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Numerical Solution of Falkner-Skan equation using non-polynomial Quartic Spline Technique

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Abstract In this work, we investigate the boundary layer flow suggested by the Falkner-Skan equation for different flow parameters. Using a stream function, the Falkner-Skan equation has been converted into a strongly nonlinear third order ordinary differential equation. To solve the obtained differential equation, we construct an efficient numerical algorithm based on nonpolynomial quartic spline function. Skin friction co-efficent has also been calculated. Finally, to validate the theoretical results obtained and to show the applicability of our constructed algorithm, we have carry out numerical experiments for various parameters with the help of graphs and tables.

1 Introduction

The boundary layer theory has been described by two groups of scholars: Falkner-Skan and Prandtl. The main difference between these two theories is due the region of flow considered. When the boundary layer flow has been considered away the boundary then it becomes Falkner-Skan boundary layer flow. While, when the boundary layer flow has been studied in the vicinity of boundary then it is called Prandtl boundary layer flow [1]. The Falkner-Skan equation is probably the most well-known nonlinear differential equations in fluid mechanics. For the Falkner-Skan problem, Weyl [2] attempted to establish an existence and uniqueness hypothesis, which Coppel [3] expanded to a more general situation and then demonstrated solution which is for exclusive specific values of certain parameters. The Falkner-Skan equation results through a similarity reduction of nonlinear partial differential equations describing boundary layer flow on a flat plate with static velocity in the direction perpendicular to a uniform mainstream. For Falkner-Skan problem, Liao [4] presented the method of homotopy analysis, which produces a convergent series solution that is uniformly valid. By applying the finite difference method, Asaithambi [5] numerically examined the Falkner-Skan problem and established the second order accuracy. The other numerical attempts also made by this author [6]. Nowadays, the Falkner-Skan equation has been examined in different contexts due to substantial applications in engineering science. The Falkner-Skan equation has subsequently been studied in a variety of scenarios due to its extensive use in engineering research such as [7, 8].

The Blasius problem, i.e. stable flow over a moving wedge, and the Sakiadis problem, i.e. steady flow over a continuous stretched sheet flowing past a quiescent ambient fluid are the two most well-known forced convection problems. And they serve as the basis for boundary-layer equations in fluid mechanics. Sakiadis [9] initiated this investigation to look at the behaviour of boundary-layer flow on continuous solid surfaces. By employing a perturbation technique, Datta [10] introduced a fresh direction to study the Blasius problem. Numerous numerical and semi-analytical solutions to this problem have occasionally been described in the literature (Abbasbandy [11], Khan et al. [12]).

There are various real-world scenarios where the stretching surface and surrounding fluid move simultaneously. As an illustration, consider cooling metallic sheets, cylinders, polymer sheets or films, etc. Due to the nonlinear component in the underlying differential equations regulating fluid motion in hydrodynamics, an accurate solution is crucial. Finding the closed-form solution to the underlying differential equations becomes difficult, if not impossible. As a result, the majority of researchers arrive to acquire the similarity solution. Numerous scholars, such as [13, 14, 15, 16, 17, 18, 19, 20]have studied these type of problems with or without heat transfer using various numerical techniques and resolved it.

On the other hand, the spline interpolation approach for numerical analysis has been the subject of substantial study during the last few decades. Different kinds of spline approaches have been applied to solve differential equations such as [19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. Over the past few years, there has been an increasing interest in utilising the quasilinearization technique [32, 33] to investigate nonlinear situations. Researchers such as Saeed and Rehman [34] and Jiwari [35] have successfully implemented quasilinearization technique for the numerical solution of nonlinear differential equations.
In this paper, we propose to study the laminar flow problem of Falkner-Skan type. The governing system of equations is non-linear partial differential equations that cannot be solved exactly. As a result, we propose a numerical method based on non-polynomial quartic spline. Using this method, one can obtain numerical solution. First, we use similarity parameter to transform the partial differential equation into third order nonlinear ordinary differential equation. Our goal is to derive and then use the non-polynomial quartic splines technique to find the approximate solution to our problem. We outline the derivation of the method along with the truncation error. Using tables and graphs, we write the conclusions.

