 2^{\jmath-1}M$ and collocation points are determined by $\omega(l) = \frac{l-0.5}{N}$, where $1 \le l \le N$, $N = 2^J,\ J \in \mathbb{N}$.

2.3 Integration of Hermite wavelet

Let

$$\begin{aligned}
\mathcal{H}p^1_\imath(\omega) &= \int_0^\omega \mathcal{H}p_\imath(\tilde{\omega})d\tilde{\omega},\\
\mathcal{H}p^2_\imath(\omega) &= \int_0^\omega \mathcal{H}p^1_\imath(\tilde{\omega})d\tilde{\omega},\\
\mathcal{H}p^3_\imath(\omega) &= \int_0^\omega \mathcal{H}p^2_\imath(\tilde{\omega})d\tilde{\omega}.
\end{aligned}$$

The above integrations can be obtained as :

$$I^1\mathcal{H}p_\imath(\omega) = \mathcal{H}p^1_\imath(\omega) = \begin{cases} (\frac{1}{\sqrt{2}})^\jmath \varrho(\frac{1}{2m+2})\{Hp_{m+1}(\omega) - Hp_{m+1}(-1)\}, & \omega \in [\kappa_1,\ \kappa_2)\\ (\frac{1}{\sqrt{2}})^\jmath \varrho(\frac{1}{2m+2})\{Hp_{m+1}(1) - Hp_{m+1}(-1)\}, & \omega \in [\kappa_2,\ 1) \end{cases}$$

$$I^2\mathcal{H}p_\imath(\omega) = \mathcal{H}p^2_\imath(\omega) = \begin{cases} (\frac{1}{\sqrt{2}})^{3\jmath} \varrho(\frac{1}{2m+4})\{Hp_{m+2}(\omega) - Hp_{m+2}(-1)\}, & \omega \in [\kappa_1,\ \kappa_2)\\ (\frac{1}{\sqrt{2}})^{3\jmath} \varrho(\frac{1}{2m+2})\{(\frac{1}{(2m+4)})\{Hp_{m+2}(1) & \\ -Hp_{m+2}(-1)\} - 2Hp_{m+1}(-1) + (\omega-1)\{Hp_{m+1}(1) & \\ -Hp_{m+1}(-1)\}\}, & \omega \in [\kappa_2,\ 1) \end{cases}$$

$$I^3\mathcal{H}p_\imath(\omega) = \mathcal{H}p^3_\imath(\omega) = \begin{cases} (\frac{1}{\sqrt{2}})^{5\jmath} \varrho(\frac{1}{2m+2})[(\frac{1}{2m+4})\{(\frac{1}{2m+6})\{Hp_{m+3}(\omega) & \\ -Hp_{m+3}(-1)\} - (1+\omega)Hp_{m+2}(-1)\} & \\ -(\omega^2+\omega+\frac{1}{2})Hp_{m+1}(-1)], & \omega \in [\kappa_1, \kappa_2)\\ (\frac{1}{\sqrt{2}})^{5\jmath} \varrho(\frac{1}{2m+2})[(\frac{1}{2m+4})\{(\frac{1}{2m+6})\{Hp_{m+3}(1) & \\ -Hp_{m+3}(-1)\} - 2Hp_{m+2}(-1)\} - 2Hp_{m+1}(-1) & \\ +(\omega-1)\{(\frac{1}{2m+4})\{Hp_{m+2}(1) - Hp_{m+2}(-1)\} & \\ -2Hp_{m+1}(-1)\} + (\frac{\omega^2}{2} - \omega + \frac{1}{2}) & \\ \{Hp_{m+1}(1) - Hp_{m+1}(-1)\}], & \omega \in [\kappa_2, 1) \end{cases}$$

where $\varrho = \sqrt{\frac{1}{2^n\sqrt{\pi}n!}}$.

