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Convergence results for Picard Normal S-iterative algorithm with applications

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Abstract. In this article, we enquire for some weak and strong convergence results for a class of mappings satisfying Condition (E) via Picard Normal S-iterative algorithm (PNSA) in the setting of uniformly convex Banach space (UCBS). Eventually, we furnish an example to substantial our attained findings and numerically compare PNSA with that of some other well-known iterative algorithms. Additionally, we discuss the existence of the solution of a nonlinear functional integral equation as an application of our result.

1 Introduction

Throughout this article, $\mathbb{N}$ stands for a set of natural numbers. Assume that S is self-map defined on nonempty subset U of Banach space $(Y, \|.\|)$ and that $F(S)$ stands for the set of all fixed points of S. The mapping S is called nonexpansive, if

$$\|Su_1 - Su_2\| \le \|u_1 - u_2\|, \quad \text{for all } \ u_1, u_2 \in U.$$

The self-map S on U is quasi-nonexpansive, if $F(S) \neq \emptyset$ and

$$\|Su_1 - u^*\| \le \|u_1 - u^*\|, \quad \text{for all } \ u_1 \in U, \, u^* \in F(S).$$

The theory for the existence of fixed points of nonexpansive mappings, at the outset, discussed by Browder [2], Göhde [6] and Kirk [9], independently. After many researchers have obtained numerous generalizations from their results.

Suzuki [22], in 2008, introduced Suzuki's generalized nonexpansive mapping which is a generalization of nonexpansive mappings.

Definition 1.1. The mapping $S : U \to U$ is known as Suzuki's generalized nonexpansive mapping (SGNM), if

$$\frac{1}{2}\|u_1 - Su_2\| \le \|u_1 - u_2\| \Rightarrow \|Su_1 - Su_2\| \le \|u_1 - u_2\|, \quad \text{for all } u_1, u_2 \in U.$$

García-Falset *et al.* [3], in 2011, extended the class of SGNM and came up with an exciting collection of satisfying Condition (E_μ) which contains the class of SGNM.

Definition 1.2. A self mapping S defined on nonempty subset U of Banach space of Y is called to satisfy condition (E_μ) on U, if there is $\mu \ge 1$ such that

$$\|u_1 - Su_2\| \le \mu\|u_1 - Su_2\| + \|u_1 - u_2\|, \quad \text{for all } u_1, u_2 \in U.$$

If for some $\mu \ge 1$, S satisfies the Condition (E_μ) on U. Then $S : U \to U$ is called to satisfy Condition (E) on U.

In the literature, there are some iterative processes that are used for elucidating fixed points for nonexpansive mappings, SGNM, and mapping satisfying condition (E_μ), $\mu \ge 1$ [15] and generalized nonexpansive mappings via Mann [11], Ishikawa [7], Noor [13], Garodia [4, 5], generalized F-iterative [16] iteration processes. Some authors implemented previously defined iterative algorithms in different spaces successfully (see [25, 26]) and some of the applications of iterative algorithms are found in [10, 18, 20, 23, 24, 27].

Agarwal *et al.* [1] introduced S-iteration algorithm as follows:

$$\begin{cases} t_1 \in U \\ t_{m+1} = (1-\alpha_m)St_m + \alpha_m Su_m \\ u_m = (1-\beta_m)t_m + \beta_m St_m, \quad \forall m \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are sequences in $(0,1)$.

Sahu [17] introduced normal S-iteration algorithm (NSA, in brief) as follows:

$$\begin{cases} t_1 \in U \\ t_{m+1} = Su_m \\ u_m = (1-\alpha_m)t_m + \alpha_m St_m, \quad \forall m \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_m\}$ is sequence in $(0,1)$.

In 2014, Kadioglu and Yildirim [8] introduced the Picard normal S-iterative algorithm (PNSA, in brief) as follows:

$$\begin{cases} t_1 \in U \\ t_{m+1} = Su_m \\ u_m = (1-\alpha_m)v_m + \alpha_m Sv_m \\ v_m = (1-\beta_m)t_m + \beta_m St_m, \quad \forall m \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are real sequences in $(0,1)$.

This research article is structured into seven sections. In Section 2, we collect some basic definitions and results which are playing key roles in this manuscript. In Section 3, we establish the strong and weak convergence results utilizing PNSA (1.3) for a mapping satisfying Condition (E) in UCBS and an example designed for such mapping is presented in Section 4. The comparison of the convergence behaviour of PNSA (1.3) with some known aforementioned iterative algorithms is presented in Section 5 and we establish existence results for solutions of the nonlinear functional integral equation as an application of our result in Section 6. The Section 7 summarizes this paper in a form of a conclusion.