2 Formation of the problem

The Laminar boundary layer theory has the possible solution as self-similarity solution, so this subject becomes high level research due to its usefulness in the laminar boundary layer flows. The Falkner-Skan type boundary layer flow is modelled by the similarity transformation as

$$\frac{\partial w}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0 \tag{2.1}$$

$$w\frac{\partial w}{\partial y} + \tilde{w}\frac{\partial w}{\partial z} = \gamma\frac{\partial^2 w}{\partial z^2} - \frac{1}{\varrho}\delta', \tag{2.2}$$

where δ' is the pressure gradient, (y, z) is the plane of flow determined by y and z coordinates, w and $\tilde{w}$ are the y and z components of velocity respectively, γ is the constant kinematic viscosity and ϱ is the fluid density. At the border of the boundary layer, the velocity $W(y)$ is subjected to the power-law relation $W(y) = W_\infty y^k$. The necessary end conditions are

$$z = 0, w(0) = W_u(y), \tilde{w}(0) = 0, \tag{2.3}$$

$$z \to \infty, w(z) \to W_\infty \tag{2.4}$$

where W_u is the stretching surface velocity which obeys the power law relation, $W_u = W_\infty y^k$. Using the similarity transformation $\varphi = (\frac{2\gamma y W(y)}{1+k}K(\omega))^{\frac{1}{2}}$, $\omega = z(\frac{(1+k)W(y)}{2\gamma y})^{\frac{1}{2}}$ and $\alpha = \frac{2k}{1+k}$ in the equation (2.2-2.4), we have

$$K'''(\omega) + K(\omega)K''(\omega) + \alpha(1 - K'^2(\omega)) = 0, \tag{2.5}$$

with relevant boundary conditions:

$$\omega = 0, K(\omega) = 0, K'(\omega) = \beta, \tag{2.6}$$

$$\omega \to \infty, K'(\omega) = 1, \tag{2.7}$$

where $\beta = -\frac{W_u}{W_\infty}$. For $\alpha \geq 0$, equation (2.5) reflects the symmetrical boundary layer flow over a wedge along with an angle $\alpha\pi$. For non-zero value of β, the boundary has a definite speed and prescribed stretch. Equations (2.5),(2.6) and (2.7) constitute a non-linear boundary value problem in an infinite domain. Our aim is to solve these equations using non-polynomial quartic spline technique.

3 Quasilinearization Technique

With this method, the model problem is turned into a sequence of linear ODEs, and the solution to the linear ODEs can be obtained by finding the limit of the sequence. Then, we will write the model problem (2.5)-(2.7) in $[0, 1]$ as:

$$K'''(\omega) = H(\omega, K(\omega), K'(\omega), K''(\omega)), 0 \leq \omega \leq 1 \tag{3.1}$$

with boundary conditions

$$K(0)=0, \quad K'(0)=\beta \quad and \quad K'(1)=1, \tag{3.2}$$

where $H(\omega, K(\omega), K'(\omega), K''(\omega))$ is continuous in $[0,1]$.
Using the quasilinearization technique in (3.1)-(3.2), we have

$$K'''_{t+1}(\omega)+p_t(\omega)K''_{t+1}(\omega)+q_t(\omega)K'_{t+1}(\omega)+r_t(\omega)K_{t+1}(\omega)=H(\omega,K_t(\omega),K'_t(\omega),K''_t(\omega))+ p_t(\omega)K''_t(\omega)+q_t(\omega)K'_t(\omega)+r_t(\omega)K_t(\omega), 0\le\omega\le 1 \tag{3.3}$$

which can be written as

$$K'''_{t+1}(\omega)+p_t(\omega)K''_{t+1}(\omega)+q_t(\omega)K'_{t+1}(\omega)+r_t(\omega)K_{t+1}(\omega)=h_t(\omega), \quad 0\le\omega\le 1 \tag{3.4}$$

subject to the boundary conditions

$$K_{t+1}(0)=0, \quad K'_{t+1}(0)=\beta \quad and \quad K'_{t+1}(1)=1, \tag{3.5}$$

where, $h_t(\omega)=H(\omega,K_t(\omega),K'_t(\omega),K''_t(\omega))+p_t(\omega)K''_t(\omega)+q_t(\omega)K'_t(\omega)+r_t(\omega)K_t(\omega)$,
$p_t(\omega)=\left(\frac{\partial H}{\partial K''}\right)_{H=H_t}$, $q_t(\omega)=\left(\frac{\partial H}{\partial K'}\right)_{H=H_t}$ and $R_t(\omega)=\left(\frac{\partial H}{\partial K}\right)_{H=H_t}$.
For our convenience, we write $K^l_{t+1}(\omega)=K^l(\omega)$, $l=0,1,2,3$, $h_t(\omega)=h(\omega)$,
$p_t(\omega)=p(\omega)$, $q_t(\omega)=q(\omega)$ and $r_t(\omega)=r(\omega)$, so that equation (3.5) becomes

