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A High Resolution Half-step Numerical Approximation for 1D Quasilinear Hyperbolic Partial Differential Equations on a Time Variable Mesh

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Abstract In this article, using half-step discretization, we recommend a novel compact 3-time level implicit approximation of order 2 in time and 4 in space on a time variable mesh for the solution of 1D quasilinear second order hyperbolic partial differential equations (HPDEs). We have used variable mesh in time direction which is more general than uniform mesh and it controls spurious oscillations in time direction. Its benefit is that the solution does not fluctuate throughout the solution domain and is relatively smoother near the dense points. The suggested approximation when applied to transmission line equation which is an important HPDE of great physical significance is revealed to be stable on a time variable mesh for all choices of grid sizes in space direction. The proposed approximation is also applicable directly to solve HPDEs irrespective of coordinates on the time variable mesh. The proposed approximation is also applicable to solve fourth order HPDEs. The computational results show good performance on six numerical benchmark problems.

1 Introduction

Hyperbolic partial differential equations (HPDEs) occur in various physical models of wave phenomena where information is propagated at finite speeds. Gravitational and seismic waves, traffic flow and acoustic waves, electromagnetic and chemical waves, water and tsunami waves, planetary and ship waves are examples of physical waves. These waves are either linear which are depicted by linear equations or nonlinear which are elucidated by nonlinear equations. Therefore, the superposition principles are not applicable for nonlinear wave equations. The approximate results of 1D second order quasilinear HPDE plays an active role in numerous sectors of engineering and physical sciences. Mathematically, nonlinear HPDEs are more difficult to handle, and there are no common techniques available for solving such HPDEs. Therefore, the application of stable numerical schemes is the only way to knob such problems. Let us consider the 1D quasilinear HPDE

$$w_{tt} = B(t,x,w)w_{xx} + Q(t,x,w,w_x,w_t), \ \ a < x < b, \ \ t > 0, \tag{1.1}$$

with conditions at $t = 0$ are

$$w(x,0) = g(x), \quad w_t(x,0) = h(x), \ \ a \le x \le b, \tag{1.2}$$

and the values at $x = a$ and $x = b$ are prescribed by

$$w(a,t) = f_0(t), \quad w(b,t) = f_1(t), \ \ t > 0. \tag{1.3}$$

It is supposed that $B(t,x,w)$ is positive in the solution domain $\Omega = \{(t,x) : a < x < b, t > 0\}$. Let $w(x,t) \in C^6, B(t,x,w) \in C^4, f_0(t)$ and $f_1(t) \in C^0$on the boundary Γ of Ω. Further, we assume that the initial values $g(x)$ and $h(x)$ are sufficiently differentiable and the initial-boundary value problem (IBVP) (1.1)-(1.3) has a unique solution. Necessary information are given in [1].

A boundary value technique was introduced to obtain the approximate solution of the HPDE in [2]. Later on, the fourth order compact implicit finite difference method (FDM) was studied in [3-4]. An approximation for the HPDE with an extended stability interval was discussed in [5]. Stability intervals for multi-dimensional 3-level schemes for HPDEs using the additive operator technique were obtained in [6]. A 3-parameter family of FDMs for solving HPDEs were studied in [7-8]. Fourth order compact FDMs for the IBVP (1.1)-(1.3) were derived in [9-10]. An order two approximation for HPDE with nonlinear term in exponential form has been introduced in [11]. A bivariate spectral collocation method was presented in [12] to solve HPDEs defined over a large time domain. Jiwari et al. [13-16] used differential quadrature algorithms to solve HPDEs. Pandit et al. [17] employed Haar wavelet technique to solve wave equations.

It is not an easy task for engineers and research scientists to attain stability intervals of a numerical approximation for the IBVP (1.1)-(1.3). Stability investigation for second order initial-value problems were discussed in [18-20]. Many physical problems such as the voltage in coaxial transmission lines,the propagation of current-signals, parallel viscous Maxwell fluid flow, and the transmission of acoustic waves are illustrated by one dimensional Telegraphic equation. Implicit stable approximations for the solution of the transmission line (or Telegraphic equation on a constant mesh established by many researchers [21-32] using the knowledge used for initial value problems. High accuracy numerical methods using three grid points for the solution of two-point nonlinear BVPs on graded mesh were discussed by many scholars [33-35]. Mohanty & Gopal [36-38], and Mohanty & Khurana [39-41] have added spline techniques based on half-step discretization for the solution of 1D nonlinear HPDEs. Singh and Lin [42] have introduced higher order variable mesh off-step method for the solution of 1D non-linear HPDEs. In recent times, Mohanty et al. [43-45] have proposed absolute stability conditions on a variable mesh for a certain initial-value problem.

It has been repeatedly demonstrated on model problems that high order methods provide tremendous practical advantages in terms of reducing the required number of storages and overall computational time for desired solution in comparison with the lower order method. This motivates us to develop high accuracy method in order to solve model problems related to second order HPDEs. To our knowledge, no high accuracy approximations on a time variable mesh based on half-step space discretization has been discussed as on date. To fill this gap, we propose a new 3-level half-step implicit methods of order of accuracy 2 in t-direction, 4 in x-direction for the solution of IBVP (1.1)-(1.3) on a variable mesh in t-direction.The suggested approximation controls the fluctuations of numerical solution throughout the computation, especially around the dense points.The paper is segregated as follows: In Section 2, we describe the formulation of the half-step methods based on a variable mesh in t-direction for the approximation of IBVP (1.1)-(1.3). In section 3, we give the complete derivation of the numerical approximations. We study the application of the proposed approximations to the uniform transmission line equation and the stability of the corresponding difference equation in Section 4. Six benchmark problems have been computed in section 5 to verify the utility of the proposed method. Final remarks and future work are suggested in Section 6.

2 Conceptualization of the method

We consider 1D HPDE in nonlinear form

$$w_{tt} = B(x,t)w_{xx} + Q(x,t,w,w_x,w_t), \tag{2.1}$$

where $B(x,t)$ is positive in the domain $\Omega = \{(x,t) : a < x < b, t > 0\}$ which is roofed by a set of points (x_i, t_n), where $\Delta x > 0$ is the uniform grid spacing in x-direction with grid points $x_i = a + i\Delta x, i = 0,1,2,...,N+1, b-a = (N+1)\Delta x$ and non-uniform grid spacing $\Delta t_n = t_n - t_{n-1} > 0$ with $t_0 < t_1 < t_2 < \cdots < t_N$. Let $\eta = \left(\frac{\Delta t_{n+1}}{\Delta t_n}\right) > 0$, so that $\Delta t_{n+1} = \eta \Delta t_n$. For $\eta = 1$, the mesh lengths in t-direction are constant throughout the computation.

We consider $B_i^n = B(x_i, t_n), {B_x}_i^n = B_x(x_i, t_n), \ldots$ etc. Again, let $W_i^n = w(x_i, t_n)$ and $w_i^n \approx W_i^n$.

We consider the subsequent approximations:

$$\overline{W}^n_{i+\frac{1}{2}} = \frac{1}{2}\left[W^n_{i+1} + W^n_i\right]. \tag{2.2}$$

$$\overline{W}^n_{i-\frac{1}{2}} = \frac{1}{2}\left[W^n_{i-1} + W^n_i\right]. \tag{2.3}$$

$$\overline{W}_{xi}^{\ n} = \frac{1}{2\Delta x}\left[W^n_{i+1} - W^n_{i-1}\right]. \tag{2.4}$$

$$\overline{W}_{xi+\frac{1}{2}}^{\ n} = \frac{1}{\Delta x}\left[W^n_{i+1} - W^n_i\right]. \tag{2.5}$$

$$\overline{W}_{xi-\frac{1}{2}}^{\ n} = \frac{1}{\Delta x}\left[W^n_i - W^n_{i-1}\right]. \tag{2.6}$$

$$\overline{W}_{xi}^{\ n+1} = \frac{1}{2\Delta x}\left[W^{n+1}_{i+1} - W^{n+1}_{i-1}\right]. \tag{2.7}$$

$$\overline{W}_{xi}^{\ n-1} = \frac{1}{2\Delta x}\left[W^{n-1}_{i+1} - W^{n-1}_{i-1}\right]. \tag{2.8}$$

$$\overline{W}_{ti}^{\ n} = \frac{1}{\eta(1+\eta)\Delta t_n}\left[W^{n+1}_i - (1-\eta^2)W^n_i - \eta^2 W^{n-1}_i\right]. \tag{2.9}$$

$$\overline{W}_{ti+1}^{\ n} = \frac{1}{\eta(1+\eta)\Delta t_n}\left[W^{n+1}_{i+1} - (1-\eta^2)W^n_{i+1} - \eta^2 W^{n-1}_{i+1}\right]. \tag{2.10}$$

$$\overline{W}_{ti-1}^{\ n} = \frac{1}{\eta(1+\eta)\Delta t_n}\left[W^{n+1}_{i-1} - (1-\eta^2)W^n_{i-1} - \eta^2 W^{n-1}_{i-1}\right]. \tag{2.11}$$

$$\overline{W}_{ti+\frac{1}{2}}^{\ n} = \frac{1}{2}\left[\bar{W}_{ti+1}^{\ n} + \bar{W}_{ti}^{\ n}\right]. \tag{2.12}$$

