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The application of Sinc and B-Spline functions to numerical solution of the time-fractional convection-diffusion equations

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Abstract This paper develops the numerical method based on the Sinc function and B-Spline scaling functions for applying the time-fractional convection-diffusion problem. Here, our approximation is combination of collocation method and transferred functions. For this purpose, we approximate the space dimension of the convection-diffusion problem by using B-spline scaling functions, and for the temporal direction, which has a fractional derivative, we use transferred DE-Sinc functions. With the help of numerical examples, we measure the efficiency and accuracy of the proposed method. In each example, the numerical results obtained with the methods of different references have been compared.

1 Introduction

In recent years, interest in fractional arithmetic has grown [1, 2, 3, 4]. Fractional differential equations have many applications in different scientific and engineering fields [5, 6]. Many phenomena in the fields of mechanics [7, 8], viscoelasticity [9], physics [10, 11, 12], biological engineering [13] and medicine [14, 15] has been developed by mathematical models that use fractional calculus.

In this paper, we consider the convection-diffusion equation with time-fraction derivative of the following form

$$ {}_0D_t^{\gamma}u(x,t) + p(x)\frac{\partial u(x,t)}{\partial x} + q(x)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in (a,b),\ 0 < t \leq T, \tag{1.1}$$

where $0 < \gamma < 1$. The initial and boundary conditions (IC and BC, respectively) are gives as

$$ u(x,0) = h(x), \qquad a \leq x \leq b. \tag{1.2}$$

$$ u(a,t) = \alpha(t), \quad u(b,t) = \beta(t), \quad 0 < t \leq T, \tag{1.3}$$

In equations (1.1), the Caputo-fractional derivative is as follows

$$ {}_oD_t^{\gamma}u(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\frac{\partial}{\partial\tau}u(x,\tau)}{(t-\tau)^{\gamma}}d\tau, \tag{1.4}$$

where $0 < \gamma < 1$. Diffusion equation with fractional left and right derivatives appear in random particle motions in physics [5, 18, 19] and Brownian motion [6]. In [20], the combination of the collocation method of Sinc and Legendre is presented for approximating the solution of the convection-diffusion equation with a fractional derivative. The combination of Sinc method and Euler's polynomials is used to solve such problems in reference [21]. In [22], the time dimension of the convection-diffusion problem was approximated using B-Spline scaling functions, and transferred Sinc functions were used for the location dimension. Also, various other methods have been used to approximate the solution of this type of problems, such as the finite difference method [13, 14, 15], the finite element method [23, 24], radial basis functions method

[25], Multipoint method [26], pseudo-spectral method [27, 28], Tau method with Chebyshev polynomials [29], Tau method with Legendre's transferred polynomials [30], spline technique [31]. The purpose of this article is to numerically solve the convection-diffusion problem with derivatives of fractional order with respect to time using methods based on Sinc and B-SSFs.
The structure of the article is as follows. we give the properties of Sinc method in section (2). In section (3), the B-spline scaling functions (B-SSFs) are presented in the finite interval. In section (4), The combination of collocation method based on B-SSFs and Sinc method to solve the convection-diffusion problem with fractional derivative with respect to time is discussed. Finally, in section (5), we examine the proposed method on two examples and compare the results with the computed results in the available methods.

2 The Sinc method

The Sinc method is used in many fields of applied sciences, approximation theory and numerical analysis. The Sinc method was first developed by Frank Stinger in 1981 [32]. In 1986, Lund applied the Sinque approximation using the Galerkin scheme for second-order ordinary differential equations [33]. In 1992, Lund and Bowers published a comprehensive book on the Sinc Galerkin and Sinc collocation methods for solving differential equations. They used second-order regular and parabolic partial derivatives equation [34]. In 1974, Mori and Takahashi introduced double exponential transformation [35] and used it to calculate integrals that have singular points at the beginning and end of the interval. In the field of equations with partial derivatives, most of the works have been done using single exponential transformation [34, 36, 37, 38]. Double exponential transformation is mostly used in the field of integral equations and ordinary differential equation. More information about the Sinc function and its history can be found in references [32, 34, 39].
The Sinc function can be defined by [27, 28]

$$Sinc(t) = \begin{cases} \frac{sin\pi t}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases},$$