$$K'''(\omega)+p(\omega)K''(\omega)+q(\omega)K'(\omega)+r(\omega)K(\omega)=h(\omega), 0\le\omega\le 1 \tag{3.6}$$

subject to boundary conditions

$$K(0)=0, \quad K'(0)=\beta \quad and \quad K'(1)=1. \tag{3.7}$$

4 Non-polynomial Quartic Spline Method

To obtain trigonometric quartic spline approximation of the equations (3.6) and (3.7), we divide the interval $[0,1]$ into M equal subintervals as follows:

$$\omega_i=ih, \quad i=0(1)M, \quad where \quad h=\frac{1}{M}.$$

Now, using the non-polynomial spline $R_i(\omega)$ we construct a numerical algorithm to interpolate the unknown function $K(\omega)$ at the grid points $\{\omega_i \ : \ i=1,2,...,M\}$ given as

$$R_i(\omega) = \xi_{1i}sin\kappa(\omega-\omega_i)+\xi_{2i}cos\kappa(\omega-\omega_i)+\xi_{3i}(\omega-\omega_i)^2+\xi_{4i}(\omega-\omega_i)+\xi_{5i}, \tag{4.1}$$

where $\xi_{1i},\xi_{2i},\xi_{3i},\xi_{4i}$ and ξ_{5i} are real finite constants and $R_i\in C^4\Delta$ has been interpolated at the mesh points ω_i which depends on the parameter κ.
The coefficients $\xi_{1i},\xi_{2i},\xi_{3i},\xi_{4i}$ and ξ_{5i} have been obtained by using the following interpolation conditions:

$$R_i(\omega_i)=K_i, \ \ R_i(\omega_{i+1})=K_{i+1}, \tag{4.2}$$

$$R'_i(\omega_i)=S_i, \ \ R'''_i(\omega_i)=L_i, \ \ R'''_i(\omega_{i+1})=L_{i+1}, \qquad i=0(1)M \tag{4.3}$$

Using (4.1), (4.2) and (4.3), we obtain the coefficients as

$$\begin{aligned}
\xi_{1i} &= \frac{1}{\kappa^3 sinkh}(L_{i+1}-coskhL_i),\\
\xi_{2i} &= -\frac{L_i}{\kappa^3},\\
\xi_{3i} &= \frac{K_{i+1}-K_i}{h^2}+\left(\frac{1-coskh}{h^2\kappa^3 sinkh}\right)L_{i+1}+\left(\frac{1-coskh-khsinkh}{h^2\kappa^3 sinkh}\right)L_i-\frac{S_i}{h},\\
\xi_{4i} &= S_i+\frac{L_i}{\kappa^2},\\
\xi_{5i} &= K_i-\frac{1}{\kappa^3 sinkh}(L_{i+1}-coskhL_i).
\end{aligned}$$

Following the continuity condition defined for spline as well as its derivatives, the relations have been obtained as:

$$S_i + S_{i-1} = -\frac{2(K_{i-1} - K_i)}{h} + \lambda_1 L_{i-1} + \lambda_1 Li, \tag{4.4}$$

$$S_i - S_{i-1} = \frac{K_{i-1} - 2K_i + K_{i+1}}{h} + \lambda_2 L_{i-1} + \lambda_3 Li + \lambda_4 L_{i+1}, \tag{4.5}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{2 - 2cos\kappa h - khsin\kappa h}{h\kappa^3 sin\kappa h}, \\
\lambda_2 &= \frac{2cos\kappa h + 2\kappa hsin\kappa h - 2 - \kappa^2 h^2}{2h\kappa^3 sin\kappa h}, \\
\lambda_3 &= -\frac{2\kappa^2 h^2 cos\kappa h - 2\kappa hsin\kappa h}{2h\kappa^3 sin\kappa h}, \\
\lambda_4 &= \frac{2 - 2cos\kappa h - \kappa^2 h^2}{2h\kappa^3 sin\kappa h}.
\end{aligned}$$

From equations (4.4-4.5), we get the relation:

$$S_i = \frac{K_{i+1} - K_{i-1}}{2h} + \lambda_5 L_{i-1} + \lambda_6 Li + \lambda_5 L_{i+1}, \tag{4.6}$$

where

$$\begin{aligned}
\lambda_5 &= \frac{2 - 2cos\kappa h - \kappa^2 h^2}{4h\kappa^3 sin\kappa h}, \\
\lambda_6 &= \frac{2\kappa^3 hcos\kappa h - 4\kappa hsin\kappa h - 4 - 4cos\kappa h}{4h\kappa^3 sin\kappa h}.
\end{aligned}$$

