Palestine Journal of Mathematics

Vol. 12(Special Issue IV)(2023) , 21–32

TRIGONOMETRY BASIS APPROXIMATED FUZZY COMPONENTS AND HIGH-RESOLUTION SCHEME FOR TWO-POINT BOUNDARY VALUE PROBLEMS

Navnit Jha and Kritika

MSC 2010 Classifications: Primary 65L10, 65L12; Secondary 03B80.

Keywords and phrases: Fuzzy transform, trigonometry basis, compact discretization, absolute error, Convergence rate.

Abstract In this paper, we present a trigonometry-based approximated fuzzy components scheme by employing the fuzzy transform and compact discretization for computing an approximate solution to $2n$ order two-point boundary value problems. The fuzzy transforms evaluate the function on the sub-domain instead of considering the limited number of nodes. The joint effect of compact discretization and fuzzy transform yields a new method that exhibits sixth-order local truncation error and renders fourth-order solution accuracies. The scheme will be assessed for error bounds such as root-mean-squared error and maximum absolute error. Various nonlinear and linear, second, fourth, and sixth-order boundary value problems will be examined to assess the utility and efficiency of the new scheme.

1 Introduction

The fuzzy transform is a novel technique to treat differential equations with a suitable membership function on an integral domain with a uniform fuzzy partition and obtain the approximated solution. Fuzzy transform assesses the solution values in the small part of the domain instead of at a particular point, which is the inherent beauty of fuzzy transform. Nevertheless, the fuzzy transform does not possess the order-preserving property of solution values, which makes it inconvenient to apply to higher-dimension problems. This drawback can be overcome by combining compact finite-difference discretization with fuzzy transform to achieve a more accurate solution in the optimal time. Fuzzy transform with compact discretization provides order preserving numerical and approximate solution values for linear and nonlinear BVPs. Briefly, we present some remarkable work presented in the past. The fuzzy transform in the context of the differential equation is described in [1, 2]. Afterward, the formulation is extended by employing modifications such as a higher degree fuzzy transform, an orthogonal basis in [3, 4]. Fuzzy transforms in the context of the higher derivatives have been described in [5] and a detailed description of the shooting technique and fuzzy transform for nonlinear boundary value problems is given in [6]. A technique by the fuzzy transform, radial basis function in a neural network for differential equation and face reorganization is described in [7, 8]. The nonlinear two-point boundary value problems (BVPs) assume an integral part in many physical and scientific models. Many models do not own the exact solution; thus, it is requisite to solve them by an approximation scheme. In recent years, there have been many such schemes: finite difference, spline collocation, wavelets, etc. The existence and uniqueness of the solutions to BVPs of higher-order is described in [9, 10, 11]. The description of cubic spline, quintic non-polynomial spline, and compact three-point finite difference discretization for nonlinear higher order two-point boundary value problems has been presented in [12, 13, 14, 15]. The homotopy analysis method for the second, fourth, and sixth order two-point BVPs was presented earlier by [16]. Our approach will take a three-point linear combination and trigonometry basis to approximate the fuzzy component. Similarly, we will approximate the fuzzy component of derivatives and forcing function. The resulting scheme will provide fourth-order accuracy, and also it will be memory efficient.

In this paper, we will obtain a high-resolution compact discretization by employing trigonometry basis functions in approximated fuzzy components for the two-point BVPs of order $k, n \in$

$\mathbb{Z}^+$:

$$u^{(2n)}(t) = \phi(t, u(t), u^{(1)}(t), \ldots, u^{(2n-1)}(t)), a < t < b, \tag{1.1}$$

$$u^{(2k)}(a) = A_k, u^{(2k)}(b) = B_k, k = 0, \ldots, n-1, \tag{1.2}$$

where $u^{(i)}$ represents ith derivative of the function $u(t)$. Suppose $\phi(t, u(t), u^{(1)}(t), \ldots, u^{(2n-1)}(t))$, is continuous inside the finite domain and satisfies the Lipschitz condition. Then, the solution of BVP (1.1)-(1.2) exists and it will be unique.

The paper is organized as follows; Section 2 provides a brief background of fuzzy transformation. Later in section 3, we will derive a high-resolution technique for the linear BVPs. The scheme in the context of nonlinear second-order BVPs is described in Section 4 and extended for fourth-order BVPs in Section 5. Afterward, Section 6 will extend the scheme for sixth-order BVPs. The scheme's utility with the help of numerical simulation is described in Section 7. Finally, we will conclude the paper with remarks and future work.

2 Preliminaries of fuzzy transform

Let M be a positive integer, and $\{t_m = a + mh, m = 0, \ldots, M+1\}, h = \frac{b-a}{M+1}$, is a set of uniformly spaced mesh points on the domain [a,b]. The fuzzy transform of the real-valued continuous function $u(t)$ on the domain $[a, b]$ is given as

$$\begin{gathered} U_m = \frac{\int_{t_{m-1}}^{t_{m+1}} u(t) A_m(x) dt}{\int_{t_{m-1}}^{t_{m+1}} A_m(t) dt}, \\ U_0 = \frac{\int_{t_0}^{t_1} u(t) A_0(t) dt}{\int_{t_0}^{t_1} A_0(t) dt} \quad \text{and} \quad U_{M+1} = \frac{\int_{t_M}^{t_{M+1}} u(t) A_{M+1}(t) dt}{\int_{t_M}^{t_{M+1}} A_{M+1}(t) dt}, \end{gathered} \tag{2.1}$$

Where

$$\begin{gathered} A_m(t) = \begin{cases} (t - t_{m-1}), & t \in [t_{m-1}, t_m], \\ (t_{m+1} - t), & t \in [t_m, t_{m+1}], \\ 0, & \text{otherwise} \end{cases} \quad m = 1, \cdots, M, \\ A_0(t) = \begin{cases} (t_1 - t), & t \in [t_0, t_1], \\ 0, & \text{otherwise}, \end{cases} \\ \text{and} \\ A_{M+1}(t) = \begin{cases} (t - t_M), & t \in [t_M, t_{M+1}] \\ 0, & \text{otherwise}, \end{cases} \end{gathered} \tag{2.2}$$

are triangular membership function. A detailed description of fuzzy transform, fuzzy partition and its properties corresponding to the differential equation is described in [1]. The approximate solution $\widehat{u}(t)$ is obtained by the inverse $\mathcal{F}$-transform, $\widehat{u}(t) = \sum_{m=0}^{M+1} U_m A_m(t)$. At the boundary points t_0 and t_{M+1}, the fuzzy components U_0 and U_{M+1} possesses $O(h)$-accuracy to corresponding exact solution, while at the interior nodes $t_m, 1 \leq m \leq M$, the fuzzy components U_m is $O(h^2)$-accurate to the solution values u_m. This can be easily seen upon employing the Newton-Cotes formula and series expansion to the fuzzy components (2.1). Further, the fuzzy approximation $\widehat{u}(t)$ to the solution $u(t)$ is computed by considering the inverse fuzzy transform

$$\widehat{u}(t) = \sum_{m=0}^{M+1} U_m A_m(t) = U_0 A_0(t) + U_{M+1} A_{M+1}(t) + \sum_{m=1}^{M} U_m A_m(t).$$

This implies,

$$\begin{bmatrix} \widehat{u}(t_0) \\ \widehat{u}(t_m) \\ \widehat{u}(t_{M+1}) \end{bmatrix} = \begin{bmatrix} U_0 A_0(t) \\ U_m A_m(t) \\ U_{M+1} A_{M+1}(t) \end{bmatrix} \approx \begin{bmatrix} u_0 \\ u_m \\ u_{M+1} \end{bmatrix} + \begin{bmatrix} O(h) \\ O(h^2) \\ O(h) \end{bmatrix}, m = 1(1)M.$$

Since Dirichlet's boundary values at the boundary nodes are specified, the inverse fuzzy transform $\widehat{u}(t)$ employed with the triangular membership function yields $O\left(h^2\right)$-accuracy to the solution at the interior nodes. That is, the direct application of fuzzy transform offers only a second-order accurate solution. We aim to extend the solution accuracy by combining the fuzzy transform with compact discretization for mildly nonlinear BVPs.