3 Method for solution of NDDE

For the sake of convenience, we will use '$\sum$' instead of $\sum_{i=1}^{n}$ throughout the paper.
Now, approximate the higher order derivative in the form of Hermite wavelet

$$\dddot{C}(\omega) \approx \sum c_i \mathcal{H}p_i(\omega). \tag{3.1}$$

Now, integrate equation (3.1) thrice from 0 to ω, we get

$$\ddot{C}(\omega) \approx \sum c_i \mathcal{I}_i^1(\omega) + \ddot{C}(0), \tag{3.2}$$

$$\dot{C}(\omega) \approx \sum c_i \mathcal{I}_i^2(\omega) + \omega\ddot{C}(0) + \dot{C}(0), \tag{3.3}$$

$$C(\omega) \approx \sum c_i \mathcal{I}_i^3(\omega) + \frac{\omega^2}{2}\ddot{C}(0) + \omega\dot{C}(0) + C(0). \tag{3.4}$$

On putting $\omega = 1$ in equation (3.4), we get

$$C(1) \approx C(0) + \dot{C}(0) + \frac{1}{2}\ddot{C}(0) + \sum c_i \mathcal{I}_i^3(1),$$
$$\ddot{C}(0) \approx 2(C(1) - C(0) - \dot{C}(0) - \sum c_i \mathcal{I}_i^3(1)).$$

On substituting $\ddot{C}(0)$ in equations (3.2), (3.3) and (3.4), we get the following equations:

$$\ddot{C}(\omega) \approx \sum c_i \mathcal{I}_i^1(\omega) + 2(\Lambda - \sum c_i \mathcal{I}_i^3(1)), \tag{3.5}$$

$$\dot{C}(\omega) \approx \sum c_i \mathcal{I}_i^2(\omega) + 2\omega(\Lambda - \sum c_i \mathcal{I}_i^3(1)) + \dot{C}(0), \tag{3.6}$$

$$C(\omega) \approx \sum c_i \mathcal{I}_i^3(\omega) + \omega^2(\Lambda - \sum c_i \mathcal{I}_i^2(1)) + C(0) + \omega\dot{C}(0), \tag{3.7}$$

where $\Lambda = C(1) - C(0) - \dot{C}(0)$.
Replace ω by $(\omega - \tau_4(\omega, C(\omega)))$, $(\omega - \tau_3(\omega, C(\omega)))$, $(\omega - \tau_2(\omega, C(\omega)))$ and $(\omega - \tau_1(\omega, C(\omega)))$ in equations (3.1), (3.5), (3.6) and (3.7) respectively, we get

$$\dddot{C}(\omega - \tau_4(\omega, C)) \approx \sum c_i \mathcal{H}p_i(\omega - \tau_4(\omega, C)), \tag{3.8}$$

$$\ddot{C}(\omega - \tau_3(\omega, C)) \approx \sum c_i \mathcal{I}_i^1(\omega - \tau_3(\omega, C)) + 2(\Lambda - \sum c_i \mathcal{I}_i^3(1)), \tag{3.9}$$

$$\dot{C}(\omega - \tau_2(\omega, C)) \approx \sum c_i \mathcal{I}_i^2(\omega - \tau_2(\omega, C)) + 2(\omega - \tau_2(\omega, C))(\Lambda - \sum c_i \mathcal{I}_i^3(1)) + C\dot{(0)}, \tag{3.10}$$

$$\begin{aligned} C(\omega - \tau_1(\omega, C)) \approx & \sum c_i \mathcal{I}_i^3(\omega - \tau_1(\omega, C)) + (\omega - \tau_1(\omega, C))^2(\Lambda - \sum c_i \mathcal{I}_i^2(1)) \\ & + C(0) + (\omega - \tau_1(\omega, C))\dot{C}(0). \end{aligned} \tag{3.11}$$

