Palestine Journal of Mathematics

Vol. ..(..)(2022) , 45–56

Alternating Direction Implicit Bi-Cubic Spline Technique For Two-Dimensional Hyperbolic Equation

Suruchi Singh, Anu Aggarwal and Swarn Singh

Math. Subject Classification: 65M60
Keywords: 2D-Hyperbolic equation, ADI, bi-cubic spline, high order, collocation, stability, tensor product

In this article, we develop a new Alternating direction implicit (ADI) collocation method based on the bi-cubic spline collocation method to find the approximate solution of 2D-hyperbolic partial differential equations. The method presented in this article uses high order bi-cubic B-spline collocation for the differential equation. It requires compact support. The second-order central finite difference method has been used in the time direction. Unconditional stability analysis of the new technique has also been also done in this article. To corroborate the potency of the new algorithm few experiments have been included in the given article. Fourth-order accuracy has been exhibited by the results obtained. Comparison analysis has been done with the literature for some examples.

1 Introduction

The second-order telegraphic equation in two dimensions defined on a regular domain $\mathcal{D}$ [18] with the forcing function f is given as:

$$u_{xx} + u_{yy} = u_{tt} + 2\alpha u_t + \beta^2 u + f \tag{1.1}$$

where $\mathcal{D} = [0,1] \times [0,1] \times (0,T)$ and $\alpha > 0$, $\beta \geq 0$ are the given constants.
The equation (1.1) is subjected to the following initial and boundary conditions:

$$u(x,y,0) = \phi_1(x,y) \quad \text{and} \quad u_t(x,y,0) = \phi_2(x,y) \tag{1.2}$$

$$u(0,y,t) = \omega_1(y,t) \quad \text{and} \quad u(1,y,t) = \omega_2(y,t) \tag{1.3}$$

$$u(x,0,t) = \omega_3(x,t) \quad \text{and} \quad u(x,1,t) = \omega_4(x,t) \tag{1.4}$$

for $x, y \in [0,1]$ and $t \geq 0$. The functions ϕ_1, ϕ_2, ω_1, ω_2, ω_3 and ω_4 are continuously differentiable functions in their respective domain.
In today's world, the applications of telegraphic equations can be experienced in the diverse discipline of sciences. To study the wave propagation of electric signals in transmission cable and atomic theory, the telegraph equation is used.
Over the past few years, loads of studies have been done to obtain various algorithms to compute the estimated solutions of the high dimensional differential equations. Finite difference methods have been used by many researchers in [1]-[5]. In [1], by using a new three-level implicit method that is unconditionally stable also, high dimensional damped wave equations have been solved by Mohanty. In the last couple of years, numerous advances have been made to develop and analyze methods based on splines [6]-[15], such as cubic splines, quadratic splines, orthogonal splines and exponential splines. The differential quadrature method based on splines is used by Ghasemi in [12]. A method based on wavelets collocation is presented in [13]. A significant amount of study using orthogonal splines for solutions of partial differential equations has been done by Bialecki and Fairweather in their research [14]. They discussed collocation methods with an orthogonal cubic spline for elliptic, parabolic, and hyperbolic partial integro-differential equations. Zadvan and Rashidinia, [16] solved multi-dimensional wave equations of order one with a nonlinear source term by using a cubic spline of non-polynomial functions. An algorithm using thin plate spline radial basis is used by Dehghan and Shokri in [17]. In [18], the RBF solutions of second-order hyperbolic equations are explored using a meshfree method which is an assimilation of boundary knot and analog method. A method based on a combination of meshless

local weak and strong forms for the solution of telegraph equation in two dimensions is done in [19]. In paper [20], the two-dimensional telegraph equation has been solved with barycentric rational interpolation whereas in [21] this equation has been solved by using the differential quadrature technique. A modified nodal bi-cubic spline collocation method is presented in [22] to solve the elliptic partial differential equation.

The alternating direction implicit method was introduced in [3],[24]. Because of its great operator-splitting ability, it is widely used for solving differential equations which are of high order.

In this study, a new methodology is developed to solve a two-dimensional telegraphic equation by using the collocation method which is based on bi-cubic B-spline in a much simpler way. For this, we introduce a tensor product of matrices resulting from one dimension. We obtain a fast method i.e. ADI method to solve a broad category of hyperbolic problems in two dimensions. For the estimation of the solution, this method makes use of compact support. The proposed method is found to be fourth-order accurate.

The paper is organized in the following way: In Section 2, a brief introduction of the bi-cubic B-spline collocation method is given. This method is used in this research to find the numerical estimation of equation (1.1). The ADI form of the newly introduced method is also suggested in this section. Section 3, has been assigned to analyze the unconditional stability of the suggested methodology. To prove the efficacy and preciseness of our present scheme we took a few examples from the literature in Section 4 and finally the article ends with the conclusion in the last Section.

2 Bi-cubic B spline collocation method

In this segment of the paper, the new algorithm which is based on the collocation of bi-cubic B-spline has been introduced in the interest of finding a better approximate solution of 2D-differential equation (1.1). The interval [0,1] is uniformly partitioned with step size $h = \frac{1}{N}$ in x and y directions. Let $\Omega_1 = \{x_i, i = 1; N, x_i - x_{i-1} = h\}$ and $\Omega_2 = \{y_j, j = 1; N, y_j - y_{j-1} = h\}$ be the partitions in x and y direction respectively. Further $\mathcal{S}_1(\Omega_1, 3)$ and $\mathcal{S}_2(\Omega_2, 3)$ denote the corresponding space of all the cubic splines [25] satisfying the given boundary conditions. The two partitions Ω_1 and Ω_2 are extended by two extra knots at each end respectively i.e. $x_{-1} = -h$ and $x_{N+1} = 1 + h$ in the x-direction and $y_{-1} = -h$ and $y_{N+1} = 1 + h$ in the y-direction. At the knots x_i's in x-direction, the cubic B-splines $\mathfrak{B}_i(x)$ are defined as ([2],[4]):