$$\overline{W}_{ti-\frac{1}{2}}^{\ n} = \frac{1}{2}\left[\bar{W}_{ti-1}^{\ n} + \bar{W}_{ti}^{\ n}\right]. \tag{2.13}$$

$$\overline{W}_{ti}^{\ n+1} = \frac{1}{\eta(1+\eta)\Delta t_n}\left[(1+2\eta)W^{n+1}_i - (1+\eta)^2 W^n_i + \eta^2 W^{n-1}_i\right]. \tag{2.14}$$

$$\overline{W}_{ti}^{\ n-1} = \frac{1}{\eta(1+\eta)\Delta t_n}\left[-W^{n+1}_i + (1+\eta)^2 W^n_i - \eta(2+\eta) W^{n-1}_i\right]. \tag{2.15}$$

$$\overline{W}_{tti}^{\ n} = \frac{1}{\eta(1+\eta)\Delta t_n^2}\left[W^{n+1}_i - (1+\eta)W^n_i + \eta W^{n-1}_i\right]. \tag{2.16}$$

$$\overline{W}_{xxi}^{\ n} = \frac{1}{\Delta x^2}\left[W^n_{i+1} - 2W^n_i + W^n_{i-1}\right]. \tag{2.17}$$

$$\begin{aligned}\overline{W}_{xxti}^{\ n} &= \frac{1}{\eta(1+\eta)\Delta t_n \Delta x^2}\left[(W^{n+1}_{i+1} - 2W^{n+1}_i + W^{n+1}_{i-1}) - (1-\eta^2)(W^n_{i+1} - 2W^n_i + W^n_{i-1})\right] \\ &\quad - \frac{1}{\eta(1+\eta)\Delta t_n \Delta x^2}\left[\eta^2(W^{n-1}_{i+1} - 2W^{n-1}_i + W^{n-1}_{i-1})\right].\end{aligned} \tag{2.18}$$

$$\overline{W}_{xtti}^{\ n} = \frac{1}{\eta(1+\eta)\Delta t_n^2 \Delta x}\left[(W^{n+1}_{i+1} - W^{n+1}_{i-1}) - (1+\eta)(W^n_{i+1} - W^n_{i-1}) + \eta(W^{n-1}_{i+1} - W^{n-1}_{i-1})\right]. \tag{2.19}$$

$$\begin{aligned}\overline{W}_{xxtti}^{\ n} &= \frac{2}{\eta(1+\eta)\Delta t_n^2 \Delta x^2}\left[(W^{n+1}_{i+1} - 2W^{n+1}_i + W^{n+1}_{i-1}) - (1+\eta)(W^n_{i+1} - 2W^n_i + W^n_{i-1})\right] \\ &\quad + \frac{2}{\eta(1+\eta)\Delta t_n^2 \Delta x^2}\left[\eta(W^{n-1}_{i+1} - 2W^{n-1}_i + W^{n-1}_{i-1})\right].\end{aligned} \tag{2.20}$$

Further, We need the following approximations for $Q(x,t,w,w_x,w_t)$

$$\overline{Q}_i^n = Q\left(x_i, t_n, W_i^n, \overline{W}_{xi}^{\,n}, \overline{W}_{ti}^{\,n}\right). \tag{2.21}$$

$$\overline{Q}_{i+\frac{1}{2}}^n = Q\left(x_{i+\frac{1}{2}}, t_n, \overline{W}_{i+\frac{1}{2}}^n, \overline{W}_{xi+\frac{1}{2}}^{\,n}, \overline{W}_{ti+\frac{1}{2}}^{\,n}\right). \tag{2.22}$$

$$\overline{Q}_{i-\frac{1}{2}}^n = Q\left(x_{i-\frac{1}{2}}, t_n, \overline{W}_{i-\frac{1}{2}}^n, \overline{W}_{xi-\frac{1}{2}}^{\,n}, \overline{W}_{ti-\frac{1}{2}}^{\,n}\right). \tag{2.23}$$

$$\overline{Q}_i^{n+1} = Q\left(x_i, t_{n+1}, W_i^{n+1}, \overline{W}_{xi}^{\,n+1}, \overline{W}_{ti}^{\,n+1}\right). \tag{2.24}$$

$$\overline{Q}_i^{n-1} = Q\left(x_i, t_{n-1}, W_i^{n-1}, \overline{W}_{xi}^{\,n-1}, \overline{W}_{ti}^{\,n-1}\right). \tag{2.25}$$

Now, we define the approximation

$$\overline{\overline{W}}_i^n = W_i^n - \frac{\eta\Delta x^2}{4(\eta^2-\eta+1)}\overline{W}_{xxi}^{\,n}. \tag{2.26}$$

Next, we define the approximation for space derivative

$$\overline{\overline{W}}_{xi}^{\,n} = \overline{W}_{xi}^{\,n} + \frac{\eta\Delta x}{4(\eta^2-\eta+1)B_i^n}\left[\left(\overline{Q}_{i+\frac{1}{2}}^n - \overline{Q}_{i-\frac{1}{2}}^n\right) - \Delta x\,\overline{W}_{xtti}^{\,n} + \Delta x\, B_{xi}^n\,\overline{W}_{xxi}^{\,n}\right]. \tag{2.27}$$

and the approximation for the time derivative

$$\overline{\overline{W}}_{ti}^{\,n} = \overline{W}_{ti}^{\,n} - \frac{\eta\Delta x^2}{4(\eta^2-\eta+1)}\overline{W}_{xxti}^{\,n}. \tag{2.28}$$

Further, we define

$$\overline{\overline{Q}}_i^n = Q\left(x_i, t_n, \overline{\overline{W}}_i^n, \overline{\overline{W}}_{xi}^{\,n}, \overline{\overline{W}}_{ti}^{\,n}\right). \tag{2.29}$$

Then the HPDE (2.1) at each grid point (x_i, t_n) is approximated by

$$\begin{aligned}
L_w \equiv\; & \overline{W}_{tti}^{\,n} - \left[B_i^n - \frac{\Delta x^2}{6}\left(\frac{B_{xi}^{\,n}}{B_i^n}\right)B_{xi}^{\,n} + \frac{(\eta-1)}{3}\Delta t_n B_{ti}^{\,n} + \frac{\Delta x^2}{12}B_{xxi}^{\,n}\right]\overline{W}_{xxi}^{\,n} \\
& - \frac{\Delta x^2}{6}\left(\frac{B_{xi}^{\,n}}{B_i^n}\right)\overline{W}_{xtti}^{\,n} - \frac{(\eta-1)}{3}\Delta t_n B_i^n \overline{W}_{xxti}^{\,n} + \frac{\Delta x^2}{12}\overline{W}_{xxtti}^{\,n} \\
= \;& \frac{1}{3}\left[\left(1 - \frac{\Delta x B_{xi}^{\,n}}{2B_i^n}\right)\overline{Q}_{i+\frac{1}{2}}^n + \left(1 + \frac{\Delta x B_{xi}^{\,n}}{2B_i^n}\right)\overline{Q}_{i-\frac{1}{2}}^n + \overline{\overline{Q}}_i^n\right]. \\
& + \frac{1}{3}\left[\frac{(\eta-1)}{\eta(\eta+1)}\left(\overline{Q}_i^{n+1} - (1-\eta^2)\overline{\overline{Q}}_i^n - \eta^2\overline{Q}_i^{n-1}\right)\right] + \overline{T}_i^n
\end{aligned} \tag{2.30}$$

where the local truncation error (LTE) $= \overline{T}_i^n = O(\Delta t_n^2 + \Delta t_n \Delta x^2 + \Delta x^4)$.

3 Deducing the numerical methods

To derive the approximation (2.30), we use variable approximations for time derivatives and Numerov type constant mesh approximations for space derivatives.
We denote:

$$W_{\alpha\beta} = \left(\frac{\delta^{\alpha+\beta}W}{\delta x^\alpha \delta t^\beta}\right)_i^n, \;\; \psi_i^n = \left(\frac{\delta Q}{\delta W}\right)_i^n, \;\; \phi_i^n = \left(\frac{\delta Q}{\delta W_x}\right)_i^n, \;\; \Phi_i^n = \left(\frac{\delta Q}{\delta W_t}\right)_i^n. \tag{3.1}$$

At (x_i, t_n), the nonlinear HPDE (2.1) takes the form

$$W_{tti}^{\,n} - B_i^n W_{xxi}^{\,n} = Q\left(x_i, t_n, W_i^n, W_{xi}^{\,n}, W_{ti}^{\,n}\right) \equiv Q_i^n. \tag{3.2}$$

It is easy to demonstrate that

$$L_w = \frac{1}{3}\left[\left(1-\frac{\Delta x {B_x}_i^n}{2B_i^n}\right)Q_{i+\frac{1}{2}}^n + \left(1+\frac{\Delta x {B_x}_i^n}{2B_i^n}\right)Q_{i-\frac{1}{2}}^n + Q_i^n\right].$$

$$+\frac{1}{3}\left[\frac{(\eta-1)}{\eta(\eta+1)}\left(Q_i^{n+1}-(1-\eta^2)Q_i^n-\eta^2 Q_i^{n-1}\right)\right]+O(\Delta t_n^2+\Delta t_n\Delta x^2+\Delta x^4) \quad (3.3)$$