2 Preliminaries

In this section, we give some essential definitions and results which help us to establish our main results. Assume that U is a nonempty, closed and convex subset of Banach space Y. For bounded sequence $\{t_m\}$ in Y, denote

- $r(t,\{t_m\}) = \limsup_{m\to\infty} \|t - t_m\|$;
- asymptotic radius of $\{t_m\}$ with respect to U by $r(U,\{t_m\}) = \inf\{r(t,\{t_m\}) : t \in U\}$;
- asymptotic center of $\{t_m\}$ with respect to U by $A(U,\{t_m\}) = \{t \in U : r(t,\{t_m\}) = r(U,\{t_m\})$.

Definition 2.1. [21] Assume that S is self-mapping defined on nonempty subset U of Banach space Y. A sequence $\{t_m\}$ in D is said to be approximate fixed point sequence (A.F.P.S.), if $\lim\limits_{m\to\infty} ||St_m - t_m|| = 0$.

Proposition 2.2. *[3] Assume that S is a function satisfying the Condition (E) defined on nonempty subset U of Banach space Y and $F(S) \neq \emptyset$, then S is quasi-nonexpansive.*

Theorem 2.3. *[3] Assume that S is a function satisfying the Condition (E) defined on compact subset U of Banach space Y, then U admits A.F.P.S. if and only if S has fixed point in U.*

Opial [14] gave the condition termed as Opial's condition which is useful in the investigation of demiclosedness principle of nonlinear mappings.

Definition 2.4. A Banach space Y is said to satisfy the Opial's condition, if for any sequence $\{t_m\}$ with $t_m \rightharpoonup t^*$ in Y such that

$$\liminf_{m\to\infty} \|t_m - t^*\| < \liminf_{m\to\infty} \|t_m - u\|$$

for all $u \in Y$ with $t^* \neq u$.

Example 2.5. The space ℓ^p $(1 < p < \infty)$ and Hilbert space satisfy the Opial's condition, but $\mathcal{L}^p[0, 2\pi]$, $(1 < p \neq 2)$ does not satisfy the Opial's condition.

Lemma 2.6. *[19, Theorem 2.3.13] Suppose that Y is a UCBS and $\{a_m\}$ is sequence in $[\theta, 1-\theta]$ for $\theta \in (0,1)$. The sequences $\{t_m\}$ and $\{u_m\}$ in Y are such that $\limsup_{m\to\infty} ||t_m - t^*|| \leq l$, $\limsup_{m\to\infty} ||u_m - t^*|| \leq l$, and $\limsup_{m\to\infty} ||a_m(t_m - t^*) + (1-a_m)(u_m - t^*)|| = l$ for some $l \geq 0$ and $t^* \in Y$. Then $\lim_{m\to\infty} ||t_m - u_m|| = 0$.*

3 Main Results

In the following section, we present strong and weak convergence results for a sequence $\{t_m\}$ generated by PNSA (1.3).

Theorem 3.1. *Suppose that S is mapping satisfying Condition (E) defined on convex and closed subset U of uniformly convex Banach space Y. Assume that $\{t_m\}$ is a sequence generated by PNSA (1.3) and $u^* \in F(S)$. Then $\lim_{m\to\infty} ||t_m - u^*||$ exists.*

Proof. Assume that $m \in \mathbb{N}$, Using Proposition 2.2 and (1.3),

$$\begin{aligned} ||v_m - u^*|| &\leq (1-\beta_m)||t_m - u^*|| + \beta_m||St_m - u^*|| \\ &\leq (1-\beta_m)||t_m - u^*|| + \beta_m||t_m - u^*|| \\ &\leq ||t_m - u^*||. \end{aligned} \tag{3.1}$$

Using Proposition 2.2, (1.3) and (3.1),

$$\begin{aligned} ||u_m - u^*|| &\leq (1-\alpha_m)||v_m - u^*|| + \alpha_m||Sv_m - u^*|| \\ &\leq (1-\alpha_m)||v_m - u^*|| + \alpha_m||v_m - u^*|| \\ &\leq ||v_m - u^*|| \\ &\leq ||t_m - u^*||. \end{aligned} \tag{3.2}$$

Using Proposition 2.2, (1.3), (3.1) and (3.2),

$$\begin{aligned} ||t_{m+1} - u^*|| &= ||Su_m - u^*|| \\ &\leq ||u_m - u^*|| \\ &\leq ||v_m - u^*|| \\ &\leq ||t_m - u^*||. \end{aligned} \tag{3.3}$$

Now, from (3.1), (3.2) and (3.3), we get

$$\max\Big\{||t_{m+1} - u^*||, ||u_m - u^*||, ||v_m - u^*||\Big\} \leq ||t_m - u^*||.$$

The inequality (3.3) shows that $\{||t_m - u^*||\}$ is non-increasing monotonic sequence and hence $\{||t_m - u^*||\}$ is bounded sequence and therefore $\lim_{m\to\infty} ||t_m - u^*||$ exists. □