The Sinc function is also called the normalized scale function. This definition of Sinc function is widely used in engineering sciences related to signal and system analysis. But here and in the discussion of using the Sinc method, the transferred function of the Sinc is used, then the k-th shifed Sinc function can be introduced by

$$Sinc\left(\frac{t-kh}{h}\right) = \begin{cases} \frac{sin\left(\pi\frac{t-kh}{h}\right)}{\pi\left(\frac{t-kh}{h}\right)}, & t \neq kh \\ 1, & t = kh, \end{cases}, \quad h > 0,\ k \in \mathbb{Z},$$

The double exponential transformation for finite interval (a, b), can be represented by [38]

$$t = \psi_{a,b}(z) = \frac{b-a}{2} tanh(\frac{\pi}{2} sinh(z)) + \frac{b+a}{2}, \tag{2.1}$$

and the inverse mapping is defined by

$$\eta(t) = log\left[\frac{1}{\pi}log(\frac{t-a}{b-t}) + \sqrt{1 + \left(\frac{1}{\pi}log(\frac{t-a}{b-t})\right)^2}\right]. \tag{2.2}$$

and Sinc collocation points are $x_k = \psi(kh)$. Then we define

$$S\left(k,h\right)\left(t\right) = \begin{cases} \frac{1}{1+e^{\eta(t)}}, & k = -M \\ Sinc\left(\frac{t-kh}{h}\right), & k = -M+1, .., M-1 \\ \frac{e^{\eta(t)}}{1+e^{\eta(t)}}, & k = M \end{cases}, \tag{2.3}$$

The approximation of the Sinc on the interval (a, b) is as follows [38]

$$f(t) = \sum_{k=-M}^{M} f(\psi(kh))S(k,h)(\eta(t)), \quad t \in (a,b). \tag{2.4}$$

Also, the sinc quadrature on the finite interval (a, b) is

$$\int_a^b f(t)dt = \int_{-\infty}^{\infty} f(\psi(\xi))(\psi'(\xi))d\xi \approx h \sum_{k=-M}^{M} f(\psi(kh))(\psi)'(kh). \tag{2.5}$$

Let $L_\beta(D_d)$ is the family of bounded analytic functions on a strip domain $\mathcal{D}_d = \{z \in \mathbb{C} :| Im(z) |< d\}$ for $d > 0$.

Theorem 2.1. *[38] If $f \in L_\beta(D_d)$, M is a positive integer and $h = \frac{log(\frac{\pi d M}{\beta})}{M}$, and $\eta : D \to D_d$ is double exponential transformation map, and $\psi = \eta^{-1}$, then there exist constants $C_1, C_2 > 0$ which is independent of M as*

$$Sup_{t\in D}|f(t) - \sum_{k=-M}^{M} f(\psi(kh))S(k,h)(\eta(t))| \le C_1\ exp(-\frac{\pi d M}{log(\frac{\pi d M}{\beta})}), \tag{2.6}$$

$$\left| \int_a^b f(t)dt - h \sum_{k=-M}^{M} f(\psi(kh))(\psi)'(kh) \right| \le C_2 exp\left(\frac{-\pi d M}{log(\frac{dM}{\beta})} \right). \tag{2.7}$$

3 B-Spline scaling functions

The history of approximating functions by special functions is from the beginning of the 19th century when Joseph Fourier proved the possibility of approximating other functions by trigonometric functions. For decades, researchers needed functions that have better features than trigonometric functions, so that in addition to having their features, they can approximate highly fluctuating signals. The structure of wavelets and scale functions is such that in addition to simple analysis, it has led to better approximations for discontinuous functions [40, 41, 42, 43].
The cardinal B-spline $N_m(t)$ is defined by integral convolution as follow

$$N_1(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & o.w \end{cases}$$

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)\,dt \ , \quad m \ge 2 \tag{3.1}$$

The compact support of positive function $N_m(x)$ is $[0, m]$. Its recursive relationship is as follows [40]

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1). \tag{3.2}$$

Here, we consider $m = 3$, for the spatial domain $[0, 1]$.
Suppose that the multiresolution spaces V_J be produced using the tensor product type, and

$$V_J = span\{\varphi_{J,k} = 2^{J/2}\varphi(2^J x - k); l \in Z\}.$$

Following [44], If the condition

$$2^J \ge 2m - 1, \tag{3.3}$$

is satisfied, then there exist at least one complete inner scaling function in $[0, 1]$. In (3.3), $J = 3$ is satistied for $m = 3$.
For $m = 3,\ J = 3$, the inner B-SSFs functions are defined by

$$\phi_{3,k}(x) = \begin{cases} \frac{1}{2}(8x-k)^2, & \frac{k}{8} \le x \le \frac{k+1}{8}, \\ \frac{3}{4} - (8x-k-\frac{3}{2})^2, & \frac{k+1}{8} \le x \le \frac{k+2}{8}, \\ \frac{1}{2}(8x-k-3)^2, & \frac{k+2}{8} \le x \le \frac{k+3}{8}, \\ 0, & o.w. \end{cases} \qquad k = 0, 1, \ldots, 5, \tag{3.4}$$