Substituting the value of S_i in equation (4.5), we obtain

$$K_{i-2} + 3K_{i-1} - 3K_i + K_{i+1} = h^3[\tau_1\,(L_{i-2} + L_{i+1}) + \tau_2\,(L_i + L_{i-1})], \quad i = 2(1)M - 1 \tag{4.7}$$

where

$$\begin{aligned}
\tau_1 &= \frac{2 - 2cos\kappa h - \kappa^2 h^2}{2\kappa^3 sin\kappa h}, \\
\tau_2 &= \frac{2\kappa^2 h^2 cos\kappa h + 2cos\kappa h - \kappa^2 h^2 - 2}{2\kappa^3 sin\kappa h}.
\end{aligned}$$

The equations in (4.7) yield $(M-2)$ linear equations involving M unknowns in K_i, $i = 1(1)M$. In order to solve the system of equations, we need two additional equations, which can be obtained as:

$$\sum_{l=0}^{2} A_l K_l + C_1 hK_0^{'} + h^3 \sum_{l=0}^{3} B_l K_l^{'''} = t_1, \quad i = 1 \tag{4.8}$$

$$\sum_{l=M-2}^{M} D_l K_l + C_M hK_M^{'} + h^3 \sum_{l=M-3}^{M} E_l K_l^{'''} = t_M, \quad i = M \tag{4.9}$$

5 Truncation Error

To obtain the truncation error of the numerical algorithm, we use Taylor's series expansion about ω_i in equation (4.7) so that

$$t_i = [1-2(\tau_1+\tau_2)]h^3K_i^{'''} + \left[\frac{-1}{2}+(\tau_1+\tau_2)\right]h^4K_i^{(4)} + \left[\frac{1}{4}-\frac{(5\tau_1+\tau_2)}{2}\right]h^5K_i^{(5)} + \left[\frac{-1}{12}+\frac{(7\tau_1+\tau_2)}{6}\right]h^6K_i^{(6)} + \left[\frac{1}{40}-\frac{(7\tau_1+\tau_2)}{24}\right]h^7K_i^{(7)} + O(h^8),$$
$$i = 2(1)M-1. \quad (5.1)$$

With the use of the aforementioned equations, the following family of methods is generated by minimising the components of various powers of h for different values of τ_1 and τ_2,

Second-order methods

For $(A_0, A_1, A_2, C_0, B_0, B_1, B_2, B_3) = (3, -4, 1, 2, \frac{-3}{2}, \frac{-4}{12}, \frac{-1}{12}, 0)$, and $(D_{M-2}, D_{M-1}, D_M, C_M, E_{M-3}, E_{M-2}, E_{M-1}, E_M) = (-3, 8, -5, 2, 0, \frac{-3}{2}, \frac{-10}{12}, \frac{-31}{12})$, the local truncation error is given as:

$$\begin{aligned} t_1 &= -\frac{1}{10}h^5K_i^{(5)} + O(h^6), \quad i = 1, \\ t_M &= \frac{1}{10}h^5K_i^{(5)} + O(h^6), \quad i = M. \end{aligned} \quad (5.2)$$

Case 1: When $(\tau_1, \tau_2) = (\frac{1}{12}, \frac{5}{12})$, the truncation error is given by

$$t_i = -\frac{1}{6}h^5K_i^{(5)} + O(h^6), \quad i = 2(1)M-1.$$

Case 2: When $(\tau_1, \tau_2) = (\frac{1}{4}, \frac{1}{4})$, the truncation error is given by

$$t_i = -\frac{1}{6}h^5K_i^{(5)} + O(h^6), \quad i = 2(1)M-1.$$

Case 3: When $(\tau_1, \tau_2) = (\frac{1}{2}, 0)$, the truncation error is given by

$$t_i = -h^5K_i^{(5)} + O(h^6), \quad i = 2(1)M-1.$$

Fourth-order methods

For $(A_0, A_1, A_2, C_0, B_0, B_1, B_2, B_3) = (3, -4, 1, 2, \frac{-8}{60}, \frac{-35}{60}, \frac{4}{60}, \frac{1}{60})$, and $(D_{M-2}, D_{M-1}, D_M, C_M, E_{M-3}, E_{M-2}, E_{M-1}, E_M) = (-3, 8, -5, 2, \frac{8}{60}, \frac{-33}{60}, 0, \frac{-157}{60})$, the local truncation error is given as:

$$\begin{aligned} t_1 &= -\frac{29}{2520}h^7K_i^{(7)} + O(h^8), \quad i = 1, \\ t_M &= \frac{677}{5040}h^7K_i^{(7)} + O(h^8), \quad i = M. \end{aligned} \quad (5.3)$$