Simplifying (2.2)-(2.20), we get

$$\overline{W}_{i+\frac{1}{2}}^n = W_{i+\frac{1}{2}}^n + \frac{\Delta x^2}{8}\frac{\delta^2 W_i^n}{\delta x^2} + O(\Delta x^3). \quad (3.4)$$

$$\overline{W}_{i-\frac{1}{2}}^n = W_{i-\frac{1}{2}}^n + \frac{\Delta x^2}{8}\frac{\delta^2 W_i^n}{\delta x^2} - O(\Delta x^3). \quad (3.5)$$

$$\overline{W}_{xi}^{\;n} = W_{xi}^{\;n} + \frac{\Delta x^2}{6}W_{30} + O(\Delta x^4). \quad (3.6)$$

$$\overline{W}_{xi+\frac{1}{2}}^{\;n} = W_{xi+\frac{1}{2}}^{\;n} + \frac{\Delta x^2}{24}W_{30} + O(\Delta x^3). \quad (3.7)$$

$$\overline{W}_{xi-\frac{1}{2}}^{\;n} = W_{xi-\frac{1}{2}}^{\;n} + \frac{\Delta x^2}{24}W_{30} - O(\Delta x^3). \quad (3.8)$$

$$\overline{W}_{xi}^{\;n+1} = W_{xi}^{\;n+1} + \frac{\Delta x^2}{6}W_{30} + O(\Delta t_n\Delta x^2). \quad (3.9)$$

$$\overline{W}_{xi}^{\;n-1} = W_{xi}^{\;n-1} + \frac{\Delta x^2}{6}W_{30} + O(\Delta t_n\Delta x^2). \quad (3.10)$$

$$\overline{W}_{ti}^{\;n} = W_{ti}^{\;n} + O(\Delta t_n^2). \quad (3.11)$$

$$\overline{W}_{ti+1}^{\;n} = W_{ti+1}^{\;n} + O(\Delta t_n^2). \quad (3.12)$$

$$\overline{W}_{ti-1}^{\;n} = W_{ti-1}^{\;n} + O(\Delta t_n^2). \quad (3.13)$$

$$\overline{W}_{ti+\frac{1}{2}}^{\;n} = W_{ti+\frac{1}{2}}^{\;n} + \frac{\Delta x^2}{8}W_{21} + O(\Delta t_n^2 + \Delta x^3). \quad (3.14)$$

$$\overline{W}_{ti-\frac{1}{2}}^{\;n} = W_{ti-\frac{1}{2}}^{\;n} + \frac{\Delta x^2}{8}W_{21} + O(\Delta t_n^2 - \Delta x^3). \quad (3.15)$$

$$\overline{W}_{ti}^{\;n+1} = W_{ti}^{\;n+1} + O(\Delta t_n^2). \quad (3.16)$$

$$\overline{W}_{ti}^{\;n-1} = W_{ti}^{\;n-1} + O(\Delta t_n^2). \quad (3.17)$$

$$\overline{W}_{tti}^{\;n} = W_{tti}^{\;n} + \frac{(\eta-1)}{3}\Delta t_n W_{03} + O(\Delta t_n^2). \quad (3.18)$$

$$\overline{W}_{xxi}^{\;n} = W_{xxi}^{\;n} + \frac{\Delta x^2}{12}W_{40} + O(\Delta x^4). \quad (3.19)$$

$$\overline{W}_{xxti}^{\;n} = W_{xxti}^{\;n} + \frac{\Delta x^2}{12}W_{41} + O(\Delta t_n^2 + \Delta t_n\Delta x^2 + \Delta x^4). \quad (3.20)$$

$$\overline{W}_{xtti}^{\;n} = W_{xtti}^{\;n} + \frac{(\eta-1)}{3}\Delta t_n W_{13} + O(\Delta t_n^2). \quad (3.21)$$

$$\overline{W}_{xxtti}^{\;n} = W_{xxtti}^{\;n} + \frac{(\eta-1)}{3}\Delta t_n W_{23} + \frac{\Delta x^2}{12}W_{42} + O(\Delta t_n^2 + \Delta t_n\Delta x^2 + \Delta x^4). \quad (3.22)$$

With the help of (3.4) – (3.17), simplifying (2.21)-(2.25), we get

$$\overline{Q}_i^n = Q_i^n + \frac{\Delta x^2}{6}W_{30}\phi_i^n + O(\Delta t_n^2 + \Delta x^4). \quad (3.23)$$

$$\overline{Q}_{i+\frac{1}{2}}^n = Q_{i+\frac{1}{2}}^n + \frac{\Delta x^2}{24}T_1 + O(\Delta t_n^2 + \Delta x^3). \quad (3.24)$$

where
$T_1 = 3W_{20}\psi_i^n + W_{30}\phi_i^n + 3W_{21}\Phi_i^n.$

$$\overline{Q}_{i-\frac{1}{2}}^n = Q_{i-\frac{1}{2}}^n + \frac{\Delta x^2}{24}T_1 + O(\Delta t_n^2 + \Delta x^3). \tag{3.25}$$

$$\overline{Q}_i^{n+1} = Q_i^{n+1} + \frac{\Delta x^2}{6}W_{30}\phi_i^n + O(\Delta t_n^2 + \Delta t_n \Delta x^2). \tag{3.26}$$

$$\overline{Q}_i^{n-1} = Q_i^{n-1} + \frac{\Delta x^2}{6}W_{30}\phi_i^n + O(\Delta t_n^2 + \Delta t_n \Delta x^2). \tag{3.27}$$

Now, we define the new approximations:

$$\overline{\overline{W}}_i^n = W_i^n + a_1\Delta x^2 \overline{W}_{xx_i}^n, \tag{3.28}$$

$$\overline{\overline{W}}_{x_i}^n = \overline{W}_{x_i}^n + b_1\Delta x\left(\overline{Q}_{i+\frac{1}{2}}^n - \overline{Q}_{i-\frac{1}{2}}^n\right) + b_2\Delta x^2\, \overline{W}_{xtt_i}^n + b_3\Delta x^2 \overline{W}_{xx_i}^n, \tag{3.29}$$

$$\overline{\overline{W}}_{t_i}^n = \overline{W}_{t_i}^n + c_1\Delta x^2 \overline{W}_{xxt_i}^n, \tag{3.30}$$

where a_1, b_1, b_2, b_3, and c_1 are variable quantities to be calculated.

With the help of (3.6), (3.11), (3.19), (3.20), (3.21), (3.24), (3.25) from (3.28), (3.29), and (3.30), we get

$$\overline{\overline{W}}_i^n = W_i^n + a_1\Delta x^2 W_{xx_i}^n + O(\Delta x^4). \tag{3.31}$$

$$\overline{\overline{W}}_{t_i}^n = W_{t_i}^n + c_1\Delta x^2 W_{xxt_i}^n + O(\Delta t_n^2 + \Delta x^4). \tag{3.32}$$

$$\overline{\overline{W}}_{x_i}^n = W_{x_i}^n + \frac{\Delta x^2}{6}T_2 + O(\Delta t_n \Delta x^2 + \Delta x^4), \tag{3.33}$$

where $T_2 = (1 - 6b_1\ B_i^n)W_{30} + 6(b_1 + b_2)W_{12} + 6(b_3 - b_1\ B_{x_i}^n)W_{20}$.

With the assistance of (3.31), (3.32) and (3.33) and simplifying (2.29)

$$\overline{\overline{Q}}_i^n = Q_i^n + \frac{\Delta x^2}{6}T_3 + O(\Delta t_n^2 + \Delta t_n\Delta x^2 + \Delta x^4), \tag{3.34}$$

where
$T_3 = 6a_1\ W_{20}\ \psi_i^n + T_2\ \phi_i^n + 6c_1\ W_{21}\ \Phi_i^n.$

With the aid of (2.30), (3.3)-(3.22), (3.24)-(3.27), (3.34), the LTE can be written as

$$\begin{aligned}\overline{T}_i^n =& -\frac{\Delta x^2}{3}\left[\left(\frac{1}{4} + \left(\frac{\eta^2-\eta+1}{\eta}\right)a_1\right)W_{20}\ \psi_i^n + \left(\frac{1}{4} + \left(\frac{\eta^2-\eta+1}{\eta}\right)c_1\right)W_{21}\ \Phi_i^n\right]\\ &- \frac{\Delta x^2}{3}\left[\frac{1}{12}\left(1 + \frac{2(\eta-1)(1-\eta^2)}{\eta(1+\eta)} + \frac{2(\eta^2-\eta+1)}{\eta}(1-6b_1\ B_i^n)\right)W_{30}\ \phi_i^n\right]\\ &- \frac{\Delta x^2}{3}\left[\left(\frac{\eta^2-\eta+1}{\eta}\right)(b_1+b_2)W_{12}\ \phi_i^n + \left(\frac{\eta^2-\eta+1}{\eta}\right)(b_3 - b_1 B_{x_i}^n)W_{20}\ \phi_i^n\right]\\ &+ O(\Delta t_n^2 + \Delta t_n\Delta x^2 + \Delta x^4).\end{aligned} \tag{3.35}$$

The proposed method (2.30) to be of $O(\Delta t_n^2 + \Delta t_n\Delta x^2 + \Delta x^4)$, the factors associated with