$$\mathfrak{B}_i(x) = \frac{1}{6h^3} \begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}], \\ h^3 + (x - x_{i-1})h^2 + (x - x_{i-1})^2 h + (x - x_{i-1})^3, & x \in [x_{i-1}, x_i], \\ h^3 + (x_{i+1} - x)h^2 + (x_{i+1} - x)^2 h + (x_{i+1} - x)^3, & x \in [x_i, x_{i+1}], \\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}], \\ 0, & otherwise. \end{cases} \tag{2.1}$$

for $i = -1, 0, 1, ..., N + 1$.

The cubic spline space $\mathcal{S}_1(\Omega_1, 3)$ will be generated by the collection of functions $\{\mathfrak{B}_i,\ i = -1 \text{ to } N + 1\}$. Using equation (2.1), the value of $\mathfrak{B}_i^{\prime}s$ and their derivatives w.r.t x calculated at the knot x_i are tabulated as follows.

Table 1. $\mathfrak{B}_i(x)$ and their derivatives w.r.t x

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\mathfrak{B}_i(x)$	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$\mathfrak{B}_i^{'}(x)$	0	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0
$\mathfrak{B}_i^{''}(x)$	0	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

Table 1 shows that three basis functions are nonzero at any particular knot x_i, namely $\mathfrak{B}_{i-1}, \mathfrak{B}_i$ and $\mathfrak{B}_{i+1}$. As defined above for the x-direction, analogously we define the collection of functions $\{\mathfrak{B}_i^*,\ i=-1 \text{ to } N+1\}$, which generate the cubic spline space $\mathcal{S}_2(\Omega_2,3)$ in the y-direction. Now let us denote by $\Omega=\{(x_i,y_j),i,j=0,...,N\}$, the partition of the domain $[0,1]\times[0,1]$ and $\mathcal{S}(\Omega,3)=\mathcal{S}_1(\Omega_1,3)\otimes\mathcal{S}_2(\Omega_2,3)$, be the bi-cubic spline space of all the functions defined on a domain Ω which satisfy the given boundary conditions. So if $U\in\mathcal{S}(\Omega,3)$ be any function, then it can be expressed as

$$U(x,y,t)=\sum_{l,m=-1}^{N+1}\zeta_{lm}(t)\mathfrak{B}_l(x)\mathfrak{B}_m^*(y) \tag{2.2}$$

where$\{\zeta_{lm}(t)\}_{l,m=-1}^{N+1}$ are time variant quantities and need to be find out.
The values of the function U and the derivatives $D_x^2U, D_y^2U, D_x^2D_y^2U, U_t$ and U_{tt} at grid points (x_i,y_j,t) are obtained respectively as:

$$U(x_i,y_j,t)=\sum_{l,m=-1}^{N+1}\zeta_{lm}(t)\mathfrak{B}_l(x_i)\mathfrak{B}_m^*(y_j), D_x^2U(x_i,y_j,t)=\sum_{l,m=-1}^{N+1}\zeta_{lm}(t)\mathfrak{B}_l''(x_i)\mathfrak{B}_m^*(y_j),$$

$$D_y^2U(x_i,y_j,t)=\sum_{l,m=-1}^{N+1}\zeta_{lm}(t)\mathfrak{B}_l(x_i)\mathfrak{B}_m^{*}{''}(y_j), D_x^2D_y^2U(x_i,y_j,t)=\sum_{l,m=-1}^{N+1}\zeta_{lm}(t)\mathfrak{B}_l''(x_i)\mathfrak{B}_m^{*}{''}(y_j),$$

$$U_t(x_i,y_j,t)=\sum_{l,m=-1}^{N+1}\dot{\zeta}_{lm}(t)\mathfrak{B}_l(x_i)\mathfrak{B}_m^*(y_j), U_{tt}(x_i,y_j,t)=\sum_{l,m=-1}^{N+1}\ddot{\zeta}_{lm}(t)\mathfrak{B}_l(x_i)\mathfrak{B}_m^*(y_j)$$

for $i,j=0,1,2,\ldots,N$ and $t\geq 0$.

So, take any numerical solution U of equation (1.1), then U belongs to the space $S(\Omega,3)$, hence from (2.2) it is linearly expressed as the combination of time dependent parameters.
Now, to find the value of U, we derive a high order collocation method drawn on bi-cubic B-spline, which is presented as :

$$\mathcal{F}U_{ij}=f_{ij}-\frac{h^2}{12}(f_{xx_{ij}}+f_{yy_{ij}}) \tag{2.3}$$

where

$$\mathcal{F}U_{ij}=\mathcal{F}_1U_{ij}+\mathcal{F}_2D_t^2U_{ij}+2\alpha\mathcal{F}_2D_tU_{ij} \tag{2.4}$$

$$\mathcal{F}_1=\mathcal{F}_{x^2}+\mathcal{F}_{y^2}+\mathcal{F}_{x^2y^2}-\beta^2,\ \ i,j=0,1,2,...,N, \tag{2.5}$$

$$\mathcal{F}_{x^2}U_{ij}=(1+\frac{\beta^2h^2}{12})D_x^2U_{ij}, \tag{2.6}$$

$$\mathcal{F}_{y^2}U_{ij}=(1+\frac{\beta^2h^2}{12})D_y^2U_{ij}, \tag{2.7}$$

$$\mathcal{F}_{x^2y^2}U_{ij}=-\frac{h^2}{6}D_x^2D_y^2U_{ij} \tag{2.8}$$

and

$$\mathcal{F}_2=-1+\frac{h^2}{12}D_x^2+\frac{h^2}{12}D_y^2-\frac{h^4}{144}D_x^2D_y^2, \qquad i,j=1,2,...,N-1, \tag{2.9}$$