When $(\tau_1, \tau_2) = (0, \frac{1}{2})$, the truncation error is given by

$$t_i = -\frac{1}{240}h^7K_i^{(7)} + O(h^8), \quad i = 2(1)M-1.$$

6 Numerical Experiments

Here, we show the performance of the presented numerical algorithm on the model problem (3.6) subject to the conditions (3.7). For this, we compute the numerical solution of the linearized model problem (3.6) and (3.7) at different grid points [0,1]. Tables are used to display the numerical results of $K'(\omega)$ for various values of the parameters α and β. Also, a graphical representation of $K'(\omega)$ is shown with the help of figures.

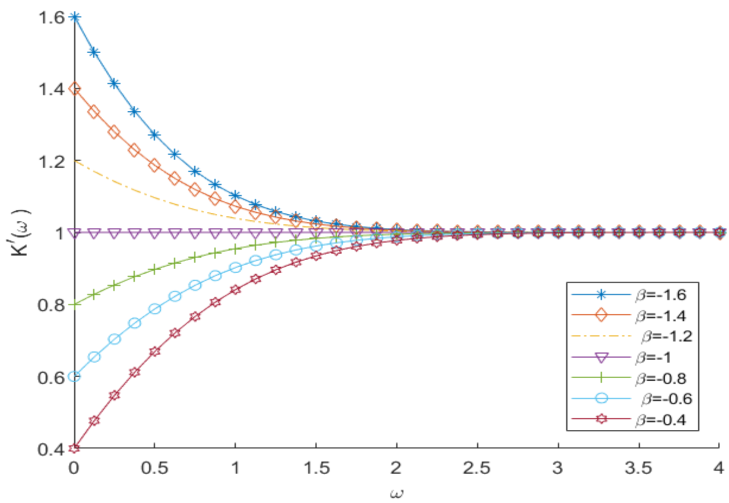

Figure 1. *Velocity component $K'(\omega)$ when the parameter β varies, taking $\alpha = 0.5$*

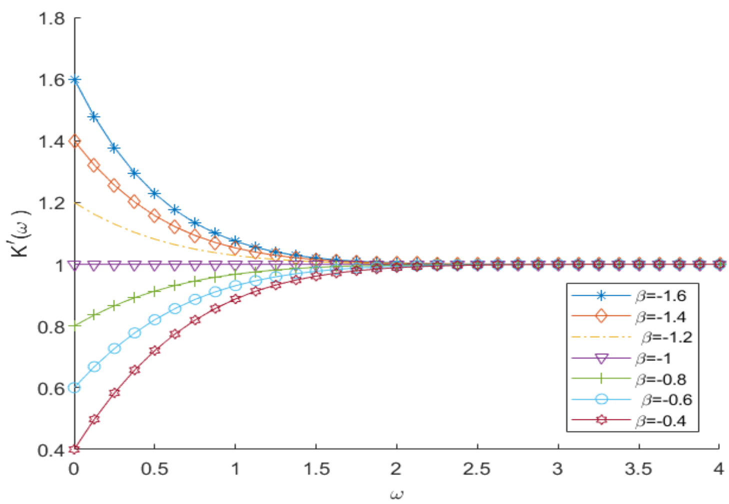

Figure 2. *Velocity component $K'(\omega)$ when the parameter β varies, taking $\alpha = 1$*

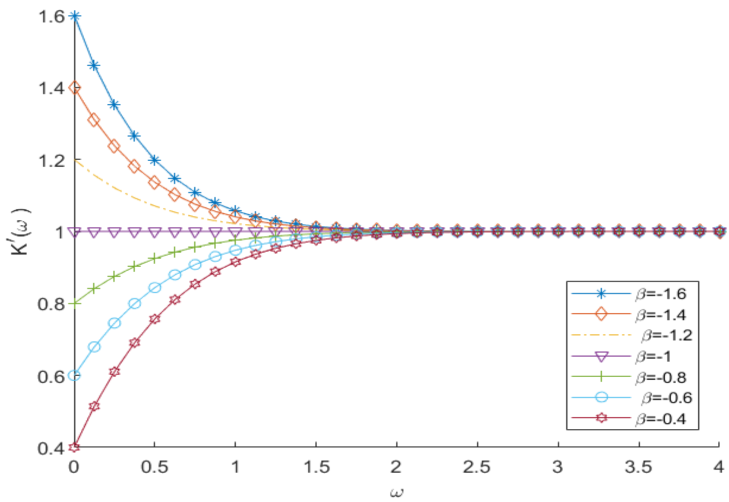

Figure 3. *Velocity component $K'(\omega)$ when the parameter β varies, taking $\alpha = 1.5$*