3 Trigonometry basis approximated fuzzy components

We will describe the fuzzy transform algorithm in the context of the following second-order differential equation

$$u^{(2)}(t) = \varphi(t). \tag{3.1}$$

Consider, the three-point discretization

$$u(t-h) + u(t+h) - 2u(t) = h^2\left(p_1\varphi(t+h) + p_0\varphi(t) + p_1\varphi(t-h)\right). \tag{3.2}$$

The discretization (3.2) is valid for $t \in [t_1, t_M]$, which are the interior nodes of the domain $[a, b]$. Taking fuzzy transform on both sides of (3.2), gives

$$\begin{aligned}\mathcal{F}[u(t-h)] + \mathcal{F}[u(t+h)] - 2\mathcal{F}[u(t)] = h^2 \big(&p_1\mathcal{F}[\varphi(t+h)] + p_0\mathcal{F}[\varphi(t)] \\ &+ p_1\mathcal{F}[\varphi(t-h)]\big).\end{aligned}$$

This implies,

$$\mathcal{F}[y(t)] + \mathcal{F}[z(t)] - 2\mathcal{F}[u(t)] = h^2\left(p_1\mathcal{F}[\chi(t)] + p_0\mathcal{F}[\varphi(t)] + p_1\mathcal{F}[\psi(t)]\right), \tag{3.3}$$

where

$$y(t) = u(t-h), z(t) = u(t+h), \chi(t) = \varphi(t+h), \text{ and } \psi(t) = \varphi(t-h).$$

Let $\mathcal{F}[y(t)] = Y_m, \mathcal{F}[z(t)] = Z_m, \mathcal{F}[u(t)] = U_m, \mathcal{F}[\chi(t)] = X_m, \mathcal{F}[\varphi(t)] = \mathbf{\Phi}_m, \mathcal{F}[\psi(t)] = \mathbf{\Psi}_m$, be the fuzzy components. Then, each fuzzy component in (3.3) is related by the

$$Y_m + Z_m - 2U_m = h^2\left(p_1 X_m + p_0 \mathbf{\Phi}_m + p_1 \mathbf{\Psi}_m\right). \tag{3.4}$$

Using the fuzzy transform properties [1], we get $Y_m = U_{m-1}, Z_m = U_{m+1}, X_m = \mathbf{\Phi}_{m+1}$, and $\mathbf{\Psi}_m = \mathbf{\Phi}_{m-1}$. Next, we will compute the approximated fuzzy components corresponding to the components present in (3.4) and derive the fuzzy components scheme. Let $u_m = u\left(t_m\right)$ represent the exact value of $u(t)$ at the node $t = t_m$. We will approximate the fuzzy component of $u(t)$ at the neighbouring nodes t_{m+1}.To approximate fuzzy components, we formulate a three-point linear combination

$$\begin{bmatrix} \overline{U}_m \\ \overline{U}_{m+1} \\ \overline{U}_{m-1} \end{bmatrix} = \begin{bmatrix} \alpha_m^{(0)} & \alpha_{m+1}^{(0)} & \alpha_{m-1}^{(0)} \\ \alpha_m^{(1)} & \alpha_{m+1}^{(1)} & \alpha_{m-1}^{(1)} \\ \alpha_m^{(2)} & \alpha_{m+1}^{(2)} & \alpha_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} u_m \\ u_{m+1} \\ u_{m-1} \end{bmatrix}, \tag{3.5}$$

where $\alpha_{m+\delta}^{(\tau)}, \tau = 0, 1, 2, \delta = 0, 1$ are free parameters. Value of free parameters can be determined by considering the relation,

$$U_m - \overline{U}_m = \frac{\int_{t_{m-1}}^{t_{m+1}} u(t)A_m(t)dt}{\int_{t_{m-1}}^{t_{m+1}} A_m(t)dt} - \left(\alpha_{m-1}^{(0)}u_{m-1} + \alpha_m^{(0)}u_m + \alpha_{m+1}^{(0)}u_{m+1}\right),$$

and evaluating it on the basis $\mathcal{B} = \{1, \sin(vt), \cos(vt)\}$. We find

$$\alpha_m^{(0)} = \frac{\left(v^2h^2+2\right)\cos(vh) - 2}{v^2h^2(\cos(vh)-1)}, \quad \alpha_{m\pm1}^{(0)} = \frac{2 - v^2h^2 - 2\cos(vh)}{2v^2h^2(\cos(vh)-1)}. \tag{3.6}$$

Similarly, the undetermined coefficients corresponding to other approximated fuzzy components can be computed, and we obtain,

$$\begin{aligned}
\alpha_m^{(1)} = \alpha_m^{(2)} &= \frac{\left\{v^2h^2 + 2\cos(vh) - 2\right\}\cos(vh)}{v^2h^2(\cos(vh)-1),} \\
\alpha_{m+1}^{(1)} = \alpha_{m-1}^{(2)} &= \frac{6\cos(vh) - 4\cos(vh)^2 - v^2h^2 - 2}{2v^2h^2(\cos(vh)-1)}, \\
\alpha_{m-1}^{(1)} = \alpha_{m+1}^{(2)} &= \frac{2 - v^2h^2 - 2\cos(vh)}{2v^2h^2(\cos(vh)-1)}.
\end{aligned} \tag{3.7}$$

The fuzzy component $\Phi_{m+\delta}$ of forcing function $\varphi(t)$ at the central and neighboring nodes $t_{m\pm\delta}, \delta = 0, \pm 1$ are approximated as

$$\overline{\Phi}_m = \alpha_{m-1}^{(0)}\varphi_{m-1} + \alpha_m^{(0)}\varphi_m + \alpha_{m+1}^{(0)}\varphi_{m+1}, \tag{3.8}$$

$$\overline{\Phi}_{m+1} = \alpha_{m-1}^{(1)}\varphi_{m-1} + \alpha_m^{(1)}\varphi_m + \alpha_{m+1}^{(1)}\varphi_{m+1}, \tag{3.9}$$

and

$$\overline{\Phi}_{m-1} = \alpha_{m-1}^{(2)}\varphi_{m-1} + \alpha_m^{(2)}\varphi_m + \alpha_{m+1}^{(2)}\varphi_{m+1}. \tag{3.10}$$

Using the relevant approximated fuzzy components (3.5), (3.8), (3.9), and (3.10) in the fuzzy relation (3.4) of the two-point BVP (3.1) and carrying out the necessary algebra to obtain,

$$\begin{aligned}
\overline{U}_{m-1} + &\overline{U}_{m+1} - 2\overline{U}_m - h^2\left(p_1\overline{\Phi}_{m-1} + p_0\overline{\Phi}_m + p_1\overline{\Phi}_{m+1}\right) = \\
&\left\{(1 - p_0 - 2p_1)\,h^2 - \frac{\alpha^2h^4}{12}\right\}u_m^{(2)} + \frac{h^4}{12}\left(1 - p_0 - 14p_1\right)u_m^{(4)} + O\left(h^6\right).
\end{aligned}$$

By equating the coefficient of $u_m^{(2)}$ and $u_m^{(4)}$ to zero, we obtain

$$p_0 = 1 - \frac{7v^2h^2}{72}, p_1 = \frac{v^2h^2}{144}. \tag{3.11}$$

As a result, we get the following high-resolution fuzzy component scheme is acquired

$$\overline{U}_{m-1} + \overline{U}_{m+1} - 2\overline{U}_m - h^2\left(p_1\overline{\Phi}_{m-1} + p_0\overline{\Phi}_m + p_1\overline{\Phi}_{m+1}\right) = O\left(h^6\right), \tag{3.12}$$

to approximate the solution of BVP (3.1), which yields $O\left(h^6\right)$ local truncation error. The purpose of using the coefficients p_0 and p_1 in the scheme (3.4) is to optimize truncation error and minimize computation. The fuzzy component scheme for second-order nonlinear BVPs will be explained in the following section, and after that, it will be extended for higher-order BVPs.