Here U_{ij} and f_{ij} symbolized U and f respectively at the grid point (x_i,y_j,t) in the domain.
The method (2.3) is identical to the matrix system, the form of which is given as:

$$\mathbb{S}(\bar{\mathcal{U}}_{tt}+2\alpha\bar{\mathcal{U}}_t)=\mathbb{D}\bar{\mathcal{U}}+\mathbb{F}, \tag{2.10}$$

where $\mathbb{D}$ and $\mathbb{S}$ are linear combinations of tensor product of $(N+1)^2$ matrices given by

$$\mathbb{D}=(1+\frac{\beta^2h^2}{12})\mathbb{B}\otimes\mathbb{A}+(1+\frac{\beta^2h^2}{12})\mathbb{A}\otimes\mathbb{B}-\beta^2\mathbb{A}\otimes\mathbb{A}-\frac{h^2}{6}\mathbb{B}\otimes\mathbb{B}, \tag{2.11}$$

$$\mathbb{S} = (\mathbb{A} - \frac{h^2}{12}\mathbb{B}) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B}) \tag{2.12}$$

where $\otimes$ denotes the tensor product of matrices. Here $\mathbb{A} = \left[\frac{1}{6}\ \frac{2}{3}\ \frac{1}{6}\right]$ and $\mathbb{B} = \left[\frac{1}{h^2}\ \frac{-2}{h^2}\ \frac{1}{h^2}\right]$ are tri-diagonal matrices. $\bar{\mathcal{U}}$ is a column matrix given by

$$\bar{\mathcal{U}} = \begin{pmatrix} \zeta_{00}(t) & \zeta_{01}(t) & \cdots & \zeta_{0N}(t) & \cdots & \zeta_{N0}(t) & \zeta_{N1}(t) & \cdots & \zeta_{NN}(t) \end{pmatrix}^t$$

The next step is the discretization of the method in the time direction. We assume δt to be the step size defined in the direction of time in a way such that $t_k = k\delta t$ for all $k = 0, 1, 2, \ldots$.. Also, the approximations of order in the direction of time are given as

$$D_t^2 U_{ij}^k = \frac{U_{ij}^{k+1} - 2U_{ij}^k + U_{ij}^{k-1}}{\delta t^2} \equiv U_{tt\,ij}^k + O(\delta t^2) \tag{2.13}$$
$$D_t U_{ij}^k = \frac{U_{ij}^{k+1} - U_{ij}^{k-1}}{2\delta t} \equiv U_{t\ ij}^k + O(\delta t^2)$$

Using above equation in equation (2.3), we get,

$$\mathcal{F}_1 U_{ij}^k + \mathcal{F}_2 \frac{(U_{ij}^{k+1} - 2U_{ij}^k + U_{ij}^{k-1})}{\delta t^2} + 2\alpha\mathcal{F}_2 \frac{(U_{ij}^{k+1} - U_{ij}^{k-1})}{2\delta t} = f_{ij}^k - \frac{h^2}{12}(f_{xx}{}_{ij}^k + f_{yy}{}_{ij}^k) \tag{2.14}$$

To get the stability of the scheme, we modify the scheme (2.14) by adding high order terms: $\mu\beta^4\delta t^4 D_t^2 U,\ -\nu\delta t^2 D_x^2 D_t^2 U,\ -\frac{h^2}{12}\mu\beta^4\delta t^4 D_y^2 D_t^2 U$ and $\frac{h^2\delta t^2}{12}\nu D_x^2 D_y^2 D_t^2 U$. So the modified scheme becomes

$$\bar{\mathcal{F}} U_{ij}^k = f_{ij}^k - \frac{h^2}{12}(f_{xx}{}_{ij}^k + f_{yy}{}_{ij}^k) \tag{2.15}$$

where

$$\bar{\mathcal{F}} U_{ij}^k = \bar{\mathcal{F}}_1 U_{ij}^k + \bar{\mathcal{F}}_2 D_t^2 U_{ij}^k + 2\alpha\bar{\mathcal{F}}_3 D_t U_{ij}^k \tag{2.16}$$

$$\bar{\mathcal{F}}_1 = \mathcal{F}_1, \tag{2.17}$$

$$\bar{\mathcal{F}}_2 = \mathcal{F}_2 + \mu\beta^4\delta t^4 - \nu\delta t^2 D_x^2 - \frac{h^2}{12}\mu\beta^4\delta t^4 D_y^2 + \frac{h^2\delta t^2}{12}\nu D_x^2 D_y^2 \tag{2.18}$$

and

$$\bar{\mathcal{F}}_3 = \mathcal{F}_2 + \nu\delta t^2(D_y^2 - \frac{h^2}{12}D_x^2 D_y^2), \tag{2.19}$$

μ & ν are free parameters to be determined.
The modified scheme given by equation (2.15) can be penned as

$$(a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}_{tt} + 2\alpha(\mathbb{A} - \frac{h^2}{12}\mathbb{B}) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}_t = \mathbb{D}\bar{\mathcal{U}} + \mathbb{F}, \tag{2.20}$$

where

$$a_0 = 1 + \mu\beta^4\delta t^4, a_1 = 1 + 12\nu\delta t^2/h^2$$

The equation (2.20) is identical to a system of ordinary differential equations of second order. So, after discretizing in the time direction, using the central difference scheme of order two, we observe

$$\left((a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\right) \frac{\bar{\mathcal{U}}^{k+1} - 2\bar{\mathcal{U}}^k + \bar{\mathcal{U}}^{k-1}}{\delta t^2}$$
$$+ 2\alpha\left((\mathbb{A} - \frac{h^2}{12}\mathbb{B}) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\right) \frac{\bar{\mathcal{U}}^{k+1} - \bar{\mathcal{U}}^{k-1}}{2\delta t} = \mathbb{D}\bar{\mathcal{U}}^k + \mathbb{F}^k$$