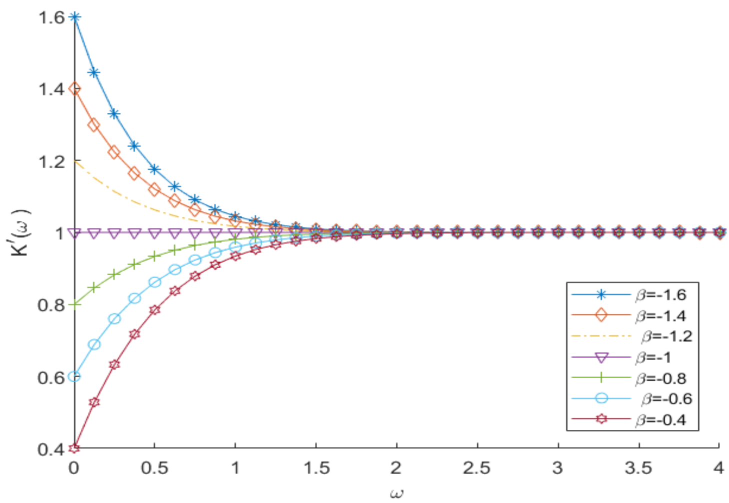

Figure 4. *Velocity component $K'(\omega)$ when the parameter β varies, taking $\alpha = 2$*

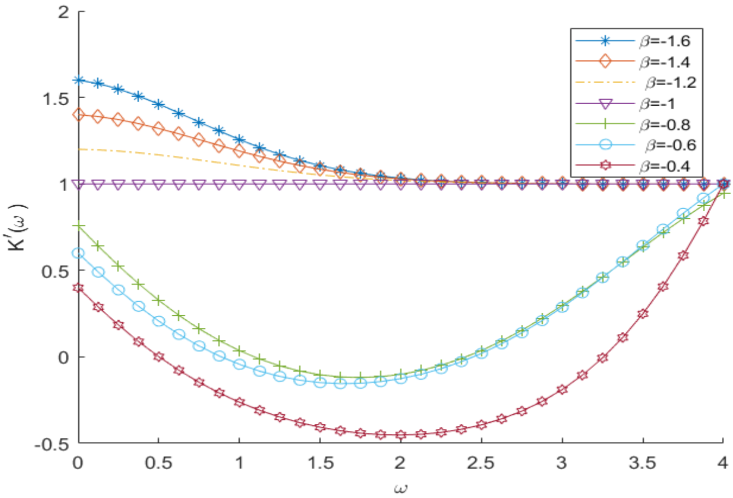

Figure 5. *Velocity component $K'(\omega)$ when the parameter β varies, taking $\alpha = -0.5$*

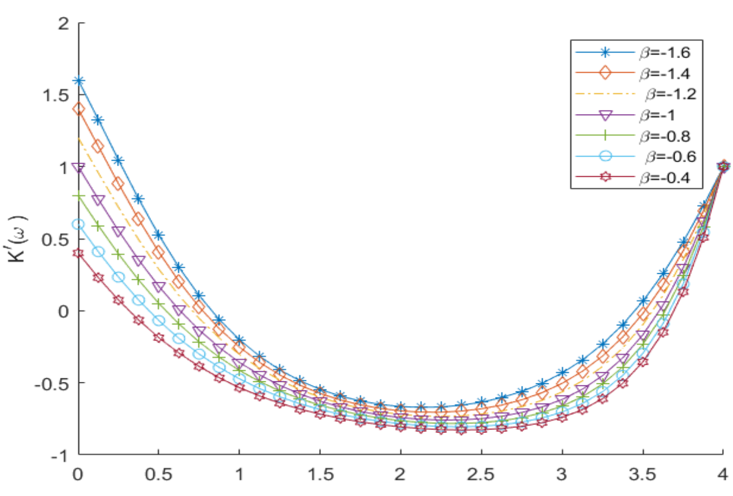

Figure 6. *Velocity component $K'(\omega)$ when the parameter β varies, taking $\alpha = -1$*