4 Trigonometry basis fuzzy transform scheme for nonlinear second-order BVPs

This section will discuss the high-resolution fuzzy transform scheme with a trigonometry basis for second-order nonlinear BVPs. Consider

$$u^{(2)}(t) = \varphi\left(t, u(t), u^{(1)}(t)\right), a < t < b, u(a) = u_a, u(b) = u_b. \tag{4.1}$$

For each interior nodal point, we define

$$\overline{\varphi}_{m\pm1} = \varphi\left(t_{m\pm1}, \overline{U}_{m\pm1}, \overline{U}_{m\pm1}^{(1)}\right). \tag{4.2}$$

Since the derivative $u^{(1)}(t)$ is also present in (4.1), we will use trigonometric basis functions to approximate its fuzzy components. Let $u_m^{(1)} = u^{(1)}(t_m)$ denote the exact value of $u^{(1)}(t)$ at the node $t = t_m$. To approximate the fuzzy component $\overline{U}_m^{(1)}$ of $u^{(1)}(t)$ at the neighboring nodes $t_{m\pm\delta}, \delta = 0, \pm 1$, consider a three-point linear combination

$$\begin{bmatrix} \overline{U}_m^{(1)} \\ \overline{U}_{m+1}^{(1)} \\ \overline{U}_{m-1}^{(1)} \end{bmatrix} = \begin{bmatrix} \beta_m^{(0)} & \beta_{m+1}^{(0)} & \beta_{m-1}^{(0)} \\ \beta_m^{(1)} & \beta_{m+1}^{(1)} & \beta_{m-1}^{(1)} \\ \beta_m^{(2)} & \beta_{m+1}^{(2)} & \beta_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} u_m \\ u_{m+1} \\ u_{m-1} \end{bmatrix}, \tag{4.3}$$

where $\beta_{m\pm\delta}^{(\tau)}, \tau = 0, 1, 2, \delta = 0, 1$ are free parameters. As discussed earlier, the value of free parameters can be determined by evaluating the expression

$$\overline{U}_m^{(1)} - U_m^{(1)} = \frac{\int_{t_{m-1}}^{t_{m+1}} \frac{du(t)}{dt} A_m(t)dt}{\int_{t_{m-1}}^{t_{m+1}} A_m(t)dt} - \left(\beta_{m-1}^{(0)} u_{m-1} + \beta_m^{(0)} u_m + \beta_{m+1}^{(0)} u_{m+1}\right),$$

at each element of the basis $\mathcal{B}$, we get a system of the linear equation which yields

$$\beta_m^{(0)} = 0, \quad \beta_{m-1}^{(0)} = -\beta_{m+1}^{(0)} = \frac{1 + \cos(2vh) - 2\cos(vh)}{vh^2 \sin(2vh)}. \tag{4.4}$$

The neighboring nodes will undergo a similar simplification, producing,

$$\begin{aligned} &\beta_m^{(1)} = -\beta_m^{(2)} = \frac{-2\sin(vh)}{vh^2}, \beta_{m-1}^{(1)} = -\beta_{m+1}^{(2)} = \frac{1 - \cos(vh)}{vh^2 \sin(vh)}, \\ &\beta_{m+1}^{(1)} = \beta_{m-1}^{(2)} = \frac{1 - 2\cos(vh)^2 + \cos(vh)}{vh^2 \sin(vh)}. \end{aligned} \tag{4.5}$$

With these coefficients, It can be easily observed that the approximated fuzzy components obtain $O(h^2)$-accuracy concerning exact fuzzy components of derivatives. Employing Taylor series expansions to the equation (3.5) and (4.3), we get

$$\begin{bmatrix} \overline{U}_{m+1} \\ \overline{U}_m \\ \overline{U}_{m-1} \end{bmatrix} = \begin{bmatrix} u_{m+1} \\ u_m \\ u_{m-1} \end{bmatrix} + \frac{h^2}{12} \begin{bmatrix} u^{(2)}(t_{m-1}) \\ u^{(2)}(t_m) \\ u^{(2)}(t_{m+1}) \end{bmatrix} + \begin{bmatrix} O(h^3) \\ O(h^4) \\ O(h^3) \end{bmatrix}, \tag{4.6}$$

$$\overline{U}_m^{(1)} = u_m^{(1)} + \frac{h^2}{12}\left(v^2 u^{(1)}(t_m) + 2u^{(3)}(t_m)\right) + O\left(h^4\right), \tag{4.7}$$

$$\overline{U}_{m+1}^{(1)} = u_{m+1}^{(1)} - \frac{h^2}{12}\left(5v^2 u^{(1)}(t_m) + 4u^{(3)}(t_m)\right) + O\left(h^3\right), \tag{4.8}$$

and

$$\overline{U}_{m-1}^{(1)} = u_{m-1}^{(1)} - \frac{h^2}{12}\left(5v^2 u^{(1)}(t_m) + 4u^{(3)}(t_m)\right) + O\left(h^3\right). \tag{4.9}$$

Employing the equations (4.6)-(4.9), in (4.2) and using Taylor's expansion, we obtain

$$\begin{aligned} \overline{\varphi}_{m+1} = \varphi_{m+1} &- \frac{h^2}{12}\left(5v^2\sigma_m^{(0)} u^{(1)}(t_m) - \rho_m^{(0)} u^{(2)}(t_m) + 4\sigma_m^{(0)} u^{(3)}(t_m)\right) \\ &- \frac{h^3}{12}\Big[v^2\left(5\sigma_m^{(1)} + \rho_m^{(0)}\right) u^{(1)}(t_m) - \left(\rho_m^{(1)} - 2v^2\sigma_m^{(0)}\right) u^{(2)}(t_m) \\ &+ 4\sigma_m^{(1)} u^{(3)}(t_m) + \sigma_m^{(0)} u^{(4)}(t_m)\Big] + O\left(h^4\right), \end{aligned} \tag{4.10}$$

$$\begin{aligned} \overline{\varphi}_{m-1} = &\varphi_{m-1} - \frac{h^2}{12}\left(5v^2\sigma_m^{(0)} u^{(1)}(t_m) + \rho_m^{(0)} u^{(2)}(t_m) + 4\sigma_m^{(0)} u^{(3)}(t_m)\right) \\ &+ \frac{h^3}{12}\Big[v^2\left(5\sigma_m^{(1)} + \rho_m^{(0)}\right) u^{(1)}(t_m) + \left(2v^2\sigma_m^{(0)} - \rho_m^{(1)}\right) u^{(2)}(t_m) \\ &+ 4\sigma_m^{(1)} u^{(3)}(t_m) + \sigma_m^{(0)} u^{(4)}(t_m)\Big] + O\left(h^4\right), \end{aligned} \tag{4.11}$$

where $\varphi_{m\pm1} = \varphi\left(t_{m\pm1}, u\left(t_{m\pm1}\right), u^{(1)}\left(t_{m\pm1}\right)\right)$, $\rho_m^{(0)} = \left.\frac{\partial\varphi}{\partial u(t)}\right|_{t=t_m}$, $\sigma_m^{(0)} = \left.\frac{\partial\varphi}{\partial u^{(1)}(t)}\right|_{t=t_m}$ and $\sigma_m^{(1)} = \left.\frac{\partial^2\varphi}{\partial t\partial u^{(1)}(t)}\right|_{t=t_m}$. To update the fuzzy components of the solution value and its differential at the central node $t = t_m$, we construct

$$\widehat{U}_m = \overline{U}_m + h^2 k_1\left(\overline{\varphi}_{m+1} + \overline{\varphi}_{m-1}\right), \tag{4.12}$$

and

$$\widehat{U}_m^{(1)} = \overline{U}_m^{(1)} + hk_1\left(\overline{\varphi}_{m+1} - \overline{\varphi}_{m-1}\right). \tag{4.13}$$