On substituting the values of equation (3.1) and equations (3.5)-(3.11), in equation (1.1) we get the following system of equations:

$$\begin{aligned} \sum c_i \mathcal{H}p_i \approx & F\big(\omega, C(0) + \omega\dot{C}(0) + \sum c_i \mathcal{I}_i^3(\omega) + \omega^2(\Lambda - \sum c_i \mathcal{I}_i^2(1)), C(0) + (\omega - \tau_1(\omega, C)) \\ & \dot{C}(0) + \sum c_i \mathcal{I}_i^3(\omega - \tau_1(\omega, C)) + (\omega - \tau_1(\omega, C))^2(\Lambda - \sum c_i \mathcal{I}_i^2(1)), \sum c_i \mathcal{I}_i^2(\omega) \\ & + 2\omega(\Lambda - \sum c_i \mathcal{I}_i^3(1)) + \dot{C}(0), \sum c_i \mathcal{I}_i^2(\omega - \tau_2(\omega, C)) + 2(\omega - \tau_2(\omega, C)) \\ & (\Lambda - \sum c_i \mathcal{I}_i^3(1)) + \dot{C}(0), \sum c_i \mathcal{I}_i^1(\omega) + 2(\Lambda - \sum c_i \mathcal{I}_i^3(1)), \sum c_i \mathcal{I}_i^1(\omega - \tau_3(\omega, C)) \\ & + 2(\Lambda - \sum c_i \mathcal{I}_i^3(1)), \sum c_i \mathcal{H}p_i(\omega), \sum c_i \mathcal{H}_i(\omega - \tau_4(\omega, C))\big). \end{aligned} \tag{3.12}$$

We determine the Hermite wavelet coefficients by solving the above system of equations. After that we put these coefficients in equation (3.7) to determine the approximate solution. While dealing with nonlinear NDDE, we employ Newton's method to solve the resulting system of nonlinear equations.

4 Convergence Analysis

This section covers the convergence study of the suggested approach.
We use the analytical version of equation (3.7) to demonstrate the convergence analysis of the suggested approach which is given below

$$C(\omega)=\sum_{\imath=1}^{\infty}c_\imath\mathfrak{J}_\imath^3(\omega)+\omega^2(\Lambda-\sum_{\imath=1}^{\infty}c_\imath\mathfrak{J}_\imath^2(1))+C(0)+\omega\dot{C}(0).$$

Theorem 4.1. *Suppose $C(\omega)$ is square integrable over $[0,1]$ in such a way that $\left|\dddot{C}(\omega)\right|\leq\alpha_0$, for all $\omega\in(0,1)$ and $\alpha_0>0$. If $\dddot{C}(\omega)=\sum_\imath^\infty c_\imath\mathcal{H}p_\imath(\omega)$, then we have the following inequality:*

$$\left|c_\imath\right|\leq 2^{-\frac{\jmath}{2}}\varrho\alpha_0\Im, \tag{4.1}$$

where $\Im$ is a constant defined in the proof below.

Proof. We have

$$\begin{aligned}\dddot{C}(\omega) &= \sum_{\imath=1}^{\infty}c_\imath\mathcal{H}p_\imath(\omega), &(4.2)\\ \left|c_\imath\right| &= \Big|\int_0^1\dddot{C}(\omega)\mathcal{H}p_\imath(\omega)d\omega\Big| \\ &\leq \sup_{\omega\in[0,1]}\left|\dddot{C}(\omega)\right|\int_0^\omega\left|\mathcal{H}p_\imath(\omega)\right|d\omega \\ &\leq \alpha_0 2^{-\frac{\jmath}{2}}\varrho\Im.\end{aligned}$$

Taking inner product of equation (4.2) and applying orthonormality condition of $\mathcal{H}p_\imath(\omega)$, we get equation (4.1).

Mean value theorem for integral has been applied and $\Im=\frac{\int_{-1}^{1}|\mathcal{H}p'_{m+1}(\ell)-\mathcal{H}p'_{m-1}(\ell)|d\ell}{2m+1}$.
Therefore, we have

$$\left|c_\imath\right|\leq 2^{-\frac{\jmath}{2}}\varrho\alpha_0\Im.$$

□

Theorem 4.2. *Let the analytic and estimate solution of equation (1.1) is denoted by $C(\omega)$ and $P_{v^J}C(\omega)$, respectively and $C(\omega)$is square integrable over $[0,1]$, $\left|\dddot{C}(\omega)\right|\leq\alpha_0$, for all $\omega\in(0,1)$ with $\alpha_0>0$. If ϵ_J is the estimation's error, then we have the following inequality:*

$$\left\|\epsilon_J\right\|_2\leq\varrho^2\alpha_0\Im^2\left(\frac{2^{-(2J+1)}}{7}+\frac{\lambda 2^{-J+1}}{3}\right).$$