This implies

$$\left((a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) + \alpha\delta t(\mathbb{A} - \frac{h^2}{12}\mathbb{B})\right) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}^{k+1} - 2(a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}^{k} \tag{2.21}$$

$$+\left((a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) - \alpha\delta t(\mathbb{A} - \frac{h^2}{12}\mathbb{B})\right) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}^{k-1}$$

$$= \delta t^2(1 + \frac{\beta^2 h^2}{12})(\mathbb{B}\otimes\mathbb{A} + \mathbb{A}\otimes\mathbb{B})\bar{\mathcal{U}}^k - \beta^2\delta t^2\mathbb{A}\otimes\mathbb{A}\bar{\mathcal{U}}^k - \frac{h^2\delta t^2}{6}\mathbb{B}\otimes\mathbb{B}\bar{\mathcal{U}}^k + \delta t^2\mathbb{F}^k$$

So, to ease the computations, the equation (2.21) is split into two equations that can be easily handled. The method (2.21) is equivalent to the two-step ADI form given by

$$\left((a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) + \alpha\delta t(\mathbb{A} - \frac{h^2}{12}\mathbb{B})\right) \otimes \mathbb{I}_y\hat{\bar{\mathcal{U}}}^{k+1} = \psi^k, \tag{2.22}$$

$$\hat{\bar{\mathcal{U}}}^{k+1} = \mathbb{I}_x \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}^{k+1}, \tag{2.23}$$

where

$$\psi^k = -\left((a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) - \alpha\delta t(\mathbb{A} - \frac{h^2}{12}\mathbb{B})\right) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}$$
$$+ 2(a_0\mathbb{A} - a_1\frac{h^2}{12}\mathbb{B}) \otimes (\mathbb{A} - \frac{h^2}{12}\mathbb{B})\bar{\mathcal{U}}^k + \delta t^2(1 + \frac{\beta^2h^2}{12})(\mathbb{B}\otimes\mathbb{A} + \mathbb{A}\otimes\mathbb{B})\bar{\mathcal{U}}^k$$
$$- \beta^2\delta t^2\mathbb{A}\otimes\mathbb{A}\bar{\mathcal{U}}^k - \frac{h^2\delta t^2}{6}\mathbb{B}\otimes\mathbb{B}\bar{\mathcal{U}}^k + \delta t^2\mathbb{F}^k.$$

Here $\mathbb{I}_x$ and $\mathbb{I}_y$ denote the identity matrices of order $N+1$. The vectors $\hat{\bar{\mathcal{U}}}_j^{k+1}$ and $\bar{\mathcal{U}}_i^{k+1}$ are given by

$$\hat{\bar{\mathcal{U}}}_j^{k+1} = \begin{pmatrix}\zeta_{0j}(t) & \zeta_{1j}(t) & \cdots & \zeta_{Nj}(t)\end{pmatrix}^t, \bar{\mathcal{U}}_i^{k+1} = \begin{pmatrix}\zeta_{i0}(t) & \zeta_{i1}(t) & \cdots & \zeta_{iN}(t)\end{pmatrix}^t$$

Thus to determine $\bar{\mathcal{U}}^{k+1}$, we solve two sets of independent one-dimensional tridiagonal system of equations, one in the direction of x and the other in the direction of y.

We will make use of equations (1.2) to compute the initial vectors $\bar{\mathcal{U}}^0$ and $\bar{\mathcal{U}}_t^0$. One can determine the vector $\bar{\mathcal{U}}^1$ at time level $k = 1$, that is $t = \delta t$, from the second order relation,

$$\bar{\mathcal{U}}^1 = \bar{\mathcal{U}}^0 + \delta t\bar{\mathcal{U}}_t^0 + O(\delta t^2).$$

Finally, the solution at any time level k can be computed from the relations (2.22) and (2.23), and the numerical approximation can be acquired by plugging $\zeta_{ij}(t)$'s in the equation (2.2).

3 Unconditional Stability

This section is assigned to examine the unconditional stability behavior of the scheme presented in this article. For that, we have considered the homogeneous part of the differential equation. The matrix stability analysis technique is used here to investigate the stable nature of the given method (2.21).

Theorem : The suggested method given by equation (2.21) is unconditionally stable.