On simplifying (4.12) and (4.13), we get

$$\widehat{U}_m = u_m + \frac{h^2}{12}\left(24k_1 + 1\right)u^{(2)}\left(t_m\right) + O\left(h^4\right), \tag{4.14}$$

$$\widehat{U}_m^{(1)} = u_m^{(1)} + \frac{h^2}{12}\left(v^2 u^{(1)}\left(t_m\right) + 2\left(12k_1 + 1\right)u^{(3)}\left(t_m\right)\right) + O\left(h^4\right). \tag{4.15}$$

By employing updates (4.14) and (4.15), we construct

$$\widehat{\varphi}_m = \varphi\left(t_m, \widehat{U}_m, \widehat{U}_m^{(1)}\right), \tag{4.16}$$

and thus, using Taylor's series expansion, we acquire

$$\begin{aligned}\widehat{\varphi}_m = \varphi_m + \frac{h^2}{12}\Big[&\sigma_m^{(0)} v^2 u^{(1)}\left(x_m\right) + \rho_m^{(0)}\left(24k_1 + 1\right)u^{(2)}\left(t_m\right)\\ &+ 2\sigma_m^{(0)}\left(12k_1 + 1\right)u^{(3)}\left(t_m\right)\Big] + O\left(h^4\right).\end{aligned} \tag{4.17}$$

Further, the approximated fuzzy components relation (3.12), when combined with (4.10), (4.11), and (4.17), yields

$$\begin{aligned}&\overline{U}_{m-1} + \overline{U}_{m+1} - 2\overline{U}_m - h^2\left(p_1\overline{\varphi}_{m+1} + p_0\widehat{\varphi}_m + p_1\overline{\varphi}_{m-1}\right)\\ &\quad = -\frac{h^4}{12}\Big[\left(24p_0k_1 + p_0 + 2p_1\right)u^{(2)}\left(t_m\right)\rho_m^{(0)} + \left(p_0 - 10p_1\right)v^2\sigma_m^{(0)}u^{(1)}\left(t_m\right)\\ &\quad + \left(24p_0k_1 + 2p_0 - 8p_1\right)u^{(3)}\left(t_m\right)\sigma_m^{(0)} + \left(12p_1 - 1\right)u^{(4)}\left(t_m\right)\\ &\quad - \left\{\left(p_0 + 2p_1 - 1\right)h^2 + \frac{v^2h^4}{12}\right\}u^{(2)}\left(t_m\right] + O\left(h^6\right).\end{aligned} \tag{4.18}$$

We obtain $p_0 = \frac{5}{6} - \frac{v^2h^2}{12}$ and $p_1 = \frac{1}{12}$ by equating coefficients of $u^{(4)}\left(t_m\right)$ and $u^{(2)}\left(t_m\right)$ to zero and solving them. On substituting values of p_0 and p_1 in (4.18), we can find the value of $k_1 = -1/20$ such that

$$\overline{U}_{m-1} + \overline{U}_{m+1} - 2\overline{U}_m - h^2\left(p_0\overline{\varphi}_{m+1} + p_1\widehat{\varphi}_m + p_0\overline{\varphi}_{m-1}\right) = O\left(h^6\right). \tag{4.19}$$

We can observe that fuzzy component relation (4.19) exhibits $O(h^6)$ local truncation error, and if we divide it by h^2, an $O\left(h^4\right)$-accurate solution can be obtained for the nonlinear BVP (1.1) at $\eta = 1$.

5 Fuzzy transform scheme for fourth-order BVPs

This section will present an algorithmic framework of the fuzzy transform scheme for fourth-order $(n = 2)$ BVPs (1.1). Consider

$$u^{(4)}(t) = \varphi\left(t, u(t), u^{(1)}(t), u^{(2)}(t), u^{(3)}(t)\right), a < t < b, \tag{5.1}$$

$$u(a) = a_1, \quad u^{(2)}(a) = a_2, \quad u(b) = b_1, \quad u^{(2)}(b) = b_2. \tag{5.2}$$

Let,

$$u^{(2)}(t) = v(t). \tag{5.3}$$

Then, the nonlinear equation (5.1) can be written as

$$v^{(2)}(t) = \varphi\left(t, u(t), u^{(1)}(t), v(t), v^{(1)}(t)\right). \tag{5.4}$$

In the subsequent representation, we denote $\overline{V}_m, \overline{V}_m^{(1)}$ as the approximated fuzzy components of $v(t)$ and $v^{(1)}(t)$, respectively, and they can be obtained by

$$\begin{bmatrix} \overline{V}_m \\ \overline{V}_{m+1} \\ \overline{V}_{m-1} \end{bmatrix} = \begin{bmatrix} \alpha_m^{(0)} & \alpha_{m+1}^{(0)} & \alpha_{m-1}^{(0)} \\ \alpha_m^{(1)} & \alpha_{m+1}^{(1)} & \alpha_{m-1}^{(1)} \\ \alpha_m^{(2)} & \alpha_{m+1}^{(2)} & \alpha_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} v_m \\ v_{m+1} \\ v_{m-1} \end{bmatrix}, \tag{5.5}$$

and

$$\begin{bmatrix} \overline{V}_m^{(1)} \\ \overline{V}_{m+1}^{(1)} \\ \overline{V}_{m-1}^{(1)} \end{bmatrix} = \begin{bmatrix} \beta_m^{(0)} & \beta_{m+1}^{(0)} & \beta_{m-1}^{(0)} \\ \beta_m^{(1)} & \beta_{m+1}^{(1)} & \beta_{m-1}^{(1)} \\ \beta_m^{(2)} & \beta_{m+1}^{(2)} & \beta_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} v'_m \\ v'_{m+1} \\ v'_{m-1} \end{bmatrix}. \tag{5.6}$$

Let us construct the approximations

$$\overline{\varphi}_{m-1} = \varphi\left(t_{m-1}, \overline{U}_{m-1}, \overline{U}_{m-1}^{(1)}, \overline{V}_{m-1}, \overline{V}_{m-1}^{(1)}\right), \tag{5.7}$$

$$\overline{\varphi}_{m+1} = \varphi\left(t_{m-1}, \overline{U}_{m-1}, \overline{U}_{m-1}^{(1)}, \overline{V}_{m-1}, \overline{V}_{m-1}^{(1)}\right). \tag{5.8}$$