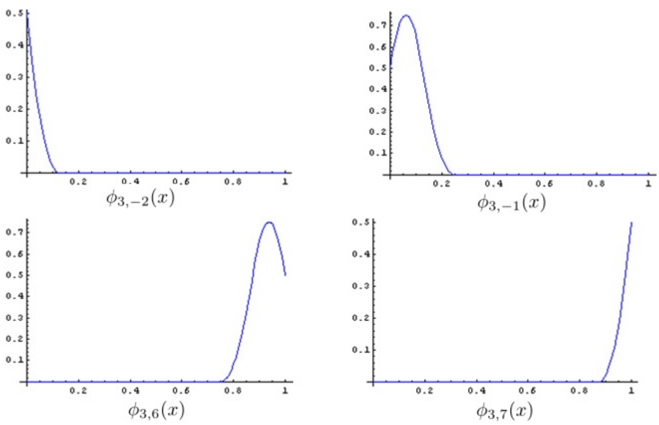

Figure 1. The boundary's B-Spline functions

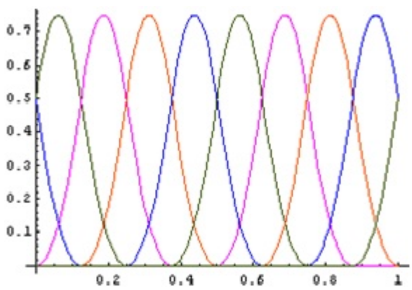

Figure 2. All of the inner, left and right boundary's B-Spline scaling functions with $J = 3$

the left boundary's B-Spline functions are given by

$$\phi_{3,-2}(x) = \begin{cases} \frac{1}{2}(8x-1)^2, & 0 \le x \le \frac{1}{8}, \\ 0, & o.w. \end{cases} \tag{3.5}$$

$$\phi_{3,-1}(x) = \begin{cases} \frac{3}{4}-(8x-\frac{1}{2})^2, & 0 \le x \le \frac{1}{8}, \\ \frac{1}{2}(8x-2)^2, & \frac{1}{8} \le x \le \frac{1}{4}, \\ 0, & o.w. \end{cases} \tag{3.6}$$

and right boundary's B-Spline functions are given by

$$\phi_{3,6}(x) = \begin{cases} \frac{1}{2}(8x-6)^2, & \frac{3}{4} \le x \le \frac{7}{8}, \\ \frac{3}{4}-(8x-\frac{15}{2})^2, & \frac{7}{8} \le x \le 1, \\ 0, & o.w. \end{cases} \tag{3.7}$$

$$\phi_{3,7}(x) = \begin{cases} \frac{1}{2}(8x-7)^2, & \frac{7}{8} \le x \le 1, \\ 0, & o.w. \end{cases} \tag{3.8}$$

Figures 1 and 2 show all of the inner and boundary's B-Spline functions with $J = 3$

4 The combination of Sinc and B-Spline scheme to the time-fractional convection-diffusion equation

According to the definitions of section 2 and 3, by applying Sinc method and B-SSFs in equations (1.1), therefore, we can write the function $u(x,t)$ as the following expansion

$$u(x,t) = \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(t))\, \phi_j(x), \tag{4.1}$$

where the coefficients d_{kj} are unknown, and by collocation nodes $\left\{x_l = \frac{l}{2^J+1}\right\}_{l=1}^{2^J+1}$ and $\{t_n = \psi(nh)\}_{n=-M}^{M}$ we get

$$u(x_l, t_n) = \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(nh)))\, \phi_j(x_l), \tag{4.2}$$

$$\frac{\partial u(x,t)}{\partial x}\Big|_{\substack{x=x_l \\ t=\psi(nh)}} = \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(nh)))\, {\phi_j}'(x_l), \tag{4.3}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2}\Big|_{\substack{x=x_l \\ t=\psi(nh)}} = \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(nh)))\, {\phi_j}''(x_l). \tag{4.4}$$