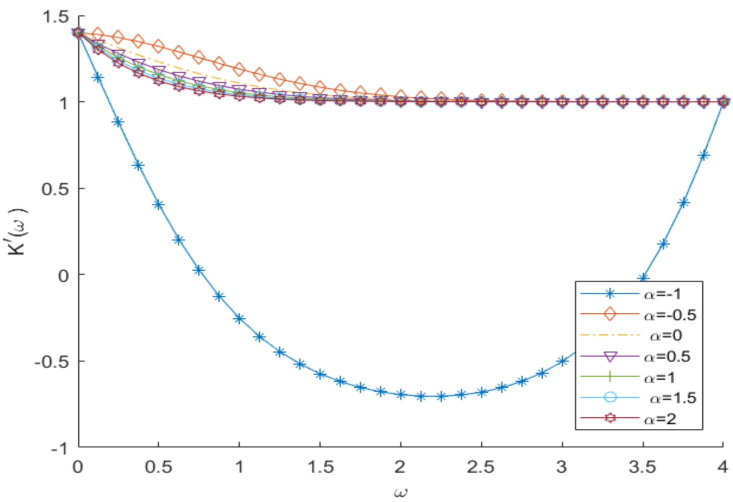

Figure 7. *Velocity component $K'(\omega)$ when the parameter α varies, taking $\beta = -1.4$*

Table 1. Comparison of coefficient $K'(\omega)$ by our method with that of exact and numerical solution of [36] for $\beta = -1.4$ and $\alpha = 1$.

ω	**ES**[36]	**NM**[36]	**OurMethod**
0	1.4	1.4	1.4
0.25	1.255714	1.255714	1.255711
0.5	1.157492	1.157492	1.157487
0.75	1.093224	1.093223	1.093218
1	1.052922	1.052920	1.052915
1.25	1.028759	1.028756	1.028751
1.5	1.014937	1.014931	1.014927
1.75	1.007405	1.007397	1.007394
2	1.003502	1.003491	1.003490
2.25	1.001575	1.001568	1.001567
2.5	1.000644	1.000668	1.000669
2.75	1.000114	1.000270	1.000271
3	1.000052	1.000102	1.000104
3.25	1.000006	1.000036	1.000037

The approximate solution of the Falkner–Skan equations (3.6)-(3.7) has been computed using our proposed method. With the help of MATLAB, the graphical representation of the solution for different values of α and β has been presented in Figures 1-7, which displays the velocity function $K'(\omega)$ when the parameters α and β vary. We observe from Figures 1-4, when $\alpha \in (0.5, 2)$, the approximate solutions $K'(\omega)$ of (3.6)-(3.7) with $\beta = -1$ shows qualitatively atypical behavior of the solutions for $\beta > -1$ and $\beta < -1$. From Figures 5-6, we observe that as we take α as negative value, the behavior of the approximate solutions $K'(\omega)$ of (3.6)-(3.7) get changed completely as compared to the solution when α is taken positive. Figure 7 shows the behavior of the approximate solutions $K'(\omega)$ when α varies with fixed $\beta = -1.4$. The numerical solutions of (3.6)-(3.7) obtained for $\beta = -1.4$ and $\alpha = 1$ are compared with the solutions given in [36] and is displayed in Table 1.

Blasius Equation

For $\alpha = 0$, equations (2.5-2.7) reduce to the Blasius equation, which is given as

$$K'''(\omega) + K(\omega)K''(\omega) = 0, \tag{6.1}$$

subject to boundary conditions:

$$\omega = 0, K(\omega) = 0, K'(\omega) = \beta \tag{6.2}$$
$$\omega \to \infty, K'(\omega) = 1. \tag{6.3}$$

This problem is associated with the boundary layer on a flat plate that has a constant velocity opposite in direction to that of a uniform mainstream. Now, we solve equations (6.1-6.3) using our proposed numerical algorithm to find the values of the function $K'(\omega)$. In Table 2, we compare our numerical results with that of the solution obtained by Numerical Method (NM)and Exact Solution (ES) given in [36]for $\beta = -1.4$.

Table 2. Comparison of the function $K'(\omega)$ for the equations (6.1-6.3) by our method with that of Exact Solution (ES) [36] and Numerical Method (NM) [36] for $\beta = -1.4$ and $\alpha = 0$.