Using the equations (5.5)-(5.6) in (5.7) and (5.8) and employing Taylor's expansion, we obtain

$$\begin{aligned}
\overline{\varphi}_{m+1} = \varphi_{m+1} &- \frac{h^2}{12}\Big(5v^2\left(\sigma_m^{(0)}u^{(1)}(t_m) + \mu_m^{(0)}v^{(1)}(t_m)\right) - \rho_m^{(0)}u^{(2)}(t_m) \\
&- \tau_m^{(0)}v^{(2)}(t_m) + 4\left(\sigma_m^{(0)}u^{(3)}(t_m) + \mu_m^{(0)}v^{(3)}(t_m)\right)\Big) \\
&- \frac{h^3}{12}\Big[v^2\left(\left(5\sigma_m^{(1)} + \rho_m^{(0)}\right)u^{(1)}(t_m) + \left(5\mu_m^{(1)} + \tau_m^{(0)}\right)v^{(1)}(t_m)\right) \\
&- \left(\rho_m^{(1)} - 2v^2\sigma_m^{(0)}\right)u^{(2)}(t_m) - \left(\tau_m^{(1)} - 2v^2\mu_m^{(0)}\right)v^{(2)}(t_m) + 4\sigma_m^{(1)}u^{(3)}(t_m) \\
&+ 4\mu_m^{(1)}v^{(3)}(t_m) + \sigma_m^{(0)}u^{(4)}(t_m) + \mu_m^{(0)}v^{(4)}(t_m)\Big] + O\left(h^4\right),
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
\overline{\varphi}_{m-1} = \varphi_{m-1} &- \frac{h^2}{12}\Big(5v^2\left(\sigma_m^{(0)}u^{(1)}(t_m) + \mu_m^{(0)}v^{(1)}(t_m)\right) - \rho_m^{(0)}u^{(2)}(t_m) \\
&- \tau_m^{(0)}v^{(2)}(t_m) + 4\left(\sigma_m^{(0)}u^{(3)}(t_m) + \mu_m^{(0)}v^{(3)}(t_m)\right)\Big) \\
&+ \frac{h^3}{12}\Big[v^2\left(\left(5\sigma_m^{(1)} + \rho_m^{(0)}\right)u^{(1)}(t_m) + \left(5\mu_m^{(1)} + \tau_m^{(0)}\right)v^{(1)}(t_m)\right) \\
&- \left(\rho_m^{(1)} - 2v^2\sigma_m^{(0)}\right)u^{(2)}(t_m) - \left(\tau_m^{(1)} - 2v^2\mu_m^{(0)}\right)v^{(2)}(t_m) + 4\sigma_m^{(1)}u^{(3)}(t_m) \\
&+ 4\mu_m^{(1)}v^{(3)}(t_m) + \sigma_m^{(0)}u^{(4)}(t_m) + \mu_m^{(0)}v^{(4)}(t_m)\Big] + O\left(h^4\right),
\end{aligned} \tag{5.10}$$

where $\varphi_{m\pm1} = \varphi\left(t_{m\pm1}, u\left(t_{m\pm1}\right), u^{(1)}\left(t_{m\pm1}\right), v\left(t_{m\pm1}\right), v^{(1)}\left(t_{m\pm1}\right)\right)$, $\rho_m^{(0)} = \left.\frac{\partial\varphi}{\partial u(t)}\right|_{t=t_m}$, $\sigma_m^{(0)} = \left.\frac{\partial\varphi}{\partial u^{(1)}(t)}\right|_{t=t_m}$, $\tau_m^{(0)} = \left.\frac{\partial\varphi}{\partial v(t)}\right|_{t=t_m}$, $\mu_m^{(0)} = \left.\frac{\partial\varphi}{\partial v^{(1)}(t)}\right|_{t=t_m}$ and $\mu_m^{(1)} = \left.\frac{\partial^2\varphi}{\partial t\partial v^{(1)}(t)}\right|_{t=t_m}$.

Next, we will update fuzzy components $\overline{U}_m, \overline{U}_m^{(1)}, \overline{V}_m$ and $\overline{V}_m^{(1)}$, by considering the following linear combinations

$$\widehat{U}_m = \overline{U}_m + h^2k_1\left(\overline{V}_{m+1} + \overline{V}_{m-1}\right), \quad \widehat{U}_m^{(1)} = \overline{U}_m^{(1)} + hk_1\left(\overline{V}_{m+1} - \overline{V}_{m-1}\right), \tag{5.11}$$

$$\widehat{V}_m = \overline{V}_m + h^2 k_1 \left(\overline{\varphi}_{m+1} + \overline{\varphi}_{m-1}\right), \quad \widehat{V}_m^{(1)} = \overline{V}_m^{(1)} + hk_1 \left(\overline{\varphi}_{m+1} - \overline{\varphi}_{m-1}\right). \tag{5.12}$$

Which yields,

$$\widehat{U}_m = u_m + h^2 k_1 \left(v_{m+1} + v_{m-1}\right) + \frac{h^2}{12} u^{(2)}\left(t_m\right) + O\left(h^4\right), \tag{5.13}$$

$$\widehat{U}_m^{(1)} = u_m^{(1)} + k_1 \left(v_{m+1} - v_{m-1}\right) + \frac{h^2}{12}\left(v^2 u^{(2)}\left(t_m\right) + 2u^{(3)}\left(t_m\right)\right) + O\left(h^4\right), \tag{5.14}$$

$$\widehat{V}_m = v_m + \frac{h^2}{12}\left(24k_1 + 1\right) v^{(2)}\left(t_m\right) + O\left(h^4\right), \tag{5.15}$$

and

$$\widehat{V}_m^{(1)} = v_m^{(1)} + \frac{h^2}{12}\left(v^2 v^{(1)}\left(t_m\right) + 2\left(12k_1 + 1\right) v^{(3)}\left(t_m\right)\right) + O\left(h^4\right). \tag{5.16}$$

Using equation (5.13)-(5.16), we can define

$$\widehat{\varphi}_m = \varphi\left(t_m, \widehat{U}_m, \widehat{U}_m^{(1)}, \widehat{V}_m, \widehat{V}_m^{(1)}\right). \tag{5.17}$$

Then, it is easy to obtain an $O\left(h^6\right)$ local truncation error for the following discretization

$$\overline{U}_{m-1} + \overline{U}_{m+1} - 2\overline{U}_m - h^2\left(p_1 \overline{V}_{m-1} + p_0 \overline{V}_m + p_1 \overline{V}_{m+1}\right) = O\left(h^6\right), \tag{5.18}$$

and

$$\overline{V}_{m-1} + \overline{V}_{m+1} - 2\overline{V}_m - h^2\left(p_1 \overline{\varphi}_{m+1} + p_0 \widehat{\varphi}_m + p_1 \overline{\varphi}_{m-1}\right) = O\left(h^6\right). \tag{5.19}$$

The boundary data $u(a) = a_1, u(b) = b_1, v(a) = a_2, v(b) = b_2$ and the Gauss-Seidel or Newton-Raphson method can be used to solve the coupled matrix system of discrete relations (5.18) and (5.19).