Proof : We rewrite the equation (2.21) as follows

$$\left[(a_0+\alpha\delta t)\mathbb{A}\otimes\mathbb{A}-(a_0+\alpha\delta t)\frac{h^2}{12}\mathbb{A}\otimes\mathbb{B}-\frac{h^2}{12}(a_1+\alpha\delta t)\mathbb{B}\otimes\mathbb{A}+\frac{h^4}{144}(a_1+\alpha\delta t)\mathbb{B}\otimes\mathbb{B}\right]\bar{\mathcal{U}}^{k+1}$$
$$+\left[(a_0-\alpha\delta t)\mathbb{A}\otimes\mathbb{A}-(a_0-\alpha\delta t)\frac{h^2}{12}\mathbb{A}\otimes\mathbb{B}-\frac{h^2}{12}(a_1-\alpha\delta t)\mathbb{B}\otimes\mathbb{A}+\frac{h^4}{144}(a_1-\alpha\delta t)\mathbb{B}\otimes\mathbb{B}\right]\bar{\mathcal{U}}^{k-1}$$
$$-2\left(a_0\mathbb{A}\otimes\mathbb{A}-\frac{h^2}{12}a_0\mathbb{A}\otimes\mathbb{B}-\frac{h^2}{12}a_1\mathbb{B}\otimes\mathbb{A}+\frac{h^4}{144}a_1\mathbb{B}\otimes\mathbb{B}\right)\bar{\mathcal{U}}^{k}$$
$$+\left(\beta^2\delta t^2\mathbb{A}\otimes\mathbb{A}-\delta t^2(1+\frac{\beta^2h^2}{12})(\mathbb{B}\otimes\mathbb{A}+\mathbb{A}\otimes\mathbb{B})+\frac{h^2\delta t^2}{6}\mathbb{B}\otimes\mathbb{B}\right)\bar{\mathcal{U}}^{k}=\delta t^2\mathbb{F}^k \tag{3.1}$$

We can write it as

$$\mathbb{P}_3\bar{\mathcal{U}}^{k+1}+\mathbb{P}_2\bar{\mathcal{U}}^{k}+\mathbb{P}_1\bar{\mathcal{U}}^{k-1}=\delta t^2\mathbb{F}^k \tag{3.2}$$

where $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are given by

$$\mathbb{P}_1=(a_0-\alpha\delta t)(\mathbb{A}\otimes\mathbb{A})-(a_0-\alpha\delta t)\frac{h^2}{12}\mathbb{A}\otimes\mathbb{B}-\frac{h^2}{12}(a_1-\alpha\delta t)\mathbb{B}\otimes\mathbb{A}+\frac{h^4}{144}(a_1-\alpha\delta t)\mathbb{B}\otimes\mathbb{B} \tag{3.3}$$

$$\begin{aligned}\mathbb{P}_2=&-2a_0\mathbb{A}\otimes\mathbb{A}+\frac{h^2}{6}a_0\mathbb{A}\otimes\mathbb{B}+\frac{h^2}{6}a_1\mathbb{B}\otimes\mathbb{A}-\frac{h^4}{72}a_1\mathbb{B}\otimes\mathbb{B}+\beta^2\delta t^2\mathbb{A}\otimes\mathbb{A}\\&-\delta t^2(1+\frac{\beta^2h^2}{12})(\mathbb{B}\otimes\mathbb{A}+\mathbb{A}\otimes\mathbb{B})+\frac{h^2\delta t^2}{6}\mathbb{B}\otimes\mathbb{B},\end{aligned} \tag{3.4}$$

and

$$\mathbb{P}_3=(a_0+\alpha\delta t)\mathbb{A}\otimes\mathbb{A}-(a_0+\alpha\delta t)\frac{h^2}{12}\mathbb{A}\otimes\mathbb{B}-\frac{h^2}{12}(a_1+\alpha\delta t)\mathbb{B}\otimes\mathbb{A}+\frac{h^4}{144}(a_1+\alpha\delta t)\mathbb{B}\otimes\mathbb{B}, \tag{3.5}$$

The scheme (3.2) is a three-time level scheme. We convert this scheme into the two-time level scheme by rewriting it as

$$\begin{pmatrix}\mathbb{P}_3 & 0\\ 0 & \mathbb{I}\end{pmatrix}\begin{pmatrix}\bar{\mathcal{U}}^{k+1}\\ \bar{\mathcal{U}}^{k}\end{pmatrix}=-\begin{pmatrix}\mathbb{P}_2 & \mathbb{P}_1\\ -\mathbb{I} & 0\end{pmatrix}\begin{pmatrix}\bar{\mathcal{U}}^{k}\\ \bar{\mathcal{U}}^{k-1}\end{pmatrix}+\begin{pmatrix}\mathbb{F}^k\\ 0\end{pmatrix} \tag{3.6}$$

Since the matrix $\begin{pmatrix}\mathbb{P}_3 & 0\\ 0 & \mathbb{I}\end{pmatrix}$ is diagonally dominant, it is invertible and so from equation (3.6), we get

$$\begin{aligned}\begin{pmatrix}\bar{\mathcal{U}}^{k+1}\\ \bar{\mathcal{U}}^{k}\end{pmatrix}&=-\begin{pmatrix}\mathbb{P}_3 & 0\\ 0 & \mathbb{I}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{P}_2 & \mathbb{P}_1\\ -\mathbb{I} & 0\end{pmatrix}\begin{pmatrix}\bar{\mathcal{U}}^{k}\\ \bar{\mathcal{U}}^{k-1}\end{pmatrix}+\begin{pmatrix}\mathbb{P}_3 & 0\\ 0 & \mathbb{I}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{F}^k\\ 0\end{pmatrix}\\&=-\begin{pmatrix}\mathbb{P}_3^{-1}\mathbb{P}_2 & \mathbb{P}_3^{-1}\mathbb{P}_1\\ -\mathbb{I} & 0\end{pmatrix}\begin{pmatrix}\bar{\mathcal{U}}^{k}\\ \bar{\mathcal{U}}^{k-1}\end{pmatrix}+\begin{pmatrix}\mathbb{P}_3 & 0\\ 0 & \mathbb{I}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{F}^k\\ 0\end{pmatrix}\end{aligned} \tag{3.7}$$

Now, the matrix $\mathbb{A}$ being diagonally dominant is invertible, so we can write

$$\mathbb{P}_3^{-1}\mathbb{P}_2=\mathbb{P}_3^{-1}(\mathbb{A}\otimes\mathbb{A})(\mathbb{A}\otimes\mathbb{A})^{-1}\mathbb{P}_2, \tag{3.8}$$

and

$$\mathbb{P}_3^{-1}\mathbb{P}_1=\mathbb{P}_3^{-1}(\mathbb{A}\otimes\mathbb{A})(\mathbb{A}\otimes\mathbb{A})^{-1}\mathbb{P}_1 \tag{3.9}$$