By applying double exponential transformation $\tau = \psi(s) = \frac{b-a}{2} tanh(\frac{\pi}{2} sinh(s)) + \frac{b+a}{2}$, and Sinc quadrture rule for fractional derivative, and assume $u'(x,t) = \frac{\partial u(x,t)}{\partial t}$ we have

$$\begin{aligned} {}_0D_t^{\gamma} u(x,t) &= \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u'(x,\tau)}{(t-\tau)^{\gamma}} d\tau = \frac{t^{1-\gamma}}{\Gamma(1-\gamma)} \int_{-\infty}^{+\infty} \frac{\pi \cosh(s) u'(x,\psi_{0,t}(s))}{\left(1+e^{-\pi \sinh(s)}\right)^{1-\gamma}\left(1-e^{-\pi \sinh(s)}\right)} ds \\ &= \frac{ht^{1-\gamma}}{\Gamma(1-\gamma)} \sum_{i=-M}^{M} \frac{\pi \cosh(ih) u'(x,\psi_{0,t}(ih))}{\left(1+e^{-\pi \sinh(ih)}\right)^{1-\gamma}\left(1-e^{-\pi \sinh(ih)}\right)} \\ &= \frac{ht^{1-\gamma}}{\Gamma(1-\gamma)} \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} \sum_{i=-M}^{M} \frac{\pi \cosh(ih) \eta'(\psi_{0,t}(ih)) S'(k,h)(\eta(\psi_{0,t}(ih)))\phi_j(x)}{\left(1+e^{-\pi \sinh(ih)}\right)^{1-\gamma}\left(1-e^{-\pi \sinh(ih)}\right)}. \end{aligned} \tag{4.5}$$

Then by substituting (4.2)-(4.5) in equation (1.1), we get

$$\begin{aligned} &\frac{h\psi(nh)^{1-\gamma}}{\Gamma(1-\gamma)} \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} \sum_{i=-M}^{M} \frac{\pi \cosh(ih) \eta'(\psi_{0,\psi(nh)}(ih)) S'(k,h)(\eta(\psi_{0,\psi(nh)}(ih)))\phi_j(x_l)}{\left(1+e^{-\pi \sinh(ih)}\right)^{1-\gamma}\left(1-e^{-\pi \sinh(ih)}\right)} \\ &+p(x_l) \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(nh)))\, {\phi_j}'(x_l) \\ &+ q(x_l) \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(nh)))\, {\phi_j}''(x_l) = f(x_l, \psi(nh)), \\ &\qquad n = -M+1, ..., M,\ l = 2, .., 2^J \ , \end{aligned} \tag{4.6}$$

and for BC and IC, we have:

$$u(a, t_n) = \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(nh)))\, \phi_j(a) = \alpha(\psi(nh)),\ n = -M, .., M, \tag{4.7}$$

$$u(b, t_n) = \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(nh)))\, \phi_j(b) = \beta(\psi(nh)), \quad n = -M, .., M, \tag{4.8}$$

$$u(x_l, 0) = \sum_{k=-M}^{M} \sum_{j=-2}^{2^J-1} d_{kj} S(k,h)(\eta(\psi(-Mh)))\, \phi_j(x_l) = h(x_l), \qquad l = 1, .., 2^J+1. \tag{4.9}$$

We solve system (4.6)-(4.9), then we obtain numerical solution of problem (1.1)-(1.3) .

5 Numerical Illustration

This section presents two examples to demonstrate the efficiency and accuracy of the presented method, and the fixed values are considered as follows:

$$h = \frac{log(\frac{\pi d M}{\beta})}{M},\ \beta = \pi,\ d = \frac{\pi}{2},$$

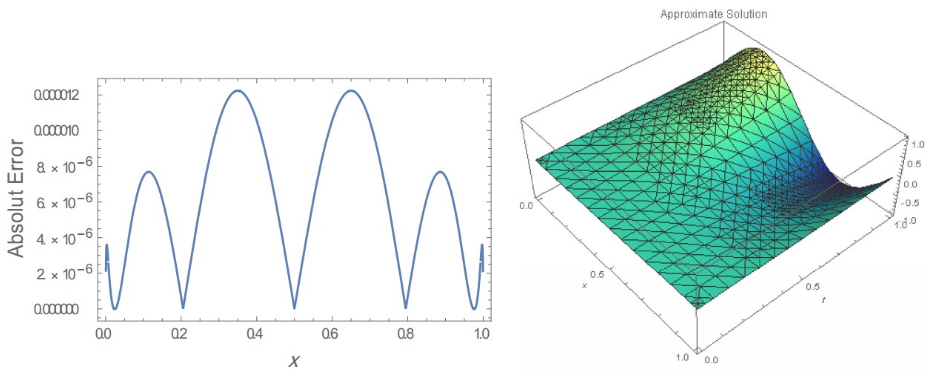

Figure 3. Graph of MAE at $t = 1$ (left), and approximate solution of $u(x, t)$ (right), with $\gamma = 0.5,\ J = 3,\ M = 20$ for Example 1.