6 Fuzzy transform scheme for sixth-order BVPs

The fuzzy component scheme (4.19) for sixth-order $(n = 3)$ nonlinear BVP will be discussed in this section. Suppose,

$$u^{(6)}(t) = \varphi\left(t, u(t), u^{(1)}(t), u^{(2)}(t), u^{(3)}(t), u^{(4)}(t), u^{(5)}(t)\right), a < t < b, \tag{6.1}$$

$$u(a) = a_1, u^{(2)}(a) = a_2, u^{(4)}(a) = a_3, u(b) = b_1, u^{(2)}(b) = b_2, u^{(4)}(b) = b_3. \tag{6.2}$$

Similar to the formulation obtained in sections 4 and 5, we will derive a fuzzy transform scheme for BVP (6.1), assuming $u^{(2)}(t) = v(t)$ and $v^{(2)}(t) = w(t)$. Hence, (6.1) can be re-written as a system of second-order differential equation

$$u^{(2)}(t) = v(t), \quad v^{(2)}(t) = w(t), \tag{6.3}$$

$$w^{(2)}(t) = \varphi\left(t, u(t), u^{(1)}(t), v(t), v^{(1)}(t), w(t), w^{(1)}(t)\right). \tag{6.4}$$

Define

$$\begin{bmatrix} \overline{V}_m \\ \overline{V}_{m+1} \\ \overline{V}_{m-1} \end{bmatrix} = \begin{bmatrix} \alpha_m^{(0)} & \alpha_{m+1}^{(0)} & \alpha_{m-1}^{(0)} \\ \alpha_m^{(1)} & \alpha_{m+1}^{(1)} & \alpha_{m-1}^{(1)} \\ \alpha_m^{(2)} & \alpha_{m+1}^{(2)} & \alpha_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} v_m \\ v_{m+1} \\ v_{m-1} \end{bmatrix}, \tag{6.5}$$

$$\begin{bmatrix} \overline{V}_m^{(1)} \\ \overline{V}_{m+1}^{(1)} \\ \overline{V}_{m-1}^{(1)} \end{bmatrix} = \begin{bmatrix} \beta_m^{(0)} & \beta_{m+1}^{(0)} & \beta_{m-1}^{(0)} \\ \beta_m^{(1)} & \beta_{m+1}^{(1)} & \beta_{m-1}^{(1)} \\ \beta_m^{(2)} & \beta_{m+1}^{(2)} & \beta_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} v_m^{(1)} \\ v_{m+1}^{(1)} \\ v_{m-1}^{(1)} \end{bmatrix}, \tag{6.6}$$

$$\begin{bmatrix} \overline{W}_m \\ \overline{W}_{m+1} \\ \overline{W}_{m-1} \end{bmatrix} = \begin{bmatrix} \alpha_m^{(0)} & \alpha_{m+1}^{(0)} & \alpha_{m-1}^{(0)} \\ \alpha_m^{(1)} & \alpha_{m+1}^{(1)} & \alpha_{m-1}^{(1)} \\ \alpha_m^{(2)} & \alpha_{m+1}^{(2)} & \alpha_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} w_m \\ w_{m+1} \\ w_{m-1} \end{bmatrix}, \tag{6.7}$$

and

$$\begin{bmatrix} \overline{W}_m^{(1)} \\ \overline{W}_{m+1}^{(1)} \\ \overline{W}_{m-1}^{(1)} \end{bmatrix} = \begin{bmatrix} \beta_m^{(0)} & \beta_{m+1}^{(0)} & \beta_{m-1}^{(0)} \\ \beta_m^{(1)} & \beta_{m+1}^{(1)} & \beta_{m-1}^{(1)} \\ \beta_m^{(2)} & \beta_{m+1}^{(2)} & \beta_{m-1}^{(2)} \end{bmatrix} \begin{bmatrix} w_m^{(1)} \\ w_{m+1}^{(1)} \\ w_{m-1}^{(1)} \end{bmatrix}. \tag{6.8}$$

Suppose $\overline{V}_m, \overline{V}_m^{(1)}, \overline{W}_m, \overline{W}_m^{(1)}$ are approximated fuzzy components of $v(t), v^{(1)}(t), w(t)$, and $w^{(1)}(t)$, respectively. Let

$$\overline{\varphi}_{m+\delta} = \varphi\left(t_{m+\delta}, \overline{U}_{m+\delta}, \overline{U}_{m+\delta}^{(1)}, \overline{V}_{m+\delta}, \overline{V}_{m+\delta}^{(1)}, \overline{W}_{m+\delta}, \overline{W}_{m+\delta}^{(1)}\right), \delta = 0, \pm 1. \tag{6.9}$$

Using the equations (6.5)-(6.8) in (6.9) and employing Taylor's expansion on $\overline{\varphi}_{m+1}, \overline{\varphi}_{m-1}$ yields

$$\begin{aligned} \overline{\varphi}_{m+1} &= \varphi_{m+1} - \frac{h^2}{12}\Big(5v^2\left(\sigma_m^{(0)}u^{(1)}(t_m) + \mu_m^{(0)}v^{(1)}(t_m) + \varsigma_m^{(0)}w^{(1)}(t_m)\right) \\ &-\rho_m^{(0)}u^{(2)}(t_m) - \tau_m^{(0)}v^{(2)}(t_m) - \lambda_m^{(0)}w^{(2)}(t_m) + 4\Big(\sigma_m^{(0)}u^{(3)}(t_m) + \mu_m^{(0)}v^{(3)}(t_m) \\ &- +\varsigma_m^{(0)}w^{(3)}(t_m)\Big)\Big) \frac{h^3}{12}\Big[v^2\Big(\left(5\sigma_m^{(1)} + \rho_m^{(0)}\right)u^{(1)}(t_m) + \left(5\mu_m^{(1)} + \tau_m^{(0)}\right)v^{(1)}(t_m) \\ &+ \left(5\zeta_m^{(1)} + \lambda_m^{(0)}\right)w^{(1)}(t_m)\Big) - \left(\rho_m^{(1)} - 2v^2\sigma_m^{(0)}\right)u^{(2)}(t_m) - \left(\tau_m^{(1)} - 2v^2\mu_m^{(0)}\right) \\ &v^{(2)}(t_m) - \left(\lambda_m^{(1)} - 2v^2\zeta_m^{(0)}\right)w^{(2)}(t_m) + 4\sigma_m^{(1)}u^{(3)}(t_m) + 4\mu_m^{(1)}v^{(3)}(t_m) \\ &+ 4\zeta_m^{(1)}w^{(3)}(t_m) + \sigma_m^{(0)}u^{(4)}(t_m) + \mu_m^{(0)}v^{(4)}(t_m) + \zeta_m^{(0)}w^{(4)}(t_m)\Big] + O\left(h^4\right), \end{aligned} \tag{6.10}$$

$$\begin{aligned} \overline{\varphi}_{m-1} &= \varphi_{m-1} - \frac{h^2}{12}\Big(5v^2\left(\sigma_m^{(0)}u^{(1)}(t_m) + \mu_m^{(0)}v^{(1)}(t_m) + \varsigma_m^{(0)}w^{(1)}(t_m)\right) \\ &-\rho_m^{(0)}u^{(2)}(t_m) - \tau_m^{(0)}v^{(2)}(t_m) - \lambda_m^{(0)}w^{(2)}(t_m) + 4\Big(\sigma_m^{(0)}u^{(3)}(t_m) + \mu_m^{(0)}v^{(3)}(t_m) \\ &+\varsigma_m^{(0)}w^{(3)}(t_m)\Big)\Big) + \frac{h^3}{12}\Big[v^2\Big(\left(5\sigma_m^{(1)} + \rho_m^{(0)}\right)u^{(1)}(t_m) + \left(5\mu_m^{(1)} + \tau_m^{(0)}\right)v^{(1)}(t_m) \\ &+ \left(5\zeta_m^{(1)} + \lambda_m^{(0)}\right)w^{(1)}(t_m)\Big) - \left(\rho_m^{(1)} - 2v^2\sigma_m^{(0)}\right)u^{(2)}(t_m) - \left(\tau_m^{(1)} - 2v^2\mu_m^{(0)}\right) \\ &v^{(2)}(t_m) - \left(\lambda_m^{(1)} - 2v^2\zeta_m^{(0)}\right)w^{(2)}(t_m) + 4\sigma_m^{(1)}u^{(3)}(t_m) + 4\mu_m^{(1)}v^{(3)}(t_m) \\ &+4\zeta_m^{(1)}w^{(3)}(t_m) + \sigma_m^{(0)}u^{(4)}(t_m) + \mu_m^{(0)}v^{(4)}(t_m) + +\zeta_m^{(0)}w^{(4)}(t_m)\Big] + O\left(h^4\right), \end{aligned} \tag{6.11}$$