Also, from (3.3)-(3.5), we get

$$\begin{aligned}(\mathbb{A}\otimes\mathbb{A})^{-1}\mathbb{P}_3=&(a_0+\alpha\delta t)\mathbb{I}\otimes\mathbb{I}-\frac{h^2}{12}(a_0+\alpha\delta t)\mathbb{I}\otimes\mathbb{A}^{-1}\mathbb{B}-\frac{h^2}{12}(a_1+\alpha\delta t)\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{I}\\&+\frac{h^4}{144}(a_1+\alpha\delta t)\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{A}^{-1}\mathbb{B},\end{aligned} \tag{3.10}$$

$$(\mathbb{A}\otimes\mathbb{A})^{-1}\mathbb{P}_1 =(a_0-\alpha\delta t)\mathbb{I}\otimes\mathbb{I}-\frac{h^2}{12}(a_0-\alpha\delta t)\mathbb{I}\otimes\mathbb{A}^{-1}\mathbb{B}-\frac{h^2}{12}(a_1-\alpha\delta t)\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{I} + \frac{h^4}{144}(a_1-\alpha\delta t)\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{A}^{-1}\mathbb{B}, \tag{3.11}$$

and

$$(\mathbb{A}\otimes\mathbb{A})^{-1}\mathbb{P}_2 =-2a_0\mathbb{I}\otimes\mathbb{I}+\frac{h^2}{6}a_0\mathbb{I}\otimes\mathbb{A}^{-1}\mathbb{B}+\frac{h^2}{6}a_1\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{I}-\frac{h^4}{72}a_1\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{A}^{-1}\mathbb{B} + \beta^2\delta t^2\mathbb{I}\otimes\mathbb{I}-\delta t^2(1+\frac{\beta^2h^2}{12})(\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{I}+\mathbb{I}\otimes\mathbb{A}^{-1}\mathbb{B})+\frac{h^2\delta t^2}{6}\mathbb{A}^{-1}\mathbb{B}\otimes\mathbb{A}^{-1}\mathbb{B} \tag{3.12}$$

Let λ be an eigenvalue of $\mathbb{A}^{-1}\mathbb{B}$, then we take λ_1 and λ_2 as the eigenvalues of the matrices $\mathbb{P}_3^{-1}\mathbb{P}_1$ and $\mathbb{P}_3^{-1}\mathbb{P}_2$ respectively, where λ_1 and λ_2 are such that they have the same set of corresponding linearly independent eigenvectors. We obtain using equations (3.8)-(3.12) that

$$\lambda_1=\frac{(a_0-\alpha\delta t)-\frac{h^2}{12}(a_0-\alpha\delta t+1-\alpha\delta t)\lambda+\frac{h^4}{144}(a_1-\alpha\delta t)\lambda^2}{(a_0+\alpha\delta t)-\frac{h^2}{12}(a_0+\alpha\delta t+1+\alpha\delta t)\lambda+\frac{h^4}{144}(a_1+\alpha\delta t)\lambda^2} \tag{3.13}$$

and

$$\lambda_2=\frac{-2a_0+\frac{h^2}{6}(a_0+a_1)\lambda-\frac{h^4}{72}a_1\lambda^2+\beta^2\delta t^2-2\delta t^2(1+\frac{\beta^2h^2}{12})\lambda+\frac{h^2\delta t^2}{6}\lambda^2}{(a_0+\alpha\delta t)-\frac{h^2}{12}(a_0+\alpha\delta t+1+\alpha\delta t)\lambda+\frac{h^4}{144}(a_1+\alpha\delta t)\lambda^2} \tag{3.14}$$

The characteristic equation of the matrix

$$\begin{pmatrix}\mathbb{P}_3^{-1}\mathbb{P}_2 & \mathbb{P}_3^{-1}\mathbb{P}_1\\ -\mathbb{I} & 0\end{pmatrix}$$

is given by

$$\Lambda^2+\lambda_2\Lambda+\lambda_1=0$$

The eigenvalues of the matrix $\mathbb{A}^{-1}\mathbb{B}$ are negative. Using equations (3.13) and (3.14), for a suitable choice of the parameters μ and ν, it can be easily observed that $1+\lambda_1+\lambda_2, 1+\lambda_1$ and $1+\lambda_1-\lambda_2$ are positive.

Thus, the proposed scheme is unconditionally stable.

4 Numerical Experiments

This section has been assigned to numerically verify the proficiency and preciseness of the suggested method in this article, the accuracy of which has already been shown to be of fourth order. We have used MATLAB for the computational work. To prove the veracity amid the exact solution of the differential equation and the numerical solution two types of errors are considered in this section and they are given by the following formulas:

$$L_\infty \text{ error} = max_{i,j}|u_{i,j}-U_{i,j}|, \text{Root Mean Square error}(RMS)=\frac{1}{N+1}\sqrt{\sum_{i,j=0}^{N}|u_{i,j}-U_{i,j}|^2}$$

The initial conditions and the boundary conditions for all the examples can be obtained from their exact solutions.

Example 1. Consider the differential equation[17]

$$u_{xx}+u_{yy}=u_{tt}+2u_t+u+2-t-x^2-y^2$$

The exact solution is given by: $u(x,y,t)=x^2+y^2+t$.
We have chosen the step size $h=1/10$ and $\delta t=1/1000$ for computation of L_∞ and RMS errors.