ω	**ES**[36]	**NM**[36]	**Meksyn′sApproach**[36]	**OurMethod**
0	1.4	1.4	1.4	1.4
0.25	1.311931	1.311931	1.311964	1.311931
0.5	1.230934	1.230934	1.230940	1.230934
0.75	1.162099	1.162099	1.162100	1.162099
1	1.107794	1.107794	1.107793	1.107792
1.25	1.067872	1.067872	1.067878	1.067868
1.5	1.040443	1.040443	1.040442	1.040440
1.75	1.022795	1.022796	1.022794	1.022790
2	1.012146	1.012147	1.012146	1.012140
2.25	1.006113	1.006114	1.006127	1.006108
2.5	1.002900	1.002906	1.002859	1.002899
2.75	1.001285	1.001302	1.001121	1.001296
3	1.000500	1.000549	1.000501	1.000543
3.25	1.000011	1.000216	1.000084	1.000210

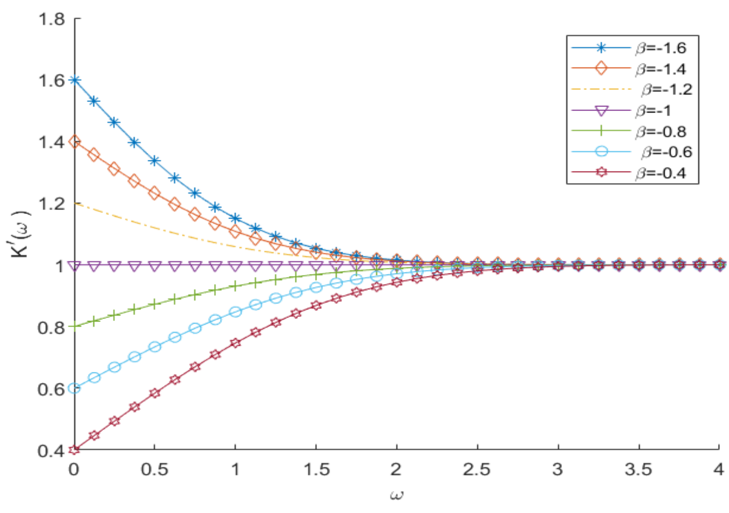

Figure 8. *Velocity component $K'(\omega)$ when the parameter β varies taking $\alpha = 0$*

7 Skin Friction

On the plate, the formulation of wall shear stress is as follows:

$$\tau_w = -\gamma \left(\frac{\partial w}{\partial z} \right)_{z=0}. \tag{7.1}$$

Using the similarity transformation, we get

$$\tau_w = -\gamma W_u \left(\frac{k}{\gamma} \right)^{\frac{1}{2}} K''(0), \qquad W_u = W_\infty y^k. \tag{7.2}$$

As a result, the coefficient for skin friction can be derived as follows:

$$C_K = \frac{g_w}{\varrho W_u^2 \tilde{G}} = -(R_e)^{-1} \left(\frac{k}{\gamma} \right)^{\frac{1}{2}} K''(0), \tag{7.3}$$

where $R_e = \frac{\varrho W_u \tilde{G}}{\gamma}$ and $\tilde{G}$ is the characteristic linear dimension.

Table 3. Comparison of skin friction coefficient $K''(0)$ by our method with that of HWCM [37] and HWQLM [37] for $\beta = 0$.

α	**HWCM**[37]	**HWQLM**[37]	**OurMethod**
0.5	0.927680	0.927680	0.927718
15	1.232587	1.232587	1.232588
1.5	1.477233	1.477233	1.477221
2.5	1.874025	1.874027	1.874020

In Table 3, we have presented the value of skin friction coefficient $K''(0)$ for different values of α and $\beta = 0$. From this table we observe that the value of skin friction co-efficent is more accurate as compared to the value obtained by HWCM [37] and HWQLM [37].

8 Discussions, Results and Conclusion

In the present paper, we have derived a numerical technique based on non polynomial quartic spline technique with the help of a stream function. First we linearize the non linear flow problem by using quasilinearisation. Then using the proposed technique, we solve the boundary layer flow presented by the Falkner-Skan equation for different flow parameters α and β.
In Figure-1-4, we discuss the behavior of the velocity component $K'(\omega)$ when the parameter β varies and $\alpha \in (0.5, 2)$. Figures 5-6, show that as we take negative α, the behavior of the approximate solutions $K'(\omega)$ is different as compared to the solution when α is taken positive. Figure 7 displays the value of $K'(\omega)$ when α varies with fixed $\beta = -1.4$. Table 1 represents the numerical solutions $K'(\omega)$ of (3.6)-(3.7) obtained for $\beta = -1.4$ and $\alpha = 1$ and Table 2 represents the numerical solutions $K'(\omega)$ for $\beta = -1.4$ and $\alpha = 0$ which are compared with the solutions given in [36]. Also, we observe that skin friction coefficient is inversely proportional to the Reynold's number. Skin friction coefficient $K''(0)$ for different value of α and β is discussed in Table-3 and is compared with values given in [37].

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