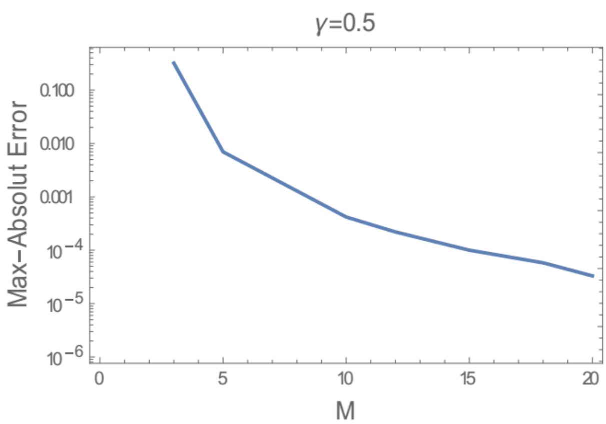

Figure 4. Graph of MAE according to $3 \leq M \leq 20$ with $J = 3$, $\gamma = 0.5$ on $(x, t) \in [0, 1] \times (0, 1]$, for Example 1.

and $J = 3$. We have compared the presented method with the methods available in valid articles to show that the B-spline-Sinc method has high accuracy. We have obtained the computational results by Mathematica 12.1 software with an 8 GB PC.

Example 5.1. Here we solve the following time-fractional convection-diffusion equation [45] by Sinc-B-Spline method

$$ {}_0D_t^{\gamma} u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = \left(\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} + 4\pi^2 t^2 \right) \sin(2\pi x), \quad 0 < t \leq 1,\ x \in (0,1), \tag{5.1} $$

with BCs

$$ u(0,t) = u(1,t) = 0, \qquad t \in [0,1], \tag{5.2} $$

and IC

$$ u(x,0) = 0, \qquad x \in [0,1], \tag{5.3} $$

and $u(x, t) = t^2 sin(2\pi x)$ is the exact solution of this problem.
Figure 3 shows the numerical solution of $u(x, t)$ and maximum absolute error (MAE) at $t = 1$ with $\gamma = 0.5,\ J = 3,\ M = 20$ by B-Spline-Sinc scheme.
The MAE according to $3 \leq M \leq 20$ with $J = 3$, $\gamma = 0.5$ on $(x, t) \in [0, 1] \times (0, 1]$, plotted in Figure 4.
The comparison of the results of presented method and the results given in [45] on $(x, t) \in [0, 1] \times (0, 1]$ with $4 \leq M \leq 20$ for $\gamma = 0.1$ is shown in Table 1.

Table 1. The MAE for Example 1 with $\gamma = 0.1$.

M	Presented method	Wavelet method [45]
4	$3.2482e-01$	$1e-01$
6	$4.8281e-02$	$5e-02$
10	$8.2157e-03$	$7e-03$
12	$6.1298e-04$	$4e-03$
16	$1.5063e-04$	$8e-04$
18	$9.3201e-05$	$4e-04$
20	$2.7569e-05$	$5e-05$

Example 5.2. Here we hold the following equation [46]

$$ {}_0D_t^{\gamma}u(x,t)+x\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)}(x^2-x^3)+(t^2+1)(2x^2-3x^3+6x-2),\ x\in(0,1),\ t\in(0,1], \tag{5.4}$$

with BCs

$$u(0,t)=u(1,t)=0,\ \ t\in(0,1], \tag{5.5}$$

and IC

$$u(x,0)=x^2-x^3,\ x\in[0,1], \tag{5.6}$$

and $u(x,t)=(x^2-x^3)(t^2+1)$ is the exact solution.