Theorem 3.2. *Suppose that S is mapping satisfying Condition (E) defined on convex and closed subset U of uniformly convex Banach space Y. Assume that $\{t_m\}$ is a sequence generated by PNSA (1.3) with $t_1 \in U$. Then $\{t_m\}$ is bounded and $\lim_{m\to\infty} ||St_m - t_m|| = 0$ if and only if $F(S)$ is nonempty.*

Proof. Suppose that $\{t_m\}$ is bounded and $\lim_{m\to\infty} ||St_m - t_m|| = 0$. We claim that $F(S) \neq \emptyset$. Assume that $u^* \in A(U, \{t_m\})$. Then

$$r(Su^*, \{t_m\}) = \limsup_{m\to\infty} ||t_m - Su^*||.$$

Now, since $S : U \to U$ satisfies the Condition (E), therefore

$$\begin{aligned} r(Su^*, \{t_m\}) &= \limsup_{m\to\infty} ||t_m - Su^*|| \\ &\leq \mu \limsup_{m\to\infty} ||St_m - t_m|| + \limsup_{m\to\infty} ||t_m - u^*|| \\ &= r(u^*, \{t_m\}). \end{aligned} \tag{3.4}$$

Since asymptotic center of the sequence $\{t_m\}$ is unique, therefore, by (3.4),

$$Su^* = u^*,$$

which shows that $u^* \in F(S)$ and hence $F(S) \neq \emptyset$.
For converse part, assume that $F(S) \neq \emptyset$ and we will prove that $\{t_m\}$ is bounded and $\lim_{m\to\infty} ||St_m - t_m|| = 0$. Assume that $u^* \in F(S)$ because $F(S) \neq \emptyset$. Then by Theorem 3.1, $\lim_{m\to\infty} ||t_m - u^*||$ exists. Assume that

$$\lim_{m\to\infty} ||t_m - u^*|| = l. \tag{3.5}$$

From Proposition 2.2 and (3.3),

$$\limsup_{m\to\infty} ||St_m - u^*|| \leq l. \tag{3.6}$$

From (3.1) and (3.3),

$$\limsup_{m\to\infty} ||v_m - u^*|| \leq \lim_{m\to\infty} ||t_m - u^*|| = l. \tag{3.7}$$

From Proposition 2.2 and (3.7),

$$\limsup_{m\to\infty} ||Sv_m - u^*|| \leq l. \tag{3.8}$$

From (3.2) and (3.3),

$$\limsup_{m\to\infty} ||u_m - u^*|| \leq \lim_{m\to\infty} ||t_m - u^*|| = l. \tag{3.9}$$

From Proposition 2.2 and (3.9),

$$\limsup_{m\to\infty} ||Su_m - u^*|| \leq l. \tag{3.10}$$

Now, from (1.3) and (3.9),

$$\begin{aligned} \limsup_{m\to\infty} ||(1-\alpha_m)(v_m - u^*) + \alpha_m(Sv_m - u^*)|| &= \limsup_{m\to\infty} ||(1-\alpha_m)v_m + \alpha_m Sv_m - u^*|| \\ &\leq \limsup_{m\to\infty} ||u_m - u^*|| \\ &\leq l. \end{aligned} \tag{3.11}$$

From (1.3) and (3.3),

$$\begin{aligned} \liminf_{m\to\infty} ||t_{m+1} - u^*|| &= \liminf_{m\to\infty} ||Su_m - u^*|| \\ &\leq \liminf_{m\to\infty} ||u_m - u^*|| \\ &= \liminf_{m\to\infty} ||(1-\alpha_m)v_m + \alpha_n Sv_m - u^*|| \\ &= \liminf_{m\to\infty} ||(1-\alpha_m)(v_m - u^*) + \alpha_m(Sv_m - u^*)||, \end{aligned}$$

and hence

$$l \leq \liminf_{m\to\infty} ||(1-\alpha_m)(v_m - u^*) + \alpha_m(Sv_m - u^*)||. \tag{3.12}$$

From (3.11) and (3.12),

$$\lim_{m\to\infty} ||(1-\alpha_m)(v_m - u^*) + \alpha_m(Sv_m - u^*)|| = l. \tag{3.13}$$

Now, (3.7), (3.8), (3.13) and Lemma 2.6 provides

$$\lim_{m\to\infty} ||Sv_m - v_m|| = 0.$$

Now, from (1.3) and (3.3),

$$\begin{aligned}\limsup_{m\to\infty} ||(1-\beta_m)(t_m - u^*) + \beta_m(St_m - u^*)|| &= \limsup_{m\to\infty} ||(1-\beta_m)t_m + \beta t_m St_m - u^*|| \\ &\leq \limsup_{m\to\infty} ||t_m - u^*|| \\ &\leq l. \end{aligned} \tag{3.14}$$