Proof. We have

$$\begin{aligned}\left\|\epsilon_J\right\|_2 &= \left\|C(\omega)-P_{v^J}C(\omega)\right\|_2 \\ &= \Big\|\sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}c_\imath\mathfrak{J}_\imath^3(\omega)-\sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}c_\imath\mathfrak{J}_\imath^2(1)\Big\|_2.\end{aligned}$$

On using Minskowski inequality, we get

$$\begin{aligned}\left\|\epsilon_J\right\|_2 &\leq \Big\|\sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}c_\imath\mathfrak{J}_\imath^3(\omega)\Big\|_2+\Big\|\sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}c_\imath\mathfrak{J}_\imath^2(1)\Big\|_2 \\ &\leq \Big\|\sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}c_\imath\mathfrak{J}_\imath^3(\omega)\Big\|_2+\Big\|\sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}c_\imath\mathfrak{J}_\imath^2(1)\Big\|_2 \\ &\leq \sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}|c_\imath|\left(\int_0^1|\mathfrak{J}_\imath^3(\omega)|^2d\omega\right)^{\frac{1}{2}}+\sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1}|c_\imath|\left(\int_0^1|\mathfrak{J}_\imath^2(1)|^2d\omega\right)^{\frac{1}{2}} \quad (4.3)\end{aligned}$$

Now, Consider the Cauchy's formula for repeating integration as

$$I^n \mathcal{H}p_\imath(\omega) = \frac{1}{(n-1)!}\int_0^\omega (\omega - s)^{n-1}\mathcal{H}p_\imath(s)ds.$$

For $n = 3$, we get

$$I^3 \mathcal{H}p_\imath(\omega) = \frac{1}{2!}\int_0^\omega (\omega - s)^2\mathcal{H}p_\imath(s)ds.$$

Now,

$$\begin{aligned}
I^3 \mathcal{H}p_m(\omega) =& \frac{1}{2!}\int_{\kappa_1}^{\kappa_2} (\omega - s)^2 \varrho 2^{\frac{\jmath}{2}}\mathcal{H}p_m(2^\jmath s - \lambda)ds \\
=& \frac{1}{2!}\int_{-1}^{1} (\omega - \frac{\varsigma+\lambda}{2^\jmath})^2 \varrho 2^{\frac{\jmath}{2}} 2^{-\jmath}\mathcal{H}p_m(\varsigma)d\varsigma \\
=& \frac{1}{2!}\int_{-1}^{1} \frac{(2^\jmath\omega - \varsigma - \lambda)^2}{2^{2\jmath}} \varrho 2^{-\frac{\jmath}{2}}\mathcal{H}p_m(\varsigma)d\varsigma \\
\le& 2^{\frac{-5\jmath-2}{2}}\varrho \frac{\max\limits_{-1\le\varsigma\le1}|\mathcal{H}p'_{m+1}(\varsigma) - \mathcal{H}p'_{m-1}(\varsigma)|}{2m+1}\int_{-1}^{1}(2^\jmath\omega - \varsigma - \lambda)^2 d\varsigma
\end{aligned}$$

$$|I^3 \mathcal{H}p_m(\omega)| \le 2^{\frac{-5\jmath-2}{2}}\varrho\Im\mathfrak{F}, \tag{4.4}$$

where $\varsigma = 2^\jmath s - \lambda$, $\Im = \frac{\max\limits_{-1\le\varsigma\le1}|\mathcal{H}p'_{m+1}(\varsigma) - \mathcal{H}p'_{m-1}(\varsigma)|}{2m+1}$ and $\mathfrak{F} = \frac{1}{3}\underset{\omega\in[0,1]}{max}|(2^\jmath\omega+1-\lambda)^3 - (2^\jmath\omega-1-\lambda)^3|$.
In similar fashion, we can get