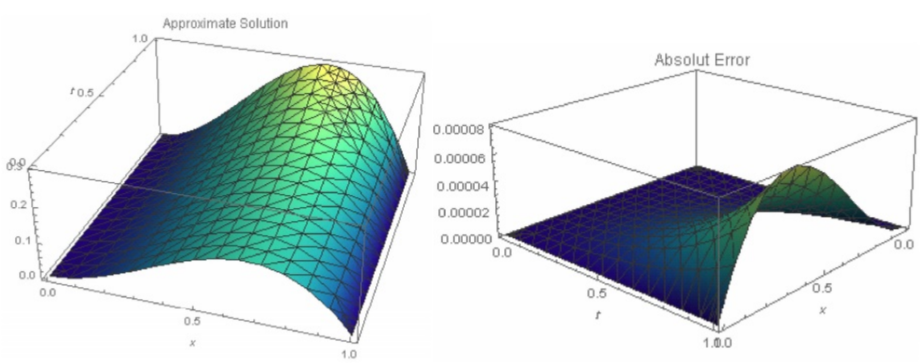

Figure 5. (left) Approximate solution, and (right) MAE, with $J=3$, $M=20$, $\gamma=0.3$ for Example 2, on $(x,t)\in[0,1]\times[0,1]$.

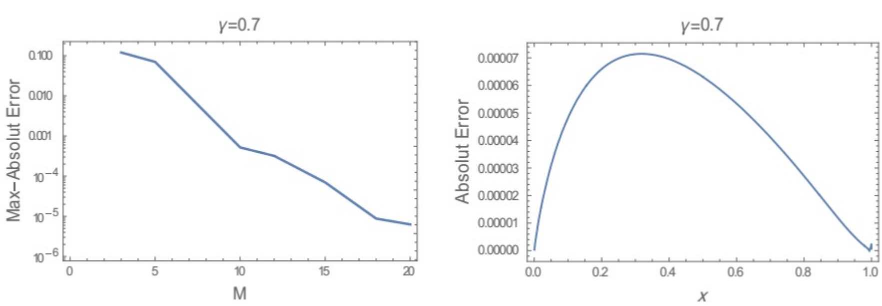

Figure 6. Maximum absolute errors according to $3 \le M \le 20$ with $J=3$, $\gamma=0.7$ on $(x,t)\in[0,1]\times[0,1]$ (left), and MAE with $M=20$, $\gamma=0.7$ at $t=1$ (right) for Example 2.

Figure 5 represents the numerical solution, and MAE, with $M = 20$, $J = 3\gamma = 0.3$ for Example 2 on $(x,t) \in [0,1] \times [0,1]$. The MAE of the presented method on $(x,t) \in [0,1] \times [0,1]$ with $\gamma = 0.7$ and MAE with $\gamma = 0.7$, $M = 20$ at $t = 1$ for Example 2 are plotted in Figure 6 .

Table 2. Comparison of the results of the presented method with $M = 20$, $J = 3$ and the Chebyshev wavelets collocation method presented in [46] with $M = 6$, $k = 2$ at different points for Example 2 with $\gamma = 0.5$.

(x,t)	Presented method	Chebyshev wavelets collocation method [46]
$(0.1, 0.1)$	$8.2130e-06$	$6.3167e-06$
$(0.2, 0.2)$	$3.4429e-05$	$9.0144e-06$
$(0.3, 0.3)$	$4.3022e-05$	$1.0906e-05$
$(0.4, 0.4)$	$7.0116e-05$	$1.2217e-05$
$(0.5, 0.5)$	$5.1213e-05$	$1.2120e-05$
$(0.6, 0.6)$	$4.9956e-05$	$1.1359e-05$
$(0.7, 0.7)$	$4.0111e-05$	$9.6580e-06$
$(0.8, 0.8)$	$2.4566e-05$	$7.0716e-06$
$(0.9, 0.9)$	$5.8874e-06$	$3.7565e-06$

The results comparison for Example 2, with $M = 20$, $J = 3$ and the Chebyshev wavelets collocation method [46] with $M = 6$, $k = 2$ at different points (x,t), for $\gamma = 0.5$, are tabulated in Table2.

6 Conclusion

This paper presented a method of transferred DE-Sinc functions and B-SSFs, which was used to approximate the solution of the fractional order convection-diffusion equation. As seen in the numerical results section, because of the rapidly convergence of the proposed method, we can obtain a very good accuracy compared to other methods. The presented examples verify the accuracy and applicability of the approach and show that accurate results can be obtained using a small number of collocation nodes. We can used the presented method to solve equations with other fractional partial derivatives such as one-dimensional fractional diffusion equation, time-dependent fractional wave diffusion equation, beam vibration equation, etc.

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