Calculation and correlations of L_∞ errors with those listed in [17] are done in Table 2. We have also compared the root mean square errors estimated at distinct time levels with results obtained in [17] and [21] in Table 3. From both tables, one can observe that the results determined by the proposed method are more accurate to the exact solution than those obtained by existing methods. A graphical representation of the numerical solution obtained by the suggested method at time $t = 1$ is laid out in figure 1.

Table 2. Comparisons of L_∞ errors obtained at distinct levels of time for Example 1.

t	1.0	2.0	3.0	5.0	7.0	10.0
Present Method	1.48(-11)	2.27(-11)	3.19(-11)	5.17(-11)	7.06(-11)	9.76(-11)
[17]	1.81(-4)	-	1.46(-4)	1.45(-4)	1.45(-4)	1.45(-4)

Table 3. RMS errors obtained at distinct levels of time for Example 1

t	1.0	2.0	3.0	5.0	7.0	10.0
Present Method	9.21(-12)	1.39(-11)	1.94(-11)	3.16(-11)	4.26(-11)	6.06(-11)
[17]	1.13(-4)	-	9.07(-5)	9.33(-5)	9.29(-5)	9.30(-5)
[21]	8.03(-5)	-	8.89(-5)	9.00(-5)	-	8.93(-5)

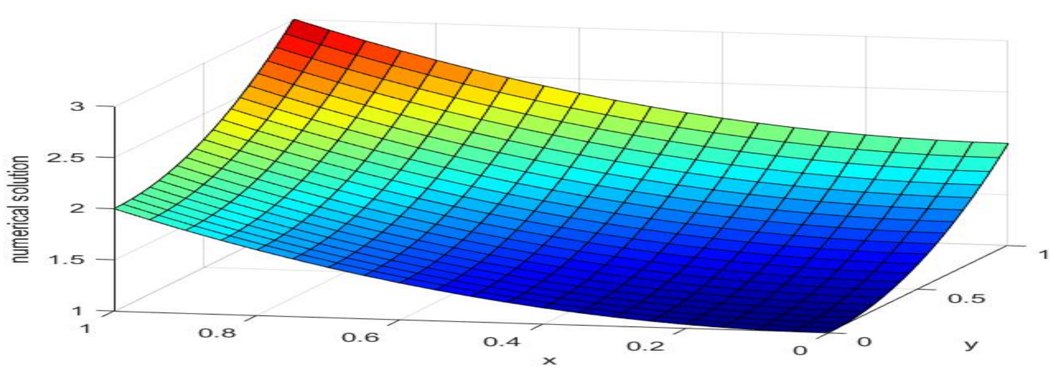

Figure 1. Graph of the numerical solution for example 1

Example 2. Consider the 2D-hyperbolic differential equation [17]

The exact solution is given by: $u(x, y, t) = e^{-t} sinh(x) sinh(y)$.

For $h = 1/10$ and $\delta t = 0.001$, L_∞ the root mean square errors are estimated. In Table 4, we have compared the errors with those obtained by Dehghan and Shokri[17] and the observations listed in [21]. It is very clear that the results calculated by the present technique are more accurate than those calculated in [17] and [21].

Table 4. Errors obtained for Example 2 for $h = 1/10$ and $\delta t = 1/1000$

t	Present Method		[17]		[21]
	L_∞	RMS	L_∞	RMS	RMS
1.0	3.56(- 8)	1.86(-8)	3.84(-5)	1.39(-5)	3.23(-5)
2.0	1.49(-8)	8.20(-9)	1.39(-5)	5.86(-5)	3.12(-5)
3.0	5.64(-9)	3.12(-9)	5.11(-6)	2.21(-5)	3.06(-5)
5.0	7.67(-10)	4.26(-10)	6.92(-7)	3.01(-5)	3.04(-5)

Example 3. Consider the 2D-hyperbolic telegraphic equation [26]

$$u_{xx} + u_{yy} = u_{tt} + 2u_t + u + 2(cost - sint)sinxsiny, \qquad 0 \le x, y \le 1, \ t \ge 0$$

The exact solution is given by : $u(x, y, t) = cos(t)sin(x)sin(y)$.
We have calculated L_∞ errors at distinct time levels by taking time step size $\delta t = 1/100$ and $h = 1/10$ in Table 5 and $\delta t = 0.001$ and $h = 0.05$ in Table 6 respectively. Errors and their comparisons with the errors calculated by Mittal *et al.* in [26] are listed in the respective tables. One can observe that the results estimated by the present method are more precise.
Figure 2 shows the 3-D plot of the numerical estimation computed by using the suggested technique at $t = 2$.

Table 5. L_∞ errors obtained at distinct levels of time for Example 3 for $h = 1/10$ and $\delta t = 1/100$

t	1.0	2.0	3.0	5.0	7.0	10.0
Present Method	5.87(-7)	4.31(-7)	8.60(-8)	4.74(-7)	4.71(-7)	4.23(-7)
CPU time(sec.)	2.00	4.25	2.22	6.08	9.90	38.92
[26]	2.27(-3)	2.87(-3)	6.08(-4)	2.99(-3)	1.87(-3)	1.51(-3)

Table 6. L_∞ errors obtained at distinct levels of time for Example 3 for $h = 1/20$ and $\delta t = 1/1000$

t	1.0	2.0	3.0	5.0	7.0	10.0
Present Method	4.96(-9)	3.70(-10)	5.01(-9)	2.81(-10)	5.06(-9)	5.29(-9)
[26]	2.49(-4)	3.22(-4)	9.93(-5)	3.32(-4)	1.76(-4)	1.35(-4)