$\Delta x^2$of LTE must be vanish and we get

$$\frac{1}{4}+\left(\frac{\eta^2-\eta+1}{\eta}\right)a_1=0. \tag{3.36}$$

$$1+\frac{2(\eta-1)(1-\eta^2)}{\eta(1+\eta)}+\frac{2(\eta^2-\eta+1)}{\eta}(1-6b_1B_i^n)=0. \tag{3.37}$$

$$\frac{1}{4}+\left(\frac{\eta^2-\eta+1}{\eta}\right)c_1=0. \tag{3.38}$$

$$\left(\frac{\eta^2-\eta+1}{\eta}\right)(b_1+b_2)=0. \tag{3.39}$$

$$\left(\frac{\eta^2-\eta+1}{\eta}\right)(b_3-b_1{B_x}_i^n)=0. \tag{3.40}$$

Consequently, on solving above set of equations (3.36)-(3.40), we obtain the values of parameters

$$a_1=c_1=\frac{-\eta}{4(\eta^2-\eta+1)},\ \ b_1=\frac{\eta}{4(\eta^2-\eta+1)B_i^n},$$

$$b_2=\frac{-\eta}{4(\eta^2-\eta+1)B_i^n},\ \ b_3=\frac{\eta B_{x_i}^n}{4(\eta^2-\eta+1)B_i^n}$$

and the LTE (3.35) reduces to $\overline{T}_i^n=O(\Delta t_n^2+\Delta t_n\Delta x^2+\Delta x^4)$.

For quasilinear differential equation (1.1), i.e. whenever $B=B(t,x,w)$, the method (2.30) requires to be amended which is done by employing following approximations:

$$B_{t_i}^n=\frac{1}{\eta(1+\eta)\Delta t_n}[B_i^{n+1}-(1-\eta^2)B_i^n-\eta^2B_i^{n-1}]+O(\Delta t_n^2), \tag{3.41}$$

$$B_{x_i}^n=\frac{1}{2\Delta x}[B_{i+1}^n-B_{i-1}^n]+O(\Delta x^2), \tag{3.42}$$

$$B_{xx_i}^n=\frac{1}{\Delta x^2}[B_{i+1}^n-2B_i^n+B_{i-1}^n]+O(\Delta x^2) \tag{3.43}$$

where

$$B_i^n=B(x_i,t_n,w_i^n),\ \ B_{i\pm1}^n=B(x_{i\pm1},t_n,w_{i\pm1}^n),\ \ B_i^{n\pm1}=B(x_i,t_{n\pm1},w_i^{n\pm1}).$$

Employing (3.41)-(3.43) into (2.30), we achieve the numerical method of $O(\Delta t_n^2+\Delta t_n\Delta x^2+\Delta x^4)$ for the quasilinear HPDE (1.1) and LTE preserves its order. For $\eta=1$, i.e. $\Delta t_n=\Delta t_{n+1}=\Delta t$, the method (2.30) set off of $O(\Delta t^2+\Delta t^2\Delta x^2+\Delta x^4)$.

Incorporating the prescribed initial and boundary values (1.2)-(1.3), the algorithm (2.30) can be revealed in a 3-diagonal matrix shape at every time level. The Gauss-elimination procedure and Newton-Raphson method [46-51] are used to solve linear and nonlinear difference equations, respectively.

4 Stability consideration

The transmission line equation is represented by

$$w_{tt}+2pw_t+q^2w=w_{xx}+g(x,t),\ \ p>0,\ q\geq 0, \tag{4.1}$$

which is defined in the region $[a<x<b]\times[t>0]$, where p,q are constants. For $q=0$, the equation (4.1) reduced to damped wave equation. Let $X_n=p^2\Delta t_n^2$, $Y_n=q^2\Delta t_n^2$ and $\lambda_n=\frac{\Delta t_n}{\Delta x}>0$.

With the application of (2.30) the HPDE (4.1) is consistent with

$$\overline{W}_{tt_i}^{\ n} - \overline{W}_{xx_i}^{\ n} - \left(\frac{\eta-1}{3}\right)\Delta t_n \overline{W}_{xxt_i}^{\ n} + \frac{\Delta x^2}{12}\overline{W}_{xxtt_i}^{\ n}$$
$$+ \frac{1}{3}\left[2p\frac{(\eta-1)}{\eta(\eta+1)}\left(\overline{W}_{t_i}^{\ n+1} - (1-\eta^2)\overline{\overline{W}}_{t_i}^{\ n} - \eta^2\overline{W}_{t_i}^{\ n-1}\right) + 2p\left(\overline{W}_{t_{i+\frac{1}{2}}}^{\ n} + \overline{W}_{t_{i-\frac{1}{2}}}^{\ n} + \overline{\overline{W}}_{t_i}^{\ n}\right)\right]$$
$$+ \frac{1}{3}\left[q^2\left(\overline{W}_{i+\frac{1}{2}}^{n} + \overline{W}_{i-\frac{1}{2}}^{n} + \overline{\overline{W}}_{i}^{n}\right) + \frac{q^2(\eta-1)}{\eta(\eta+1)}\left(W_i^{n+1} - (1-\eta^2)\overline{\overline{W}}_i^{n} - \eta^2 W_i^{n-1}\right)\right]$$
$$+ \frac{\sum g}{3} + O(\Delta t_n^2 + \Delta t_n \Delta x^2 + \Delta x^4), \tag{4.2}$$

where
$g_i^n = g(x_i, t_n)$ and $\sum g = \left(g_{i+\frac{1}{2}}^n + g_{i-\frac{1}{2}}^n + g_i^n\right) + \frac{(\eta-1)}{\eta(\eta+1)}\left(g_i^{n+1} - (1-\eta^2)g_i^n - \eta^2 g_i^{n-1}\right)$.

Let us signify: $\delta_x W_i^n = W_{i+\frac{1}{2}} - W_{i-\frac{1}{2}}$ be the central difference approximation in x-direction. This implies $\delta_x^2 W_i^n = W_{i+1}^n - 2W_i^n + W_{i-1}^n$. To simplify (4.2), we use the following:

$$\overline{W}_{t_{i+\frac{1}{2}}}^{\ n} + \overline{W}_{t_{i-\frac{1}{2}}}^{\ n} + \overline{\overline{W}}_{t_i}^{\ n} = \left[\frac{(6+\delta_x^2)}{\eta(1+\eta)2\Delta t_n} - \frac{\eta\delta_x^2}{4\eta(1+\eta^3)\Delta t_n}\right]\left[W_i^{n+1} - (1-\eta^2)W_i^n - \eta^2 W_i^{n-1}\right]. \tag{4.3}$$

$$\overline{W}_{i+\frac{1}{2}}^{n} + \overline{W}_{i-\frac{1}{2}}^{n} + \overline{\overline{W}}_{i}^{n} = \left[\frac{(6+\delta_x^2)}{2} - \frac{\delta_x^2}{4(1-\eta+\eta^2)}\right]W_i^n. \tag{4.4}$$

$$\overline{W}_{t_i}^{\ n+1} - (1-\eta^2)\overline{W}_{t_i}^{\ n} - \eta^2\overline{W}_{t_i}^{\ n-1} = \frac{2}{\Delta t_n}\left[W_i^{n+1} - (1+\eta)W_i^n + \eta W_i^{n-1}\right]$$
$$+ \frac{(1-\eta)\delta_x^2}{4(1-\eta+\eta^2)\Delta t_n}\left[W_i^{n+1} - (1-\eta^2)W_i^n - \eta^2 W_i^{n-1}\right]. \tag{4.5}$$

$$W_i^{n+1} - (1-\eta^2)\overline{\overline{W}}_i^{n} - \eta^2 W_i^{n-1} = W_i^{n+1} - (1-\eta^2)W_i^n - \eta^2 W_i^{n-1} + \frac{\eta(1-\eta^2)}{4(1-\eta+\eta^2)}\delta_x^2 W_i^n. \tag{4.6}$$

Multiplying $\frac{\eta(1+\eta)}{2}\Delta t_n^2$ throughout (4.2), using the relations (4.3) – (4.6) and simplifying, we get

$$\left[1 + \frac{\delta_x^2}{12} + \frac{2(\eta-1)}{3}\sqrt{X_n}\right]\left[W_i^{n+1} - (1+\eta)W_i^n + \eta W_i^{n-1}\right]$$
$$+ \left[\sqrt{X_n}\left(1+\frac{\delta_x^2}{12}\right) + \left(\frac{\eta-1}{6}\right)Y_n - \left(\frac{\eta-1}{6}\right)\lambda_n^2\delta_x^2\right]\left[W_i^{n+1} - (1-\eta^2)W_i^n - \eta^2 W_i^{n-1}\right]$$
$$- \frac{\eta(1+\eta)}{2}\lambda_n^2\delta_x^2 W_i^n + \frac{\eta(1+\eta)}{2}Y_n\left(1+\frac{\delta_x^2}{12}\right)W_i^n$$
$$= \frac{\eta(1+\eta)\Delta t_n^2}{6}\sum g + O(\Delta t_n^4 + \Delta t_n^3\Delta x^2 + \Delta t_n^2\Delta x^4) \tag{4.7}$$