where $\varphi_{m\pm1} = \varphi\left(t_{m\pm1}, u\left(t_{m\pm1}\right), u^{(1)}\left(t_{m\pm1}\right), v\left(t_{m\pm1}\right), v^{(1)}\left(t_{m\pm1}\right), w\left(t_{m\pm1}\right), \ w^{(1)}\left(t_{m\pm1}\right)\right)$, $\rho_m^{(0)} = \left.\frac{\partial\varphi}{\partial u(t)}\right|_{t=t_m}, \sigma_m^{(0)} = \left.\frac{\partial\varphi}{\partial u^{(1)}(t)}\right|_{t=t_m}, \ \tau_m^{(0)} = \left.\frac{\partial\varphi}{\partial v(t)}\right|_{t=t_m}, \ \mu_m^{(0)} = \left.\frac{\partial\varphi}{\partial v^{(1)}(t)}\right|_{t=t_m}, \ \lambda_m^{(0)} = \left.\frac{\partial\varphi}{\partial w(t)}\right|_{t=t_m}, \varsigma_m^{(0)} = \left.\frac{\partial\varphi}{\partial w^{(1)}(t)}\right|_{t=t_m}$ and $\zeta_m^{(0)} = \left.\frac{\partial\varphi}{\partial t\partial w^{(1)}(t)}\right|_{t=t_m}$.
Next, we will update fuzzy components of $u(t_m), u'(t_m), v(t_m)$, and $v'(t_m)$ at the central mesh point $t = t_m$, we consider

$$\widehat{U}_m = \overline{U}_m + h^2k_1\left(\overline{V}_{m+1} + \overline{V}_{m-1}\right), \quad \widehat{U}_m^{(1)} = \overline{U}_m^{(1)} + hk_1\left(\overline{V}_{m+1} - \overline{V}_{m-1}\right), \tag{6.12}$$

$$\widehat{V}_m = \overline{V}_m + h^2k_1\left(\overline{W}_{m+1} + \overline{W}_{m-1}\right), \quad \widehat{V}_m^{(1)} = \overline{V}_m^{(1)} + hk_1\left(\overline{W}_{m+1} - \overline{W}_{m-1}\right), \tag{6.13}$$

$$\widehat{W}_m = \overline{W}_m + h^2 k_1 \left(\overline{\varphi}_{m+1} + \overline{\varphi}_{m-1}\right), \quad \widehat{W}_m^{(1)} = \overline{W}_m^{(1)} + h k_1 \left(\overline{\varphi}_{m+1} - \overline{\varphi}_{m-1}\right). \tag{6.14}$$

As a result, a system of difference equations with $O(h^6)$ local truncation error is obtained as

$$\overline{U}_{m-1} + \overline{U}_{m+1} - 2\overline{U}_m - h^2\left(p_1 \overline{V}_{m-1} + p_0 \overline{V}_m + p_1 \overline{V}_{m+1}\right) = O\left(h^6\right), \tag{6.15}$$

$$\overline{V}_{m-1} + \overline{V}_{m+1} - 2\overline{V}_m - h^2\left(p_1 \overline{W}_{m+1} + p_0 \overline{W}_m + p_1 \overline{W}_{m-1}\right) = O\left(h^6\right), \tag{6.16}$$

$$\overline{W}_{m-1} + \overline{W}_{m+1} - 2\overline{W}_m - h^2\left(p_1 \overline{\varphi}_{m+1} + p_0 \widehat{\varphi}_m + p_1 \overline{\varphi}_{m-1}\right) = O\left(h^6\right). \tag{6.17}$$

The discrete system (6.15)-(6.17) combined with the boundary data (6) can be solved by the iterative method and acquired fourth-order accurate solution values.

7 Numerical simulation

The utility and efficiency of the numerical schemes are analysed by stimulations on a wide range of singular and non-singular higher order BVPs. The essence of numerical approximations lies in their solution accuracy, memory efficiency, and CPU time. The computation of error metrics such as maximum absolute error, root mean squared error and their order of convergence are determined using the formula

$$\ell_\infty^{u^{(2n)}} = \max_{1\le i\le M}\left|u^{(2n)}(t_i) - u_i^{(2n)}\right|, \ell_2^{u^{(2n)}} = \left(\frac{1}{M}\sum_{i=1}^{M}\left|u^{(2n)}(t_i) - u_i^{(2n)}\right|^2\right)^{1/2}, \tag{7.1}$$

$$O_\infty^{u^{(2n)}} = \log_2\left(\frac{\ell_\infty^{u^{(2n)}}|_{M-grids}}{\ell_\infty^{u^{(2n)}}|_{2M-grids}}\right), O_2^{u^{(2n)}} = \log_2\left(\frac{\ell_2^{u^{(2n)}}|_{M-grids}}{\ell_2^{u^{(2n)}}|_{2M-grids}}\right), n = 1,2,3. \tag{7.2}$$

The simulations in the context of linear equations are performed using a Gauss-Seidel solver, and nonlinear equations are solved by the Newton-Raphson iterative method with the error tolerance of 10^{-10}. The boundary condition will be taken from the exact solution unless specified otherwise.

Example 1. Consider the linear second-order differential equation

$$u^{(2)}(t) + \frac{\eta}{t}u^{(1)}(t) - \frac{\eta}{t^2}u(t) = \frac{(16t^8 + 4(\eta+3)t^4 - \eta)e^{t^4}}{t^2}, \tag{7.3}$$

possessing the analytical solution $u(t) = e^{t^4}$. Errors and their order of convergence for $\eta = 2, v = 0.001$ and $\eta = 2, v = 5.8$ are assessed at the various number of nodal points in Table 1, efficiency and convergence rate of the scheme.

Example 2. Consider the nonlinear second-order differential equation

$$u^{(2)}(t) = +\frac{\eta}{t}u^{(1)}(t) - \frac{\eta}{t^2}u(t) + R_e u(t)u^{(1)}(t) + g(t), \tag{7.4}$$

the function $g(t)$ is evaluated using the exact solution $u(t) = t^2\cosh t$. Errors and their order of convergence for $\eta = 2, v = 0.5$, and $R_e = 1000$ (Reynold's number), are obtained at the various meshes points are presented in Table 2 corroborating the fourth-order convergence.

Example 3. Consider the linear fourth-order differential equation

$$u^{(4)}(t) = \frac{-2\eta}{t}u^{(3)}(t) + \frac{\eta(2-\eta)}{t^2}u^{(2)}(t) + \frac{\eta(2-\eta)}{t^3}u^{(1)}(t) = g(t), \tag{7.5}$$

The forcing function $g(t)$ is obtained from the exact solution $u(t) = t^4\cosh t$. Errors and their order of convergence for $\eta = 2, v = 0.5$, nodal points are presented in Table 3.

Example 4. Consider the nonlinear fourth-order differential equation

$$u^{(4)}(t) = \eta u^{(2)}(t) - \frac{(u(t))^2}{2} + u(t) + g(t), \tag{7.6}$$

the function $g(t)$ is evaluated for analytical solution $u(t) = \eta \sinh t$. Errors and their order of convergence for $\eta = 2, v = 0.01$, are calculated for the various meshes points are presented in Table 4.