From (1.3) and (3.3),

$$\begin{aligned}\liminf_{m\to\infty} ||t_{m+1} - u^*|| &= \liminf_{m\to\infty} ||Su_m - u^*|| \\ &\leq \liminf_{m\to\infty} ||u_m - u^*|| \\ &\leq \liminf_{m\to\infty} ||v_m - u^*|| \\ &= \liminf_{m\to\infty} ||(1-\beta_m)t_m + \beta_m St_m - u^*|| \\ &= \liminf_{m\to\infty} ||(1-\beta_m)(t_m - u^*) + \beta_m(St_m - u^*)||,\end{aligned}$$

and hence,

$$l \leq \liminf_{m\to\infty} ||(1-\beta_m)(t_m - u^*) + \beta_m(St_m - u^*)||. \tag{3.15}$$

From (3.14) and (3.15),

$$\lim_{m\to\infty} ||(1-\beta_m)(t_m - u^*) + \beta_m(St_m - u^*)|| = l. \tag{3.16}$$

Now, from (3.3), (3.6), (3.16) and Lemma 2.6, we have $\lim_{m\to\infty} ||St_m - t_m|| = 0$. □

The following Theorem presents the weak convergence result for a sequence $\{t_m\}$ generated by PNSA (1.3) using Opial's property.

Theorem 3.3. *Suppose that S is mapping satisfying Condition (E) defined on convex and closed subset U of uniformly convex Banach space Y. Assume that $\{t_m\}$ is a sequence generated by PNSA (1.3). Suppose that $F(S) \neq \emptyset$ and Y satisfies Opial's property. Then sequence $\{t_m\}$ generated by (1.3) weakly converges to element of $F(S)$.*

Proof. We have $\lim_{m\to\infty} ||St_m - t_m|| = 0$ and sequence $\{t_m\}$ generated by PNSA (1.3) is bounded, due to Theorem 3.1, therefore Y is reflexive and it implies that there is subsequence $\{t_{m_k}\}$ of $\{t_m\}$ such that $\{t_{m_k}\}$ weakly converges to some $u^* \in U$. Now, due to Opial's property, the sequence $\{t_m\}$ weakly converges to $u^* \in U$. □

The following Theorem presents the strong convergence result for sequence $\{t_m\}$ generated by PNSA (1.3) using Opial's property.

Theorem 3.4. *Suppose that S is mapping satisfying Condition (E) defined on convex and closed subset U of uniformly convex Banach space Y. Assume that $t_1 \in U$. Also assume that $F(S) \neq \emptyset$ and closed. Then the sequence $\{t_m\}$ generated by PNSA (1.3) strongly converges to element of $F(S)$, if $\liminf_{m\to\infty} d(t_m, F(S)) = 0$, where $d(u^*, F(S))$ represents the distance of u^* from the set $F(S)$.*

Proof. Assume that $\liminf_{m\to\infty} d(t_m, F(S)) = 0$. Then there is a subsequence $\{u_m\}$ of $\{t_m\}$ such that

$$\liminf_{m\to\infty} d(u_m, F(S)) = 0.$$

Assume that $\{u_{m_k}\}$ is subsequence of $\{u_m\}$ such that $||u_{m_k} - v_k|| \leq \frac{1}{2^k}$ $\forall\, k \geq 1$, where $\{v_k\}$ is sequence of fixed points of mapping S. Now, by Theorem 3.1,

$$||u_{m_{k+1}} - v_k|| \leq ||u_{m_k} - v_k|| \leq \frac{1}{2^k}.$$

Now from (3), we set

$$||v_{k+1} - v_k|| \leq ||v_{k+1} - u_{m_{k+1}}|| + ||u_{m_{k+1}} - v_k|| \leq \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}$$

which ensure that $\{v_k\}$ is Cauchy sequence in $F(S)$. Now, since $F(S)$ is closed and $\{v_k\}$ converges to some fixed point of mapping S, say $v \in F(S)$. Therefore,

$$||u_{m_k} - v|| \leq ||u_{m_k} - v_k|| + ||v_k - v||,$$

as $m \to \infty$, $\{u_{m_k}\}$ strongly converges to $v \in F(S)$ and from Theorem 3.1, $\lim_{m\to\infty} ||t_m - v||$ exists and consequently $\{u_m\}$ strongly converges to $v \in F(S)$. □

The following Theorem presents the strong convergence result for sequence $\{t_m\}$ generated by PNSA (1.3) using Condition (I).