The scheme (4.7) is stable under certain restrictions on the choices of grid sizes. In order to avoid these difficulties, we can modify the scheme (4.7) as

$$\left[1 + \frac{\delta_x^2}{12} + \frac{2(\eta-1)}{3}\sqrt{X_n} + \gamma_1 Y_n - \gamma_2\lambda_n^2\delta_x^2\right]\left[W_i^{n+1} - (1+\eta)W_i^n + \eta W_i^{n-1}\right] - \frac{\eta(1+\eta)}{2}\lambda_n^2\delta_x^2 W_i^n$$
$$+ \left[\sqrt{X_n}\left(1+\frac{\delta_x^2}{12}\right) + \left(\frac{\eta-1}{6}\right)Y_n - \left(\frac{\eta-1}{6}\right)\lambda_n^2\delta_x^2\right]\left[W_i^{n+1} - (1-\eta^2)W_i^n - \eta^2 W_i^{n-1}\right]$$
$$+ \frac{\eta(1+\eta)}{2}Y_n\left(1+\frac{\delta_x^2}{12}\right)W_i^n = \frac{\eta(1+\eta)\Delta t_n^2}{6}\sum g + O(\Delta t_n^4 + \Delta t_n^3\Delta x^2 + \Delta t_n^2\Delta x^4) \tag{4.8}$$

where γ_1, γ_2 are parameters independent of step lengths to be find out and the extra added higher order term $\left[\gamma_1 Y_n - \gamma_2 \lambda_n^2 \delta_x^2\right] \left[W_i^{n+1} - (1+\eta)W_i^n + \eta W_i^{n-1}\right]$ does not affect the accuracy of the modified scheme.

Let $\epsilon_i^n = W_i^n - w_i^n$ be an error defined at each (x_i, t_n). Then the error equation is expressed as

$$\left[1 + \frac{\delta_x^2}{12} + \frac{2(\eta-1)}{3}\sqrt{X_n} + \gamma_1 Y_n - \gamma_2 \lambda_n^2 \delta_x^2\right]\left[\epsilon_i^{n+1} - (1+\eta)\epsilon_i^n + \eta\epsilon_i^{n-1}\right] - \frac{\eta(1+\eta)}{2}\lambda_n^2\delta_x^2\epsilon_i^n$$
$$+ \left[\sqrt{X_n}\left(1 + \frac{\delta_x^2}{12}\right) + \left(\frac{\eta-1}{6}\right)Y_n - \left(\frac{\eta-1}{6}\right)\lambda_n^2\delta_x^2\right]\left[\epsilon_i^{n+1} - (1-\eta^2)\epsilon_i^n - \eta^2\epsilon_i^{n-1}\right]$$
$$+ \frac{\eta(1+\eta)}{2}Y_n\left(1 + \frac{\delta_x^2}{12}\right)\epsilon_i^n = O(\Delta t_n^4 + \Delta t_n^3 \Delta x^2 + \Delta t_n^2 \Delta x^4) \tag{4.9}$$

Neglecting error term and using $\epsilon_i^n = \zeta^n . \exp(\theta i\sqrt{-1})$ in the equation (4.9), the characteristic equation is obtained as

$$A_1\zeta^2 + B_1\zeta + C_1 = 0, \tag{4.10}$$

where

$$A_1 = \left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right) + \frac{2(\eta-1)}{3}\sqrt{X_n} + \gamma_1 Y_n + 4\gamma_2\lambda_n^2 sin^2\frac{\theta}{2}\right]$$
$$+ \left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right)\sqrt{X_n} + \left(\frac{\eta-1}{6}\right)Y_n + \frac{2(\eta-1)}{3}\lambda_n^2 sin^2\frac{\theta}{2}\right], \tag{4.11}$$

$$B_1 = -(1+\eta)\left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right) + \frac{2(\eta-1)}{3}\sqrt{X_n} + \gamma_1 Y_n + 4\gamma_2\lambda_n^2 sin^2\frac{\theta}{2}\right]$$
$$- (1-\eta^2)\left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right)\sqrt{X_n} + \left(\frac{\eta-1}{6}\right)Y_n + \frac{2(\eta-1)}{3}\lambda_n^2 sin^2\frac{\theta}{2}\right]$$
$$+ 2\eta(1+\eta)\lambda_n^2 sin^2\frac{\theta}{2} + \left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right)\frac{\eta(1+\eta)}{2}Y_n, \tag{4.12}$$

$$C_1 = \eta\left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right) + \frac{2(\eta-1)}{3}\sqrt{X_n} + \gamma_1 Y_n + 4\gamma_2\lambda_n^2 sin^2\frac{\theta}{2}\right]$$
$$- \eta^2\left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right)\sqrt{X_n} + \left(\frac{\eta-1}{6}\right)Y_n + \frac{2(\eta-1)}{3}\lambda_n^2 sin^2\frac{\theta}{2}\right]. \tag{4.13}$$

For $|\zeta| < 1$, the conditions are $A_1 + B_1 + C_1 > 0,\ A_1 - C_1 > 0$ and $A_1 - B_1 + C_1 > 0$.

Now the condition

$$A_1 + B_1 + C_1 = 2\eta(1+\eta)\lambda_n^2 sin^2\frac{\theta}{2} + \left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right)\frac{\eta(1+\eta)}{2}Y_n > 0, \tag{4.14}$$

is fulfilled for all $\theta \in (0, 2\pi), p > 0, q \geq 0$ excluding $\theta = 0, 2\pi$ and $q = 0$.
The constraint

$$A_1 - C_1 = (1-\eta)\left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right) + \left(\gamma_1 - \frac{(1+\eta^2)}{6}\right)Y_n + 4\left(\gamma_2 - \frac{(1+\eta^2)}{6}\right)\lambda_n^2 sin^2\frac{\theta}{2}\right]$$
$$+ \frac{\left[4\eta + (1+\eta^2)cos^2\frac{\theta}{2}\right]\sqrt{X_n}}{3} > 0 \tag{4.15}$$

must be satisfied for all $p > 0,\ q \geq 0$ provided $0 < \eta \leq 1, \gamma_1 \geq \frac{1+\eta^2}{6},\ \gamma_2 \geq \frac{1+\eta^2}{6}$.
Finally, the condition

$$A_1 - B_1 + C_1 = 2(1+\eta)\left[\left(1 - \frac{1}{3}sin^2\frac{\theta}{2}\right) + \frac{(1-\eta)}{3}\sqrt{X_n}cos^2\frac{\theta}{2} + \left(\gamma_1 - \frac{(2\eta^2-\eta+2)}{12}\right)Y_n\right]$$
$$+ 4\left[\left(\gamma_2 - \frac{(2\eta^2-\eta+2)}{12}\right)\lambda_n^2 sin^2\frac{\theta}{2} + \frac{\eta}{12}Y_n sin^2\frac{\theta}{2}\right] > 0, \tag{4.16}$$

must be satisfied for all $p > 0, q \geq 0$ provided $0 < \eta \leq 1$ and $\gamma_1 \geq \frac{2\eta^2-\eta+2}{12},\ \gamma_2 \geq \frac{2\eta^2-\eta+2}{12}$.

For $q = 0$ and $\theta = 0$ or 2π, the equation (4.10) becomes

$$\left[1+\left(\frac{1+2\eta}{3}\right)\sqrt{X_n}\right]\zeta^2-\left[(1+\eta)\left(1+\frac{2(\eta-1)}{3}\sqrt{X_n}\right)+(1-\eta^2)\sqrt{X_n}\right]\zeta$$
$$+\eta\left[1+\frac{2(\eta-1)}{3}\sqrt{X_n}-\eta\sqrt{X_n}\right]=0. \tag{4.17}$$

If ζ_1 and ζ_2 are two roots of (4.17), we have the relations

$$\zeta_1+\zeta_2 = 1+\frac{\eta\left[1-(\frac{2+\eta}{3})\sqrt{X_n}\right]}{1+(\frac{1+2\eta}{3})\sqrt{X_n}}, \tag{4.18}$$

$$\zeta_1.\zeta_2 = \frac{\eta\left[1-(\frac{2+\eta}{3})\sqrt{X_n}\right]}{1+(\frac{1+2\eta}{3})\sqrt{X_n}}. \tag{4.19}$$

On solving (4.18) and (4.19), we get $\zeta_1 = 1,\ \zeta_2 = \frac{\eta\left[1-(\frac{2+\eta}{3})\sqrt{X_n}\right]}{1+(\frac{1+2\eta}{3})\sqrt{X_n}}$. In this situation also $|\zeta| \leq 1$, subject to $0 < \eta \leq 1$.

Because $\frac{1+\eta^2}{6} > \frac{2-\eta+2\eta^2}{12}$, the conditions (4.14)-(4.16) are contended for all varying angle $\theta,\ p > 0,\ q \geq 0$ subject to $0 < \eta \leq 1,\ \gamma_1 \geq \frac{1+\eta^2}{6},\ \gamma_2 \geq \frac{1+\eta^2}{6}$. Thus for $p > 0,\ q \geq 0$, $0 < \eta \leq 1,\ \gamma_1 \geq \frac{1+\eta^2}{6},\ \gamma_2 \geq \frac{1+\eta^2}{6}$, the approximation (4.8) is stable for all values of $\Delta t_n > 0$, $\Delta x > 0$.