$$|I^2 \mathcal{H}p_m(1)| \le 2^{\frac{-3\jmath+2}{2}}(\lambda - 2^\jmath)\varrho\Im, \tag{4.5}$$

where $\Im = \frac{\max\limits_{-1\le\varsigma\le1}|\mathcal{H}p'_{m+1}(\varsigma) - \mathcal{H}p'_{m-1}(\varsigma)|}{2m+1}$.
Substituting equations (4.1),(4.4) and (4.5) in equation (4.3) and after simplifying, we get

$$\begin{aligned}
\left\|\epsilon_J\right\|_2 &\le \sum_{\jmath=J+1}^{\infty}\sum_{\imath=2^J}^{2^{J+1}-1} \varrho^2\alpha_0\Im^2\left(\frac{1}{2^{(3\jmath+1)}} + \frac{(\lambda-2^\jmath)}{2^{(2\jmath-1)}}\right) \\
&\le \sum_{\jmath=J+1}^{\infty}\left\{\varrho^2\alpha_0\Im^2\left(\frac{1}{2^{(3\jmath+1)}} + \frac{(\lambda-2^\jmath)}{2^{(2\jmath-1)}}\right)\right\}(2^{J+1} - 1 - 2^J + 1) \\
&\le 2^J\sum_{\jmath=J+1}^{\infty}\varrho^2\alpha_0\Im^2\left(\frac{1}{2^{(3\jmath+1)}} + \frac{(\lambda-2^\jmath)}{2^{(2\jmath-1)}}\right) \\
\left\|\epsilon_J\right\|_2 &\le \varrho^2\alpha_0\Im^2\left(\frac{2^{-(2J+1)}}{7} + \frac{\lambda 2^{-J+1}}{3}\right).
\end{aligned} \tag{4.6}$$

□

From equation (4.6) it is possible to claim that the error and resolution level J are correlated to each other in a reverse fashion, which infers that as $J \to \infty$, then $\left\|\epsilon_J\right\| \to 0$.
Hence, we draw the conclusion that when resolution levels are increased, the estimate solution approaches to the exact solution.

5 Numerical Examples

Problem 1. Let:

$$\ddot{C} + \ddot{C}(\sqrt{sin\ \omega}) - \omega C = f(\omega), \tag{5.1}$$

which satisfies the BCs:

$$C(0) = 0,\ \dot{C}(0) = 1,\ C(1) = 0.$$

The exact solution is

$$C(\omega) = \omega(1 - \omega)e^{\omega}.$$

The source function $f(\omega)$ can be calculated with the help of exact solution.

We have employed Hermite wavelet series method (HWSM) for solving equation (5.1). The obtained root mean square errors (RMSE) and maximum absolute errors (MAE) for various convergence parameters $(M,\ J)$ are reported in Table1. The table clearly demonstrates that the RMSE and MAE progressively decline as we enhance the values of the convergence parameters $(M,\ J)$. A comparison of exact solution with HWSM solution is displayed in Figure 1, while the effect of the convergence parameters on the behaviour of absolute errors is shown in Figures 2a and 2b.

Table 1. Error analysis of Problem 1

N	(M, J)	RMSE	MAE
2	$(2, 1)$	$6.12669937e-05$	$4.88296275e-05$
8	$(3, 1)$	$2.25249042e-10$	$1.38766775e-10$
16	$(4, 2)$	$4.20599746e-11$	$1.66392316e-11$
32	$(5, 3)$	$1.60497734e-13$	$6.61901089e-14$