Theorem 3.5. *Suppose that S is mapping satisfying Condition (E) and Condition (I) defined on convex and closed subset U of uniformly convex Banach space Y. Assume that $t_1 \in U$. Assume that $t_1 \in U$. Also, assume that $F(S) \neq \emptyset$ and closed. Then the sequence $\{t_m\}$ generated by PNSA (1.3) strongly converges to element of $F(S)$.*

Proof. Since S satisfies the Condition (I), therefore

$$||t_m - St_m|| \geq g(d(t_m, F(S))). \tag{3.17}$$

Due to Theorem 3.1, we have

$$\liminf_{m\to\infty} ||St_m - t_m|| = 0. \tag{3.18}$$

From (3.17) and (3.18),

$$\liminf_{m\to\infty} g(d(t_m, F(S))) = 0.$$

By the property of function $g : [0, \infty] \to [0, \infty]$,

$$\liminf_{m\to\infty} d(t_m, F(S)) = 0.$$

Now, due to Theorem 3.4, the sequence $\{t_m\}$ strongly converges to fixed point of mapping S. □

4 Numerical Example

Example 4.1. Assume that $U = [-1, 2] \subseteq \mathbb{R}$ with usual norm. The mapping $S : U \to U$ is given by

$$Su = \begin{cases} -\frac{u}{7}, & \text{if } u \in [-1, 0) \\ -u, & \text{if } u \in [0, 1]\backslash\{\frac{1}{7}\} \\ 0, & \text{if } u = \frac{1}{7}. \end{cases}$$

If $u_1 = \frac{1}{7}$ and $u_2 = 1$, then
$\frac{1}{2}||u_1 - Su_1|| = \frac{1}{14}$ and $||u_1 - u_2|| = \frac{6}{7}$, therefore

$$\frac{1}{2}||u_1 - Su_1|| < ||u_1 - u_2||.$$

Here $||Su_1 - Su_2|| = 1$, therefore

$$||Su_1 - Su_2|| > ||u_1 - u_2||,$$

which shows that S is not SGNM. We will prove that S is a function satisfying Condition (E). For this, we can consider the following cases:
Case I : If $u_1, u_2 \in [-1, 0)$,

$$\begin{aligned} ||u_1 - Su_2|| &\leq ||u_1 - Su_1|| + ||Su_1 - Su_2|| \\ &= ||u_1 - Su_1|| + \frac{1}{7}||u_1 - u_2|| \\ &\leq ||u_1 - Su_1|| + ||u_1 - u_2||. \end{aligned}$$

Case II : If $u_1, u_2 \in [0, 1]\backslash\{\frac{1}{7}\}$,

$$\begin{aligned} ||u_1 - Su_2|| &\leq ||u_1 - Su_1|| + ||Su_1 - Su_2|| \\ &= ||u_1 - Su_1|| + ||u_1 - u_2||. \end{aligned}$$

Case III : If $u_1 \in [-1, 0)$ and $u_2 = \frac{1}{7}$,

$$\begin{aligned} ||u_1 - Su_2|| &= ||u_1|| \leq \frac{8}{7}||u_2|| + ||u_2 - \frac{1}{7}|| \\ &= ||u_1 - Su_1|| + ||u_1 - u_2||. \end{aligned}$$

Case IV : If $u_1 \in [-1, 0)$ and $u_2 \in [0, 1]\backslash\{\frac{1}{7}\}$,

$$\begin{aligned} ||u_1 - Su_2|| &= ||u_1 + u_2|| \leq ||u_1|| + ||u_2|| \\ &\leq \frac{8}{7}||u_1|| + ||u_1 - u_2|| \\ &= ||u_1 - Su_1|| + ||u_1 - u_2||. \end{aligned}$$

Case V : If u_1‘ $\in [0, 1]\backslash\{\frac{1}{7}\}$ and $u_2 = \frac{1}{7}$,

$$\begin{aligned} ||u_1 - Su_2|| &= ||u_1|| \leq 2||u_1|| + ||u_1 - \frac{1}{7}|| \\ &= ||u_1 - Su_1|| + ||u_1 - u_2||. \end{aligned}$$

Therefore the mapping S satisfies condition (E) and its fixed point is 0.

5 Numerical Results

In this section, a comparison of the convergence behaviour of PNSA (1.3) with NSA (1.2), S-iterative algorithm (1.1) and Picard iterative algorithm for a mapping satisfying Condition (E) defined in Example 4.1 is presented. We select the different set of parameters of α_m, β_m, γ_m and stopping criteria $||t_m - u^*|| \leq 10^{-11}$. The influence of initial values of PNSA (1.3), NSA (1.2), S-iterative algorithm (1.1) and Picard iterative algorithm is examined in Table 1 using $\alpha_m = \frac{2m}{3m+21}$, $\beta_m = \frac{3m}{4m+51}$.

Observations: Here one can note that it is exhibited in Table 1, Table 2 and Figure 1, PNSA (1.3) is faster than NSA (1.2), S-iterative algorithm (1.1) and Picard iterative algorithm for a different set of parameters and initial values for a mapping satisfying Condition (E) defined in Example 4.1.