5 Numerical results

With the aid of (2.4), (2.7)-(2.9), (2.14)-(2.18), (2.21), (2.24)-(2.25), a time-variable method of $O(\Delta t_n^2 + \Delta x^2)$ may be written as

$$\overline{W}_{tt_i}^{\ n}-\left[B_i^n+\frac{(\eta-1)}{3}\Delta t_n B_{t_i}^n\right]\overline{W}_{xx_i}^{\ n}-\frac{(\eta-1)}{3}\Delta t_n B_i^n\overline{W}_{xxt_i}^{\ n}$$
$$=\bar{Q}_i^n+\frac{2(\eta-1)}{3\eta(\eta+1)}\left[\overline{Q}_i^{n+1}-(1-\eta^2)\overline{Q}_i^n-\eta^2\overline{Q}_i^{n-1}\right]+O(\Delta t_n^2+\Delta x^2). \tag{5.1}$$

The following 6 important problems are solved. We can compute the homogeneous right-hand side functions, initial & boundary values from the analytical solution as trial procedure. The Gauss-elimination procedure (tri-diagonal solver) is used to solve directly linear difference equations, whereas for nonlinear HPDEs the resulting nonlinear difference equations are solved using Newton-Raphson method [46-51]. We have chosen zero vectors as the initial approximations for all nonlinear cases. For this choice, the method indeed converges to the exact solution for sufficiently small values of Δt_n and Δx. MATLAB codes are used to obtain all the numerical results.

The methods (2.30), (4.8), and (5.1) are all 3-level schemes. For computation, it is mandatory to obtain the approximate solution at $t = \Delta t_1$.

The values of $w,\ w_t$ at $t = 0$ are given in explicit manner. This implies, the values of $(w, w_x, w_{xx}, ..., w_t, w_{xt}, w_{xxt}, \dots)_i^0$ are calculated automatically at $t = 0$.

At $t = \Delta t_1$, an approximate value of w can be calculated using the formula

$$W_i^1 = W_i^0 + \Delta t_1 (W_t)_i^0 + \frac{\Delta t_1^2}{2}(W_{tt})_i^0 + O(\Delta t_1^3). \tag{5.2}$$

From Eq. (1.1), we calculate

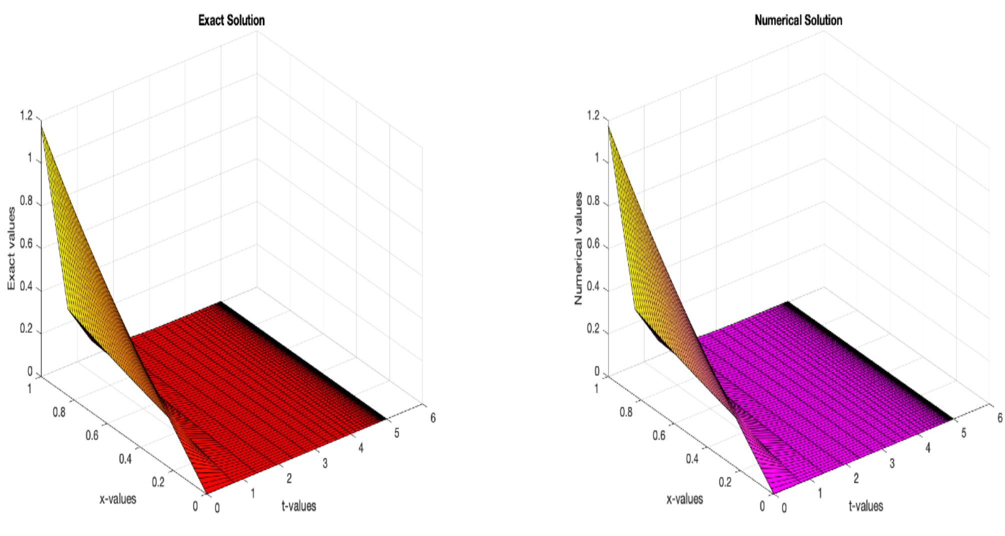
$$(W_{tt})_i^0 = \left[B(t,x,W)W_{xx} + Q(t,x,W,W_x,W_t)\right]_i^0. \tag{5.3}$$

With the aid of initial values of w, w_t, w_x, w_{xx}, from (5.3) it is easy to calculate $(W_{tt})_i^0$ and then from (5.2) the numerical solution of w at $t = \Delta t_1$. The maximum absolute errors (MAEs) and corresponding CPU time are computed at final time level in each case.

We solve the Eq. (4.1) in the region $0 < x < 1, t > 0$. The exact solution in closed form is given by $w(x,t) = \exp(-2t)\ \sinh x$. In Table 1, at $t = 5$ the MAEs are tabulated for $\Delta x = 1/64, \gamma_1 = 1.0, \gamma_2 = 1.5$ for $\eta = 0.85$. Figs. $1a$ & $1b$ portrayed the analytical & numerical approximation at $t = 5.0$ for $(p,q) = (20,10)$ and $\eta = 0.85$.

Table 1. The MAEs at $t = 5$ for $\eta = 0.85, \gamma_1 = 1.0, \gamma_2 = 1.5$

(p, q)	Proposed Method	Method (5.1)
$(20, 2)$	1.1882(-03)	1.2001(-02)
CPU time in secs	0.4956	0.3304
$(50, 5)$	1.2694(-04)	1.5043(-03)
CPU time in secs	0.4841	0.3227
$(15, 6)$	1.0507(-05)	1.3357(-04)
CPU time in secs	0.4561	0.3040
$(20, 10)$	5.5523(-06)	5.6859(-05)
CPU time in secs	0.4486	0.2990
$(15, 3)$	2.9946(-07)	2.2451(-06)
CPU time in secs	0.4586	0.3057



(a) Plot of exact solution

(b) Plot of numerical solution

Figure 1. Graph of exact vs. numerical solutions at $t = 5$ for $(p,q) = (20,10),\ \eta = 0.85,\ \gamma_1 = 1.0,\ \gamma_2 = 1.5$ of Problem 1.

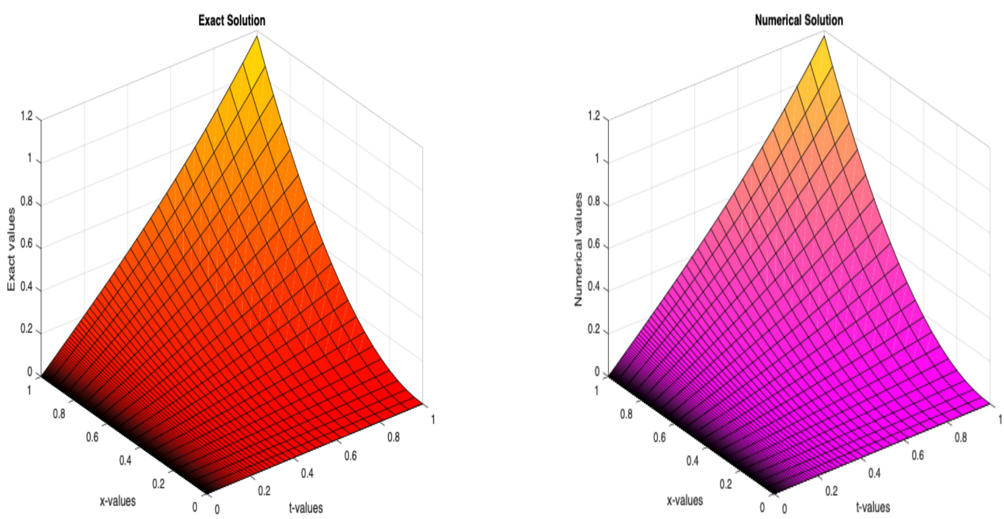
(*Wave equation in polar form*)

$$\frac{\delta^2 w}{\delta t^2} = \frac{\delta^2 w}{\delta r^2} + \frac{\alpha}{r}\frac{\delta w}{\delta r} + f(r,t),\ \ 0 < r < 1,\ t > 0 \tag{5.4}$$

The Eq. above is solved with the assistance of technique used in [9] and [42], where $w(r,t) = \cosh r\ \sinh t$. In Table 2, the MAEs are compiled for $n = 200, \Delta x = 1/20$, and for $\alpha = 1$ & 2.The analytical & numerical solution curves are displayed in Figs. $2a$ and $2b$ for $n = 200, \alpha = 1$ and $\eta = 1.07$.

Table 2. The MAEs for $\Delta x = 1/20, n = 200$

η	$\alpha = 1$		$\alpha = 2$	
	$ProposedMethod$	$Method(5.1)$	$ProposedMethod$	$Method(5.1)$
0.94 CPU time in secs	3.1916(-07) 0.0221	3.0787(-06) 0.0147	5.1823(-07) 0.0202	4.4045(-06) 0.0134
0.96 CPU time in secs	2.6347(-07) 0.0217	1.3338(-06) 0.0145	4.1649(-07) 0.0196	1.8994(-06) 0.0130
0.98 CPU time in secs	2.6419(-07) 0.0211	3.4578(-07) 0.0140	2.2361(-08) 0.0190	4.8937(-07) 0.0124
1.03 CPU time in secs	2.6390(-07) 0.0214	1.0344(-06) 0.0144	2.1164(-07) 0.0198	1.1038(-06) 0.0132
1.06 CPU time in secs	2.6322(-07) 0.0224	3.9817(-06) 0.0148	4.5184(-07) 0.0205	4.2473(-06) 0.0136
1.07 CPU time in secs	3.1832(-07) 0.0232	5.3647(-06) 0.0152	4.8834(-07) 0.0209	5.7222(-06) 0.0139



(a) Plot of exact solution (b) Plot of numerical solution

Figure 2. Graph of exact vs. numerical solutions for $n = 200,\ \alpha = 1,\ \eta = 1.07$ of Problem 2.