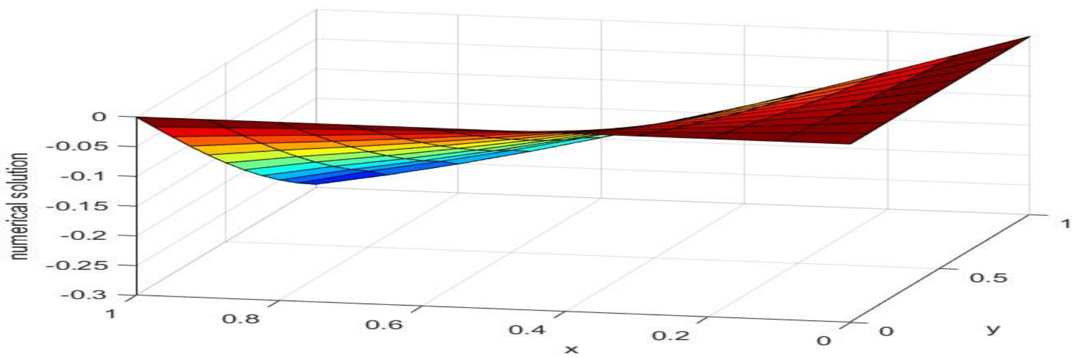

Figure 2. 3-D plot of the numerical estimation for $t = 2$ for example 3

Example 4. Consider the 2D-telegraphic equation [18]

$$u_{xx} + u_{yy} = u_{tt} + 12u_t + u + f, \qquad 0 \le x, y \le 1, \quad t \ge 0$$

where f is the forcing function given by

$$f(x, y, t) = \frac{12}{(1 + x + y + t)} + log(1 + x + y + t) + 1/(1 + x + y + y + t)^2.$$

The exact solution of the equation is : $u(x, y, t) = log(x + y + 1 + t)$.
For this example, the L_∞ errors are listed in Table 7 for $h = 1/10$ and $\delta t = 1/100$. Comparison of the errors are done with the errors obtained by Dehghan and Salehi in [18]. Calculations of errors at different time levels with $h = 1/20$ and $\delta t = 1/1000$ and their comparison with the errors obtained in paper [26] are tabulated in Table 8. From the two tables, it is evident that our method produces more accurate approximations to the exact solution.

Table 7. L_∞ errors obtained for Example 4 for $\delta t = 1/100$ and $h = 1/10$

t	0.5	1.0	2.0	3.0	5.0	7.0	10.0
Present Method	6.29(-6)	2.70(-6)	1.57(-6)	1.35(-7)	8.90(-8)	9.22(-10)	4.42(-11)
[18]	4.51(-4)	1.15(-5)	1.92(-5)	2.86(-5)	-	-	-

Table 8. L_∞ errors obtained for Example 4 for $\delta t = 1/1000$ and $h = 1/20$

t	1.0	2.0	3.0	4.0	5.0
Present Method	4.18(-8)	2.61(-8)	2.43(-9)	1.35(-9)	1.46(-9)
[26]	3.30(-3)	1.13(-3)	4.35(-4)	5.41(-4)	3.48(-4)

Example 5. Consider the differential equation [23]

$$u_{xx} + u_{yy} = u_{tt} + 2\alpha u_t + \beta^2 u + (2\alpha + 1 - \beta^2)e^{-(x+y-t)}, \quad 0 \le x, y \le 1, t \ge 0$$

which for $\alpha = 1$ and $\beta = 1$ reduces to

$$u_{xx} + u_{yy} = u_{tt} + 2u_t + u + 2e^{-(x+y-t)}, \qquad 0 \le x, y \le 1\,, t \ge 0$$

The exact solution of the differential equation is given as $u(x, y, t) = e^{(x+y-t)}$.
In Table 9, L_∞ and RMS errors are calculated at distinct time levels for this experiment. The

errors are computed by taking step size $h = 1/10$ and $\delta t = 1/100$. Comparison of the obtained errors are done with the errors obtained in [23] and are found to be better.

Table 9. Errors obtained for Example 5 with $h = 1/10$ and $\delta t = 1/100$

	Present Method			[23]	
t	L_∞	RMS	CPU time(sec.)	L_∞	RMS
1	2.70(-6)	1.81(-6)	2.67	1.40(-3)	3.50(-3)
2	1.57(-6)	9.05(-7)	4.23	1.30(-3)	2.00(-3)
3	1.35(-7)	6.18(-8)	6.10	2.45(-4)	5.97(-4)
5	8.90(-8)	4.94(-8)	10.84	4.92(-5)	9.56(-5)
7	9.22(-10)	4.28(-10)	14.96	3.81(-6)	1.03(-5)
10	4.42(-11)	2.27(-11)	39.40	2.23(-7)	5.13(-7)

5 Conclusion

During this course of study, we have solved the 2D-hyperbolic telegraphic equation by a bi-cubic B-spline collocation method. It has been found that the method suggested in this article has an accuracy of fourth-order, also it requires a compact stencil which results in reducing the computational work. The discussion on unconditional stability is also discussed in this article. Comparisons of the obtained results with those available in the literature prove the excellency of the given method, in terms of both exactness and computational efficacy. It should be noted that this is a highly accurate method, which is justified by solving many numerical examples.

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Author information

Suruchi Singh, Department of Mathematics, Aditi Mahavidyalaya, University of Delhi, INDIA.
E-mail: `suruchi@aditi.du.ac.in`

Anu Aggarwal, Department of Mathematics, Lakshmibai College, University of Delhi, INDIA.
E-mail: `anujain@lb.du.ac.in`

Swarn Singh, Department of Mathematics, Sri Venkateswara College, University of Delhi, INDIA.
E-mail: `ssingh@svc.ac.in`