Table 1. Comparison of number of iterations of PNSA (1.3) with other iterative algorithms for Example 4.1

Initial Value	**PNSA**	**NSA**	**S-iteration**	**Picard**
-1	11	13	19	23
-0.8	11	13	19	23
-0.6	11	13	18	22
-0.4	11	13	18	22
-0.2	10	12	18	22
0.2	11	13	19	23
0.4	11	13	19	23
0.6	11	13	19	23
0.8	11	13	19	24
1	11	13	19	24

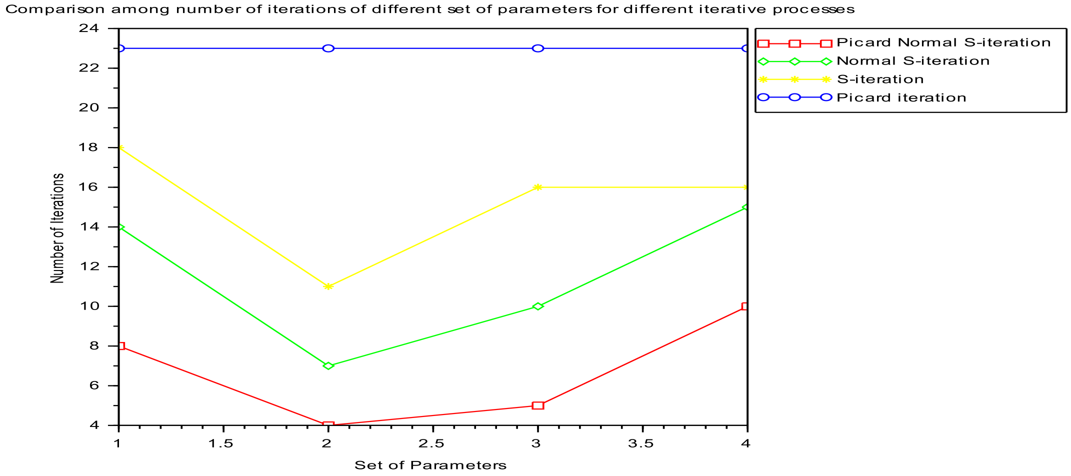

Figure 1. Comparison among number of iterations of PNSA (1.3) for initial value 0.4 for Example 4.1

Table 2. Comparison of number of iterations of PNSA (1.3) with other iterative algorithms for different parameters for Example 4.1

	Initial Values									
Iteration	-1	-0.8	-0.6	-0.4	-0.2	0.2	0.4	0.6	0.8	1
For $\alpha_m = \frac{m^2}{2m^2+71}$, $\beta_m = \frac{m^3}{3m^3+11}$										
PNSA	8	8	8	8	8	8	8	8	8	8
NSA	14	14	14	14	13	14	14	14	14	14
S-iteration	18	18	18	17	17	17	18	18	19	19
Picard	23	23	22	22	22	23	23	23	24	24
For $\alpha_m = \frac{1}{(3m+7)^{\frac{1}{8}}}$, $\beta_n = \frac{1}{(7m+2)^{\frac{1}{9}}}$										
PNSA	4	4	4	4	4	4	4	5	5	5
NSA	7	7	7	7	7	7	7	7	8	8
S-iteration	10	10	9	9	9	10	11	11	11	11
Picard	23	23	22	22	22	23	23	23	24	24
For $\alpha_m = \frac{4m}{7m+11}$, $\beta_m = \frac{m}{3m+55}$										
PNSA	5	5	5	5	5	5	5	6	6	6
NSA	10	10	9	9	9	10	10	10	10	10
S-iteration	16	16	16	15	15	15	16	16	17	17
Picard	23	23	22	22	22	23	23	23	24	24
For $\alpha_m = \frac{3m}{6m-1}$, $\beta_m = \frac{1}{(2m+7)^{\frac{2}{13}}}$										
PNSA	10	10	10	10	10	10	10	11	11	11
NSA	16	16	16	15	15	15	15	15	16	16
S-iteration	17	17	17	17	16	16	16	16	16	16
Picard	23	23	22	22	22	23	23	23	24	24

6 Application

In the following section, we discuss the application to substantiate one of our obtained results. Consider $C[0,1]$ as space of continuous functions defined on $\mathcal{E} = [0,1] \subset \mathbb{R}_+$ and nonlinear functional integral equation (FIE, in brief)

$$\varphi(u) = f(u) + \lambda_1 \int_0^u \kappa(u,s) g(s,\varphi(s))ds + \lambda_2 \int_0^1 \overline{\kappa}(u,s)\overline{g}(s,\varphi(s))ds \tag{6.1}$$

for all $u, s \in \mathcal{E}$, λ_1, λ_2 are positive constants and $f : \mathcal{E} \longrightarrow \mathbb{R}_+$, $g, \overline{g} : \mathcal{E} \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $\kappa, \overline{\kappa} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{R}_+$. We have the following conditions.