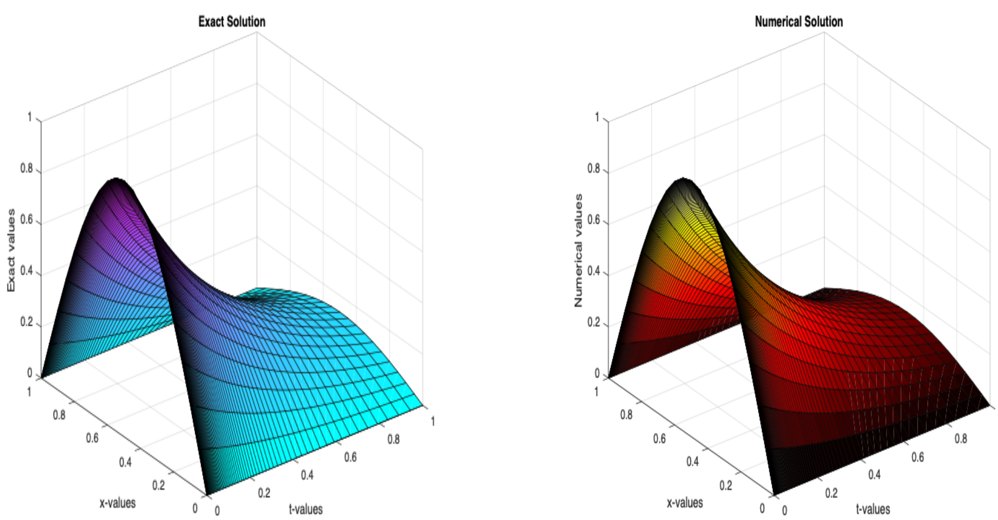
(*Van der Pol equation*)

$$\frac{\delta^2 w}{\delta t^2} = \frac{\delta^2 w}{\delta x^2} + \alpha(w^2 - 1)\frac{\delta w}{\delta t} + f(t,x), \quad t > 0,\ 0 < x < 1 \tag{5.5}$$

where $w(x,t) = \exp(-\alpha t)\sin(\pi x)$. In Table 3, for $n = 200, \Delta x = 1/20$, and for $\alpha = 1$ & 2, the MAEs are computed. Figures 3a & 3b correspond to the analytical & numerical solution curves for $n = 200, \alpha = 2, \eta = 1.06$.

Table 3. The MAEs for $\Delta x = 1/20, n = 200$

η	$\alpha = 1$		$\alpha = 2$	
	$Proposed Method$	$Method(5.1)$	$Proposed Method$	$Method(5.1)$
0.93	5.3432(-06)	1.4896(-03)	4.7700(-05)	8.0056(-04)
CPU time in secs	0.0450	0.0299	0.0419	0.0279
0.95	3.5545(-06)	1.4874(-03)	2.3910(-05)	7.7861(-04)
CPU time in secs	0.0444	0.0296	0.0415	0.0275
0.98	2.1082(-06)	1.4851(-03)	4.6629(-06)	7.6154(-04)
CPU time in secs	0.0440	0.0292	0.0411	0.0272
1.03	2.4042(-06)	1.4846(-03)	9.3143(-06)	7.6722(-04)
CPU time in secs	0.0446	0.0298	0.0416	0.0278
1.06	4.0307(-06)	1.4860(-03)	3.3156(-05)	7.9109(-04)
CPU time in secs	0.0452	0.0305	0.0422	0.0281
1.08	7.3634(-06)	1.4874(-03)	5.71533(-05)	8.1503(-04)
CPU time in secs	0.0458	0.0308	0.0428	0.0284



(a) Plot of exact solution

(b) Plot of numerical solution

Figure 3. Graph of exact vs. numerical solutions for $n = 200,\ \alpha = 2,\ \eta = 1.06$ of Problem 3.

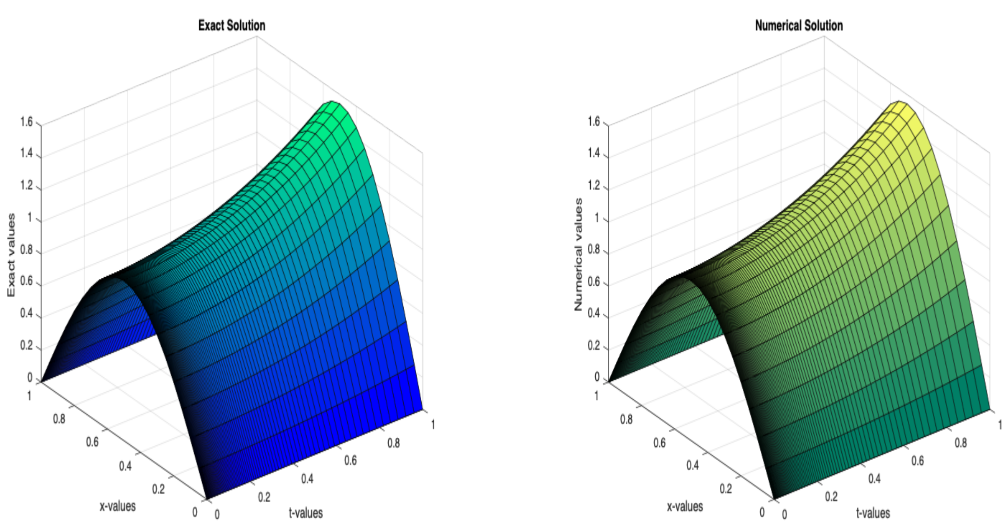
(*Dissipative nonlinear wave equation*)

$$\frac{\delta^2 w}{\delta t^2} = \frac{\delta^2 w}{\delta x^2} - 2w\frac{\delta w}{\delta t} + f(t,x), \quad t > 0,\ 0 < x < 1 \tag{5.6}$$

where $w(x,t) = \sin(\pi x)\ \cosh t$. The MAEs are compiled in Table 4 for $n = 200, \Delta x = 1/20$ and $1/25$. Figs. 4a & 4b portrayed the analytical & numerical solution curves for $n = 200, \eta = 1.04, \Delta x = 1/20$.

Table 4. The MAEs for $n = 200$

η	$\Delta x = 1/20$		$\Delta x = 1/25$	
	$Proposed Method$	$Method(5.1)$	$Proposed Method$	$Method(5.1)$
0.92 CPU time in secs	3.3406(-05) 0.0338	2.1295(-03) 0.0225	3.9743(-05) 0.0390	1.3446(-03) 0.0260
0.94 CPU time in secs	1.7025(-05) 0.0334	2.1458(-03) 0.0221	1.8468(-05) 0.0385	1.3592(-03) 0.0257
0.96 CPU time in secs	5.9008(-06) 0.0328	2.1576(-03) 0.0218	7.3557(-06) 0.0378	1.3706(-03) 0.0253
0.98 CPU time in secs	4.2371(-07) 0.0321	2.1649(-03) 0.0212	1.1299(-06) 0.0374	1.3774(-03) 0.0250
1.02 CPU time in secs	2.0618(-06) 0.0324	2.1635(-03) 0.0215	3.6558(-06) 0.0379	1.3756(-03) 0.0252
1.04 CPU time in secs	1.4884(-05) 0.0331	2.1502(-03) 0.0219	1.6513(-05) 0.0388	1.3624(-03) 0.0255
1.06 CPU time in secs	3.5964(-05) 0.0339	2.1284(-03) 0.0223	3.7651(-05) 0.0392	1.3408(-03) 0.0259
1.08 CPU time in secs	6.4696(-05) 0.0344	2.0987(-03) 0.0229	6.2088(-05) 0.0398	1.3115(-03) 0.0263



(a) Plot of exact solution

(b) Plot of numerical solution

Figure 4. Graph of exact vs. numerical solutions for $n = 200$, $\eta = 1.04$, $\Delta x = 1/20$ of Problem 4.

(*Quasilinear equation*)

$$\frac{\delta^2 w}{\delta t^2} = (1+w^2)\frac{\delta^2 w}{\delta x^2} + w\frac{\delta w}{\delta x} + f(t,x), \ \ t>0, \ 0<x<1 \tag{5.7}$$

where $w(x,t) = \exp(-2t)\sin(\pi x)$. In Table 5, the MAEs are compiled for $n = 200, \Delta x = 1/20$ and $1/25$. Figures 5a & 5b represented the numerical and analytical solution curves for $n = 200, \eta = 1.06, \Delta x = 1/20$.