‘

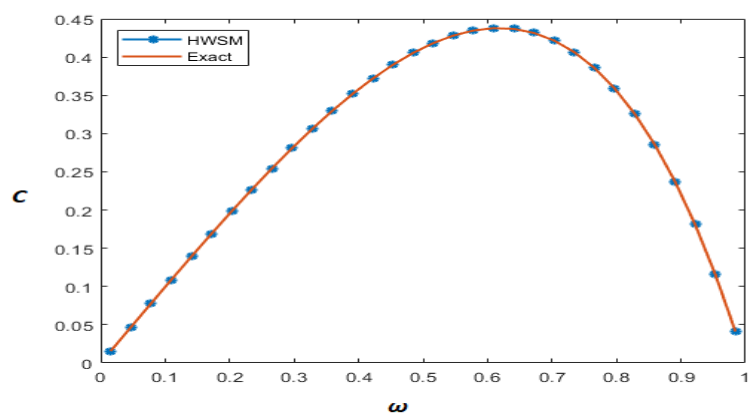


Figure 1. Graph of exact and HWM solutions of Problem 1 for $M = 8$ and $J = 3$

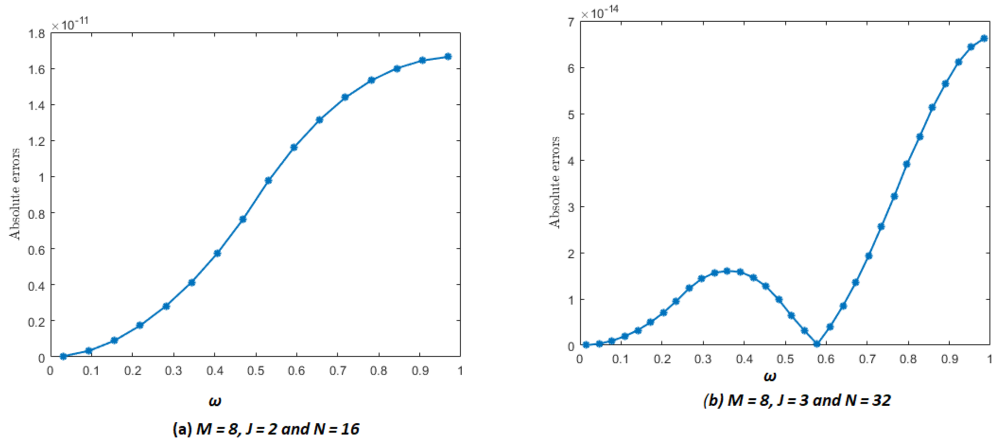


Figure 2. Effect of the convergence parameters M and J on the behaviour of the absolute errors

Problem 2. Let:

$$\dddot{C}(\omega - sin(\omega)) + \ddot{C}(\omega) + \dot{C}(\omega - sin(\omega)) + 2e^{-3C(\omega - sin(\omega))} = f(\omega), \tag{5.2}$$

which satisfies the BCs:

$$C(0) = 0,\ \dot{C}(0) = 1,\ C(1) = \ln(2).$$

The exact solution is

$$C(\omega) = \ln(1 + \omega).$$

The source function $f(\omega)$ can be calculated with the help of exact solution.

We have applied HWSM for solving equation (5.2). The obtained RMSE and MAE for various convergence parameters $(M,\ J)$ are reported in Table 2. The table clearly demonstrates that the RMSE and MAE progressively decline as we enhance the values of the convergence parameters $(M,\ J)$. A comparison of exact solution with HWSM solution is displayed in Figure 3, while the effect of the convergence parameters on the behaviour of absolute errors is shown in Figures 4a and 4b.

Table 2. Error analysis of Problem 2

N	(M, J)	RMSE	MAE
2	$(2, 1)$	$1.02024306e-03$	$7.91301587e-04$
8	$(3, 1)$	$6.73726635e-06$	$3.90098448e-06$
16	$(4, 2)$	$9.47481042e-08$	$4.29584032e-08$
32	$(5, 3)$	$1.47686240e-09$	$4.81675810e-10$