($\mathbf{K_1}$) The function $f : \mathcal{E} \longrightarrow J$ is continuous.

($\mathbf{K_2}$) The functions $\kappa, \overline{\kappa} : \mathcal{E} \times J \longrightarrow \mathbb{R}_+$ are continuous such that for all $u, s \in \mathcal{E}$

$$\int_0^u \kappa(u,s)ds \le K_1 \qquad \text{and} \qquad \int_0^1 \overline{\kappa}(u,s)ds \le K_2.$$

($\mathbf{K_3}$) The functions $g, \overline{g} : \mathcal{E} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and there are two constants L_1, L_2 such that for all $u \in \mathcal{E}; \varphi_1, \varphi_2 \in C$,

$$|g(u,\varphi_1) - g(u,\varphi_2) \le L_1|\varphi_1(u) - \varphi_2(u)|$$
$$|\overline{g}(u,\varphi_1) - \overline{g}(u,\varphi_2) \le L_2|\varphi_1(u) - \varphi_2(u)|$$

($\mathbf{K_4}$) $\lambda_1 L_1 K_1 + \lambda_2 L_2 K_2 = 1$.

The following result represents the existence of the solution of FIE (6.1).

Theorem 6.1. *Assume that U is compact subset of $Y = C[0,1]$, where supremum norm is defined by $||\varphi_1 - \varphi_2|| = \sup_{u\in\mathcal{E}} |\varphi_1(u) - \varphi_2(u)|$ and assumptions from ($\mathbf{K_1}$) to ($\mathbf{K_4}$) are true. The mapping $S : U \to U$ is defined by*

$$S\phi(u) = f(u) + \lambda_1 \int_0^u \kappa(u,s)g(s,\phi(s))ds + \lambda_2 \int_0^1 \overline{\kappa}(u,s)\overline{g}(s,\phi(s))ds.$$

Then S admits A.F.P.S. if and only if nonlinear FIE (6.1) has solution in Y.

Proof. Assume that $\varphi_1, \varphi_2 \in U$. Then

$$\begin{aligned}
|\varphi_1(u) - S\varphi_2(u)| &= \Big|\varphi_1(u) - \Big(f(u) + \lambda_1 \int_0^u \kappa(u,s)g(s,\varphi_1(s))ds + \lambda_2 \int_0^1 \overline{\kappa}(u,s)\overline{g}(s,\varphi_1(s))ds\Big) \\
&\quad + \Big(f(u) + \lambda_1 \int_0^u \kappa(u,s)g(s,\varphi_1(s))ds + \lambda_2 \int_0^1 \overline{\kappa}(u,s)\overline{g}(s,\varphi_1(s))ds \\
&\quad - f(u) - \lambda_1 \int_0^t \kappa(u,s)g(s,\varphi_2(s))ds + \lambda_2 \int_0^1 \overline{\kappa}(u,s)\overline{g}(s,\varphi_2(s))ds\Big)\Big| \\
&\le |\varphi_1(u) - S\varphi_1(u)| + \lambda_1 \int_0^u \kappa(u,s)\big|g(s,\varphi_1(s)) - g(s,\varphi_2(s))\big|ds \\
&\quad + \lambda_2 \int_0^1 \overline{\kappa}(u,s)\big|\overline{g}(s,\varphi_1(s)) - \overline{g}(s,\varphi_2(s))\big|ds \\
&\le |\varphi_1(u) - S\varphi_1(u)| + \lambda_1 \int_0^u \kappa(u,s)L_1\big|\varphi_1(s) - \varphi_2(s)\big|ds \\
&\quad + \lambda_2 \int_0^1 \overline{\kappa}(u,s)L_2\big|\varphi_1(s) - \varphi_2(s)\big|ds.
\end{aligned}$$

On taking supremum of both the sides, we get

$$\begin{aligned}
||\varphi_1 - S\varphi_2|| &\le ||\varphi_1 - S\varphi_1|| + (\lambda_1 K_1 L_1 + \lambda_2 K_2 L_2)||\varphi_1 - \varphi_2|| \\
&= ||\varphi_1 - S\varphi_1|| + ||\varphi_1 - \varphi_2||.
\end{aligned}$$

This shows that the S is mapping satisfying the Condition (E) on U with $\mu = 1$. By Theorem 2.3, nonlinear FIE (6.1) has solution in U. □

Corollary 6.2. *Assume that U is compact subset of $Y = C[0,1]$, where supremum norm is defined by $||\varphi_1 - \varphi_2|| = \sup_{u\in\mathcal{E}} |\varphi_1(u) - \varphi_2(u)|$ and assumptions from* ($\mathbf{K_1}$) *to* ($\mathbf{K_4}$) *are true. The mapping $S : U \to U$ is defined by*

$$S\varphi(u) = f(u) + \lambda_1 \int_0^u \kappa(u,s) g(s,\varphi(s)) ds.$$

Then the nonlinear FIE

$$\varphi(u) = f(u) + \lambda_1 \int_0^u \kappa(u,s) g(s,\phi(s)) ds$$

has solution in Y if and only if S admits A.F.P.S.