Table 5. The MAEs for $n = 200$

	$\Delta x = 1/20$		$\Delta x = 1/25$	
η	$ProposedMethod$	$Method(5.1)$	$ProposedMethod$	$Method(5.1)$
0.92 CPU time in secs	9.0421(-03) 0.0430	2.4201(-02) 0.0286	5.7682(-03) 0.0492	5.8054(-02) 0.0328
0.94 CPU time in secs	2.2259(-03) 0.0424	2.3751(-02) 0.0282	2.9284(-03) 0.0488	2.3064(-02) 0.0324
0.96 CPU time in secs	2.1786(-03) 0.0418	2.3282(-02) 0.0278	2.1646(-03) 0.0482	2.2598(-02) 0.0320
0.98 CPU time in secs	2.0197(-03) 0.0414	2.1674(-02) 0.0274	2.0068(-03) 0.0473	2.1008(-02) 0.0316
1.02 CPU time in secs	6.8477(-04) 0.0416	7.2930(-03) 0.0276	6.7854(-04) 0.0484	7.0505(-03) 0.0318
1.04 CPU time in secs	9.0145(-03) 0.0422	8.9228(-02) 0.0280	8.9560(-03) 0.0490	8.8964(-02) 0.0322
1.06 CPU time in secs	1.5085(-02) 0.0428	1.4867(-01) 0.0284	1.4660(-02) 0.0495	1.4859(-01) 0.0326
1.08 CPU time in secs	5.0831(-02) 0.0436	1.6415(-01) 0.0288	5.5835(-02) 0.0498	8.4005(-01) 0.0332

(*Fourth-order nonlinear HPDE*)

$$\left(\frac{\delta^2}{\delta t^2} - \frac{\delta^2}{\delta x^2}\right)^2 w = w\frac{\delta w}{\delta x} + f(x,t), \ \ t>0, \ 0<x<1. \tag{5.8}$$

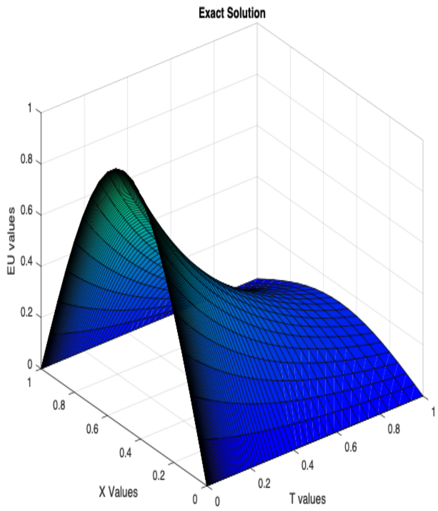
The initial and boundary values associated with (5.8) are given by

$$w(x,0) = \cos(\pi x), \ w_t(x,0) = -\pi\cos(\pi x),$$
$$w_{tt}(x,0) = \pi^2\cos(\pi x), \ w_{ttt}(x,0) = -\pi^3\cos(\pi x), \quad 0 \le x \le 1. \tag{5.9}$$
$$w(0,t) = \exp(-\pi t), \ w_{xx}(0,t) = -\pi^2\exp(-\pi t),$$
$$w(1,t) = -\exp(-\pi t), \ w_{xx}(1,t) = \pi^2\exp(-\pi t), \quad t>0. \tag{5.10}$$

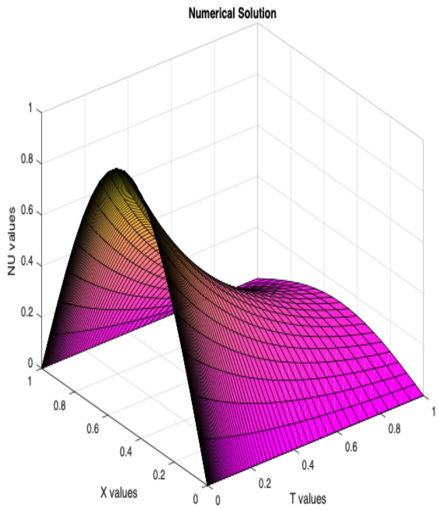
To solve (5.8), we substitute

$$\left(\frac{\delta^2}{\delta t^2} - \frac{\delta^2}{\delta x^2}\right) w = v, \ \ t>0, \ 0<x<1. \tag{5.11}$$

$$\left(\frac{\delta^2}{\delta t^2} - \frac{\delta^2}{\delta x^2}\right) v = w\frac{\delta w}{\delta x} + f(x,t), \ \ t>0, \ 0<x<1. \tag{5.12}$$



(a) Plot of exact solution



(b) Plot of numerical solution

Figure 5. Graph of exact vs. numerical solutions for $n = 200,\ \eta = 1.06,\ \Delta x = 1/20$ of Problem 5.

Note that, the grid lines and the coordinate axes are parallel to each other. This implies, the tangential and normal derivatives of w are known on the boundary. Therefore, the initial and boundary values associated with the system of equations (5.11)-(5.12) can be re-written as

$$w(x,0) = \cos(\pi x),\ w_t(x,0) = -\pi\cos(\pi x),$$
$$v(x,0) = 2\pi^2\cos(\pi x),\ v_t(x,0) = -2\pi^3\cos(\pi x), \quad 0 \le x \le 1. \tag{5.13}$$
$$w(0,t) = \exp(-\pi t),\ v(0,t) = 2\pi^2\exp(-\pi t),$$
$$w(1,t) = -\exp(-\pi t),\ v(1,t) = -2\pi^2\exp(-\pi t), \quad t > 0. \tag{5.14}$$

Employing the approximation (2.30), the coupled systems (5.11)-(5.12) can be solved.

$w(x,t) = \exp(-\pi t)\cos(\pi x)$ is the exact solution of Eq. (5.8). In Table 6, the MAEs are reported for $n = 200, \Delta x = 1/20$, and $1/15$. Figures 6a and 6b correspond to the graphs of exact vs approximate solution for $n = 200, \eta = 1.04, \Delta x = 1/20$.

6 Final remarks

In the present article, using 3 variable grid points in t-direction and two half-step constant mesh points in x-direction, we have considered two new stable approximations of $O(\Delta t_n^2 + \Delta x^2)$ and of $O(\Delta t_n^2 + \Delta t_n \Delta x^2 + \Delta x^4)$ for the solution of the IBVP (1.1)-(1.3). The proposed scheme is revealed to be stable for all choices of grid sizes when applied to the transmission line equation. We have numerically solved six remarkable HPDEs to illustrate the utility of the suggested approximations. The effectiveness of the projected approximation is demonstrated from the computational results.

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Table 6. The MAEs for $n = 200$

η	$\Delta x = 1/20$		$\Delta x = 1/15$	
	$ProposedMethod$	$Method(5.1)$	$ProposedMethod$	$Method(5.1)$
0.94	1.4521(-05)	6.0737(-04)	1.4138(-04)	9.3127(-04)
CPU time in secs	0.1685	0.1023	0.1248	0.0832
0.96	1.2859(-05)	6.5769(-04)	1.2927(-04)	9.9499(-04)
CPU time in secs	0.1674	0.1017	0.1238	0.0825
0.98	1.2184(-05)	6.8723(-04)	1.2252(-04)	1.0359(-03)
CPU time in secs	0.1662	0.1011	0.1227	0.0819
1.02	1.4335(-04)	4.8316(-03)	1.4416(-04)	4.9185(-03)
CPU time in secs	0.1668	0.1015	0.1229	0.0822
1.04	4.4393(-03)	1.4312(-02)	4.4651(-03)	1.4398(-02)
CPU time in secs	0.1678	0.1020	0.1241	0.0828
1.06	7.8739(-03)	2.4816(-02)	7.8634(-03)	2.4936(-02)
CPU time in secs	0.1689	0.1025	0.1251	0.0836

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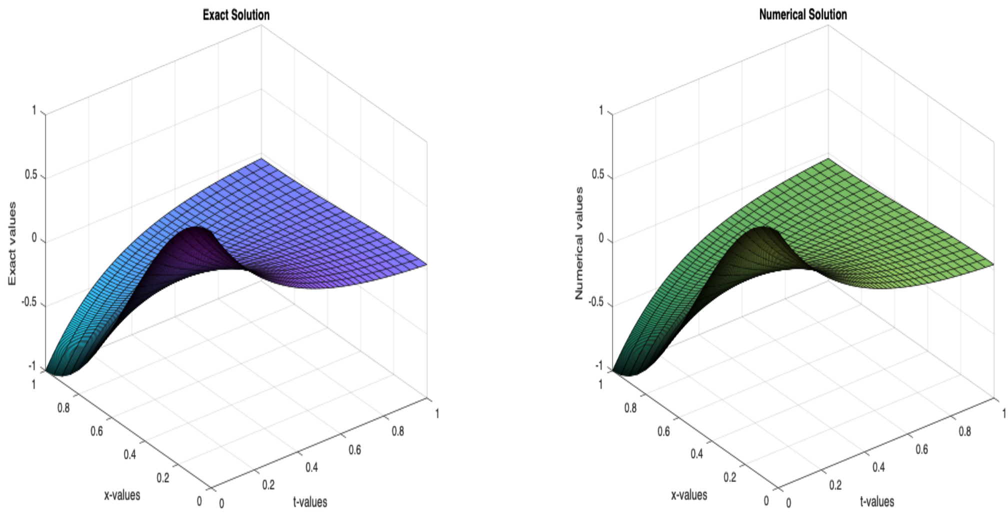

(a) Plot of exact solution (b) Plot of numerical solution

Figure 6. Graph of exact vs. numerical solutions for $n = 200,\ \eta = 1.04,\ \Delta x = 1/20$ of Problem 6.

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