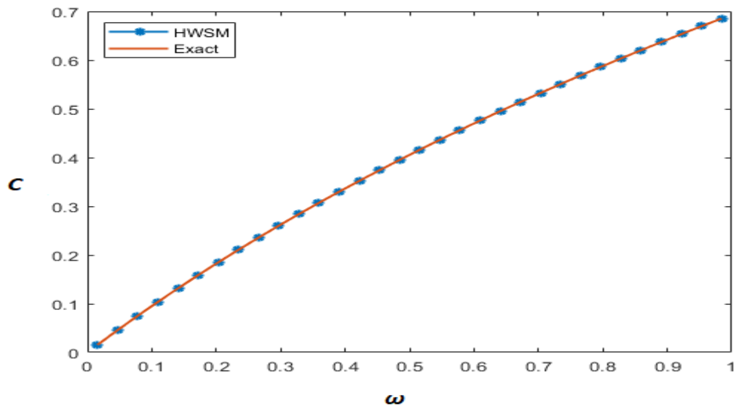
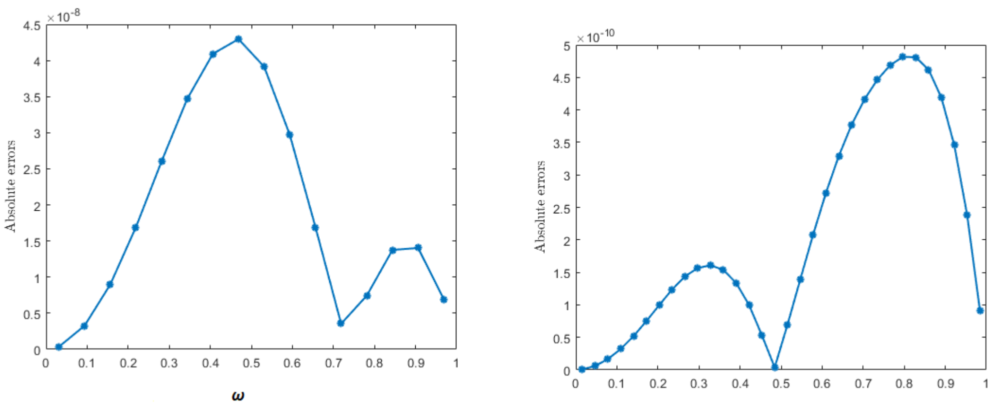


Figure 3. Graph of exact and HWM solutions of Problem 2 for $M = 8$ and $J = 3$



(a) M = 8, J = 2 and N = 16

(b) M = 8, J = 3 and N = 32

Figure 4. Effect of the convergence parameters M and J on the behaviour of the absolute errors

Conclusion

In this paper, we have used HWSM to get the numerical solution of third order NDDEs. Integral operator technique has been employed, i.e., the highest order derivative is estimated with regard to the Hermite wavelet basis and integration of the same is utilized to estimate the unknown variable and its lower order derivatives. The advantage of this technique is that we don't need to deal with the BCs separately. These conditions are automatically taken into consideration. We approximate delay term directly by using Hermite wavelet. We have tabulated the obtained MAE and RMSE in Tables 1 and 2. These tables demonstrate that the suggested method produces good results and converges very fast to the exact solution. Moreover, the proposed method is easy to execute. Figures 1 and 3, demonstrate the graphical comparison between the exact and HWSM solutions, Figures 2a, 2b, 4a, and 4b depict that how absolute errors behave. For all computational work we have used MATLAB 2021, intel i5 and windows 10.

References

[1] B. Ahmad, and J. J. Nieto and A. Alsaedi and M. El-Shahed, A study of nonlinear Langevin equation involving two fractional orders in different intervals ,*Nonlinear Analysis: Real World Applications* **13(2)**, 599–606 (2012).

[2] T. Y. Na, Computational methods in engineering boundary value problems, *New York, Academic Press, Inc.(Mathematics in Science and Engineering* **145**, (1979).

[3] H. Schlichting and J. Kestin, Boundary layer theory, *Springer*, **121**, (1961).

[4] E. Momoniat, Symmetries, first integrals and phase planes of a third-order ordinary differential equation from thin film flow, *Mathematical and Computer Modelling* **49 (1-2)**, 215–225 (2009).

[5] M. A. Khan and I. A. Tirmizi and E. H. Twizell and S. Islam, Non polynomial spline approach to the solution of a system of third-order boundary-value problems, *Applied Mathematics and Computation* **168 (1)**, 152–163 (2005).