Proof. On setting $\overline{\kappa}(t,s) \equiv 0$, we get the desired result. □

The following Corollary is the result of Pandey *et al.* [15, Theorem 6.1]

Corollary 6.3. *Assume that U is compact subset of $Y = C[0,1]$, where supremum norm is defined by $||\varphi_1 - \varphi_2|| = \sup_{u\in\mathcal{E}} |\varphi_1(u) - \varphi_2(u)|$ and assumptions from* ($\mathbf{K_1}$) *to* ($\mathbf{K_4}$) *are true. The mapping $S : U \to U$ is defined by*

$$S\varphi(u) = f(u) + \lambda_2 \int_0^1 \overline{\kappa}(u,s) \overline{g}(s,\varphi(s)) ds.$$

Then the nonlinear FIE

$$\varphi(u) = f(u) + \lambda_2 \int_0^1 \overline{\kappa}(u,s) \overline{g}(s,\varphi(s)) ds$$

has solution in Y if and only if S admits A.F.P.S.

Proof. On setting $\kappa(t,s) \equiv 0$, we get the desired result. □

Example 6.4. Let us consider the following nonlinear FIE : For $u, s \in [0,1]$,

$$\phi(u) = (u^2+4) + \frac{3}{20}\int_0^u \left[(u^3+1)(2s+4)\right]\frac{|\varphi(s)|}{3} ds + \frac{5}{12}\int_0^1 \left[u^2(2s+5)\right]\frac{|\varphi(s)|}{5} ds. \tag{6.2}$$

If we take

$$\lambda_1 = \frac{3}{20},\ \lambda_2 = \frac{5}{12},\ f(u) = u^2+4;$$

$$\kappa(u,s) = (u^3+1)(2s+4), \qquad \overline{\kappa}(u,s) = u^2(2s+5);$$

$$g(s,\varphi(s)) = \frac{|\varphi(s)|}{3} \qquad \overline{g}(s,\varphi(s)) = \frac{|\varphi(s)|}{5}.$$

Then nonlinear FIE (6.2) will be in form of (6.1).
It is clear that function $f(u) = u^2 + 4, \forall u \in [0,1]$ is continuous.
For each $u, s \in [0,1]$,

$$\int_0^1 \kappa(u,s) = \int_0^1 (u^3+1)(2s+4) ds \le 10;$$

$$\int_0^1 \overline{\kappa}(u,s) = \int_0^u u^2(2s+5) ds \le 6.$$

For $\varphi_1, \varphi_2 \in U; s \in [0,1]$,

$$\begin{aligned} \big|g(s,\varphi_1(s)) - g(s,\varphi_2(s))\big| &= \Big|\frac{|\varphi_1(s)|}{3} - \frac{|\varphi_2(s)|}{3}\Big| \\ &= \frac{1}{3}\Big||\varphi_1(s)| - |\varphi_2(s)|\Big| \\ &\le \frac{1}{3}\Big|\varphi_1(s) - \varphi_2(s)\Big| \end{aligned}$$

$$\begin{aligned} \big|\overline{g}(s,\varphi_1(s)) - \overline{g}(s,\varphi_2(s))\big| &= \Big|\frac{|\varphi_1(s)|}{5} - \frac{|\varphi_2(s)|}{5}\Big| \\ &= \frac{1}{5}\Big||\varphi_1(s)| - |\varphi_2(s)|\Big| \\ &\le \frac{1}{5}\Big|\varphi_1(s) - \varphi_2(s)\Big| \end{aligned}$$

Since all assumptions of Theorem 6.1 are satisfied with $K_1 = 10, K_2 = 6$ and $\lambda_1 K_1 L_1 + \lambda_2 K_2 L_2 = 1$. Therefore nonlinear FIE (6.2) has a solution.

7 Conclusion

It concludes that we have established convergence theorems for mapping satisfying Condition (E) via PNSA (1.3) in UCBS. Further, a comparison of the rate of convergence of PNSA (1.3) for such mappings is done and it is observed that PNSA (1.3) is faster, numerically, than well-known iteration processes such as Picard iterative algorithm, NSA (1.2) and S-iterative algorithm (1.1). As existence results for nonlinear FIE discussed in Pandey *et al.* [15], so one can note that our obtained result improves those of due to Pandey *et al.* [15].

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