Example 5. Consider the linear fourth-order differential equation

$$\begin{aligned} u^{(6)}(t) = \frac{-3\eta}{t}u^{(5)}(t) - \frac{3\eta(\eta-2)}{t^2}u^{(3)}(t) \\ + \frac{3\eta(\eta-2)(\eta-4)}{t^4}u^{(2)}(t) - \frac{3\eta(\eta-2)(\eta-4)}{t^5}u'(t) + g(t), \end{aligned} \tag{7.7}$$

the function $g(t)$ is computed using the exact solution $u(t) = t^6 \sinh t$. The errors and their order of convergence for $\eta = 2, v = 0.05$, are reported in Table 5. The numerical simulations in each case exhibit order preserving solution values and close to the exact solution.

8 Conclusion

Fuzzy transform is a novel technique to approximate the solution of nonlinear BVPs. It provides linear piecewise approximating polynomials solutions with at most second-order accuracy. However, the approximation of fuzzy components enables the hybrid fuzzy scheme to achieve fourth-order accuracy. The simulation of the approximation technique is carried out on a variety of problems, including both singular and non-singular categories of linear and nonlinear BVPs of the second, fourth, and sixth orders. The advantage of the present scheme is implementing a non-polynomial (trigonometry) basis to approximate the fuzzy components, which helps to determine an order-preserving solution scheme. Extension of the high-resolution fuzzy transform scheme for partial differential equations with nonlinear gradients is an open problem.

References

[1] I. Perfilieva, *Fuzzy transforms: Transactions on rough sets II,* Springer, Berlin, Heidelberg, 63-81 (2004).

[2] A. Khastan, Z. Aliiani and I. Perfilieva, Fuzzy transform to approximate solution of two point boundary value problems, *Math. Meth. Appl. Sci.* **40**, 6147-6154 (2017).

[3] I. Perfilieva, M. Daňková and B. Bede, Towards a higher degree F-transform, *Fuzzy Sets Syst.* **180**, 3-19 (2011).

[4] I. Perfilieva, Fuzzy transforms: Theory and applications, *Fuzzy Sets Syst.* **157**, 993-1023 (2006).

[5] I. Perfilieva and V. Kreinovich, Fuzzy transforms of higher order approximate derivatives: A theorem, *Fuzzy Sets Syst.* **180**, 55-68 (2011).

[6] I. Perfilieva, P. Števuliáková and R. Valášek, F-transform-based shooting method for nonlinear boundary value problems, *Soft Comput.* **21**, 3493-3502 (2017).

[7] M. Štěpnička and O. Polakovič, A neural network approach to the fuzzy transform, *Fuzzy Sets Syst.* **160**, 1037-1047 (2009).

[8] S.B. Roh, S.K. Oh, J.H. Yoon and K. Seo, Design of face recognition system based on fuzzy transform and radial basis function neural networks, *Soft Comput.* **23**(13), 4969-4985 (2019).

[9] R. P. Agarwal and P. R. Krishnamoorthy, Boundary value problems for nth order ordinary differential equations, *Bull. Inst. Math., Acad. Sin.* **7**, 211–230 (1979).

[10] A. R. Aftabizadeh, Existence and uniqueness theorems for fourth-order boundary value problems, *J. Math. Anal. Appl.* **116**, 415–426 (1986).

[11] Z. Wei, A necessary and sufficient condition for 2nth order singular super-linear m-point boundary value problems, *J. Math. Anal. Appl.* **327**, 930–947 (2007).

[12] N. Jha, R.K. Mohanty and V. Chauhan, The convergence of geometric mesh cubic spline finite difference scheme for nonlinear higher order two-point boundary value problems, *Int. J. Comput. Math.* **527924**, 1-12, (2014).

[13] N. Jha, High order accurate quintic nonpolynomial spline finite difference approximations for the numerical solution of non-linear two point boundary value problems, *Int. J. Model. Simul. Sci. Comput.* **5**(1), 1-16 (2014).

[14] N. Jha, L.K. Bieniasz, A fifth (six) order accurate, three-point compact finite difference scheme for the numerical solution of sixth order boundary value problems on geometric meshes, *J. Sci. Comput.* **64**, 898-913 (2015).

[15] N. Jha, R.K. Mohanty, V. Chauhan, Efficient algorithms for fourth and sixth order two-point non-linear boundary value problems using non-polynomial spline approximations on geometric mesh, *Comput. Appl. Math.* **35**, 389-404 (2016).

[16] S. Liang and D. J. Jeffrey, An efficient analytical approach for solving fourth order boundary value problems, *Comput. Phys. Commun.* **180**, 2034–2040 (2009).

Author information

Navnit Jha and Kritika, Department of Mathematics, South Asian University, New Delhi, India 110021, India.
E-mail: `navnitjha@sau.ac.in`, `kritikagpt95@gmail.com`

Table 1. Error metrics and computational convergence order in Example 1

M	ℓ_2^u	O_∞^u	ℓ_∞^u	O_∞^u
$\eta = 2, \quad \nu = 0.001$				
8	$1.39e-03$	$\cdots$	$1.94e-03$	$\cdots$
14	$1.89e-04$	3.6	$2.70e-04$	3.5
26	$1.83e-05$	3.8	$2.67e-05$	3.7
50	$1.44e-06$	3.9	$2.12e-06$	3.9
$\eta = 2, \quad \nu = 5.8$				
8	$1.45e-04$	$\cdots$	$1.87e-04$	$\cdots$
14	$1.72e-05$	3.8	$2.90e-05$	3.3
26	$1.66e-06$	3.8	$3.24e-06$	3.5
50	$1.32e-07$	3.9	$2.67e-07$	3.8

Table 2. Error metrics and computational convergence order in Example 2

M	ℓ_2^u	O_2^u	ℓ_∞^u	O_∞^u
8	$5.36e-02$	$\cdots$	$1.34e-01$	$\cdots$
14	$2.54e-03$	5.3	$4.15e-03$	6.2
26	$2.61e-04$	3.7	$5.05e-04$	3.4
50	$2.20e-05$	3.8	$5.15e-05$	3.5

Table 3. Error metrics and computational convergence order in Example 3

M	$\ell_\infty^{u^{(2)}}$	$O_\infty^{u^{(2)}}$	ℓ_∞^u	O_∞^u
8	$1.24e-03$	$\cdots$	$6.95e-05$	$\cdots$
14	$2.27e-04$	3.0	$9.32e-06$	3.6
26	$2.89e-05$	3.8	$8.90e-07$	3.9
50	$2.89e-06$	3.5	$4.67e-08$	4.5

Table 4. Error metrics and computational convergence order in Example 4

M	$\ell_\infty^{u^{(2)}}$	$O_\infty^{u^{(2)}}$	ℓ_∞^u	O_∞^u
26	$1.61e+01$	$\cdots$	$3.26e-02$	$\cdots$
50	$1.22e-00$	3.9	$2.47e-03$	3.9
98	$8.50e-02$	4.0	$1.71e-04$	4.0
194	$5.63e-03$	4.0	$1.14e-05$	4.0

Table 5. Error metrics and computational convergence order in Example 5

M	$\ell_\infty^{u^{(4)}}$	$O_\infty^{u^{(4)}}$	$\ell_\infty^{u^{(2)}}$	$O_\infty^{u^{(2)}}$	ℓ_∞^u	O_∞^u
8	$7.45e-01$	$\cdots$	$1.09e-01$	$\cdots$	$5.52e-03$	$\cdots$
14	$9.79e-02$	3.6	$1.43e-02$	3.6	$7.15e-04$	3.7
26	$9.33e-03$	3.8	$1.36e-03$	3.8	$6.86e-05$	3.8