[6] J. Lin, Y. Zhang and C-S. Liu, Solving nonlinear third-order three-point boundary value problems by boundary shape functions methods, *Advances in Difference Equations* **2021 (1)**, 1–23 (2021).

[7] S. K. Vanani and A. Aminataei, On the numerical solution of neutral delay differential equations using multiquadric approximation scheme, *Bulletin of the Korean Mathematical Society* **45 (4)**, 663–670 (2008).

[8] A. A. Timothy and A. P Olutola, Uniform stability and boundedness of solutions of nonlinear delay differential equations of the third order, *Mathematical Journal of Okayama University* **55 (1)**, 157–166 (2013).

[9] A. T. Ademola, A. Peter Olutola, O. M. Ogunlaran and O. A. Ezekiel, Uniform stability, boundedness and asymptotic behaviour of solutions of some third order nonlinear delay differential equations, *Differential Equations and control processes* **4** , 53708542 (2013).

[10] Z. Gui, Existence of positive periodic solutions to third-order delay differential equations, *Electronic Journal of Differential Equations* **2006 (91)**, 1–7 (2007)

[11] C. Tunç, Existence of periodic solutions to nonlinear differential equations of third order with multiple deviating arguments, *International Journal of Differential Equations* **2012**, 406835 (2012).

[12] H. O. Tejumola and B. Tchegnani, Stability, boundedness and existence of periodic solutions of some third and fourth order nonlinear delay differential equations, *J. Nigerian Math. Soc* **19**, 9–19 (2000).

[13] M.P Alam, T. Begum and A. Khan, A new spline algorithm for solving non-isothermal reaction diffusion model equations in a spherical catalyst and spherical biocatalyst, *Chemical Physics Letters* **754**, 137651 (2020).

[14] M.P Alam, T. Begum and A. Khan, A high-order numerical algorithm for solving Lane–Emden equations with various types of boundary conditions, *Computational and Applied Mathematics* **40 (6)**, 1–28 (2021).

[15] N. Senu, K. C. Lee, A. Ahmadian and S. N. I. Ibrahim, Numerical solution of delay differential equation using two-derivative Runge-Kutta type method with Newton interpolation, *Alexandria Engineering Journal* **61 (8)**, 5819–5835 (2022).

[16] H. Y. Seong and Z. A. Majid, Solving second order delay differential equations using direct two-point block method, *Ain Shams Engineering Journal* **8 (1)**, 59–66 (2017).

[17] E. Babolian and F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, *Applied Mathematics and computation* **188 (1)** , 417–426 (2007).

[18] C. Cattani and A. Kudreyko, Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind, *Applied Mathematics and Computation* **215 (12)** , 4164–4171 (2010).

[19] M. Faheem, A. Khan and A. Raza, A high resolution Hermite wavelet technique for solving space–time-fractional partial differential equations, *Mathematics and Computers in Simulation* **194** , 588–609 (2022).

[20] A. Khan, M. Faheem and A. Raza, Solution of third-order Emden–Fowler-type equations using wavelet methods, *Engineering Computations* **38 (6)**, 2850–2881 (2021).

[21] A. Raza and A. Khan, Haar wavelet series solution for solving neutral delay differential equations, *Journal of King Saud University-Science* **31 (4)**, 1070–1076 (2019).

[22] M. Faheem, A. Raza and A. Khan, Collocation methods based on Gegenbauer and Bernoulli wavelets for solving neutral delay differential equations, *Mathematics and Computers in Simulation* **180** , 72–92 (2021).

[23] C. K. Chui, An introduction to wavelets, *Academic press* (1992).

[24] L. Debnath and F. A. Shah, Wavelet transforms and their applications, *Springer* (2002).

[25] U. Saeed and M. Rehman, Hermite wavelet method for fractional delay differential equations, *Journal of Difference Equations*, **2014**, 359093 (2014).

[26] M. Faheem, A. Raza and A. Khan, Wavelet collocation methods for solving neutral delay differential equations *International Journal of Nonlinear Sciences and Numerical Simulation, doi.org/10.1515/ijnsns-2020-0103*, (2021).

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