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Bounds on Neutral Phase Speed for the Stratified Shear Flows

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Abstract. We consider Taylor Goldstein problem in β- plane under Boussineq approximation, which deals with incompressible, inviscid stratified shear flows. In this paper, first we obtained upper and lower bounds of neutral phase speed. Second , we obtained the bound for complex phase speed which depends on parameter like vorticity function, minimum and maximum of velocity profile and stratification parameter. Also, we obtained a criterion for stability and illustrated with examples.

1 Introduction :

The stability analysis of stratified shear flows under normal mode approach has been studied extensively (see [15], [1] & [11]). Parallel shear flow problem is a classical hydrodynamic stability problem and attracts many researchers. [6] considered inviscid homogeneous parallel shear flow in β- plane, which is the standard homogeneous shear flow problem. [6] derived Rayleigh inflexion point theorem. [3] derived upper bound for the growth rate. [9] proved the phase velocity lies inside the upper half of the semi circle which is the extension of [4] semi circle.
For stratified shear flows known as Taylor-Goldstein problem, [4] derived semi- circle theorem and [8] derived sufficient condition for stability. [5] extended works of [4] and derived semiellipse theorem depends on stratification parameter. [2] derived parabolic instability region depending on certain condition. [12] proved Howard's conjecture. [14] extended semiellipse theorem to extended Taylor-Goldstein problem. [14] proved that neutral waves are bounded. [13] derived instability region for extended Taylor-Goldstein problem. [10] obtained sharper estimate for growth rate and derived long wave stability criterion.
In this paper, we consider incompressible, inviscid, density varying fluid streaming in the horizontal direction in β- plane under Boussineq approximation known as Taylor-Goldstein problem in β- plane. For this problem, we derived upper and lower bound for neutral waves and obtained and semielliptical instability region where major axis depends on stratification parameter, vorticity function and curvature . Also, we derived a condition for stability.

2 Bounds on Neutral Phase speed:

The Taylor Goldstein problem in β- plane is given by

$$D^2(\phi) + \left[\frac{N^2}{(U-c)^2} - \frac{D^2(U)-\beta}{U-c} - k^2\right]\phi = 0, \tag{2.1}$$

with boundary conditions

$$\phi(z_1) = 0 = \phi(z_2). \tag{2.2}$$

where U is the basic velocity profile, ϕ is the eigen function, $c = c_r + ic_i$ phase velocity, $k > 0$ is the wave number, β coriolis parameter given by $\beta = \left(\frac{2\Omega}{a}\right) cos\theta$, where a is the radius of earth, Ω is the earth's rotation rate,θ is the latitude [7], [9].

Theorem 2.1. *The upper and lower bound for neutral waves $c = c_r$ with $c_i = 0$ is given by*

$$U_{\min}(z_s) + \left[\frac{\left(D^2(U)-\beta\right)(z_s)}{2k^2}\right]_{\min} - \left[\frac{\sqrt{\left[\left(D^2(U)-\beta\right)(z_s)\right]^2 + 4k^2N^2}}{2k^2}\right]_{\max} \le c \le$$

$$U_{\max}(z_s) + \left[\frac{\left(D^2(U)-\beta\right)(z_s)}{2k^2}\right]_{\min} + \left[\frac{\sqrt{\left[\left(D^2(U)-\beta\right)(z_s)\right]^2 + 4k^2N^2}}{2k^2}\right]_{\max}$$

Proof. Multiplying (2.1) by conjugate of ϕ, integrating over $[z_1, z_2]$and using (2.2), we get

$$\int_{z_1}^{z_2} |D(\phi)|^2\,dz + k^2\int_{z_1}^{z_2} |\phi|^2\,dz + \int_{z_1}^{z_2}\frac{\left(D^2(U)-\beta\right)}{(U-c)}|\phi|^2\,dz - \int_{z_1}^{z_2}\frac{N^2}{(U-c)^2}|\phi|^2\,dz = 0. \quad (2.3)$$

Since first term is positive, dropping the term, we get

$$\int_{z_1}^{z_2}\left[k^2 + \frac{\left(D^2(U)-\beta\right)}{(U-c)} - \frac{N^2}{(U-c)^2}\right]|\phi|^2\,dz \le 0;$$

i.e., $\int_{z_1}^{z_2}\left[k^2(U-c)^2 + \left(D^2(U)-\beta\right)(U-c) - N^2\right]\frac{|\phi|^2}{(U-c)^2}dz \le 0.$

There exist a point $z = z_s \in [z_1, z_2]$ such that

$$k^2(U(z_s)-c)^2 + \left(D^2(U)-\beta\right)(z_s)(U(z_s)-c) - N^2 \le 0;$$

$$k^2c^2 - \left[2k^2U(z_s) + \left(D^2(U)-\beta\right)(z_s)\right]c$$
$$+\left[k^2U^2(z_s) + \left(D^2(U)-\beta\right)(z_s)U(z_s) - N^2\right] \le 0.$$

Solving for c, we get

$$\begin{array}{l} U_{\min}(z_s) + \left[\frac{\left(D^2(U)-\beta\right)(z_s)}{2k^2}\right]_{\min} - \left[\frac{\sqrt{\left[\left(D^2(U)-\beta\right)(z_s)\right]^2+4k^2N^2}}{2k^2}\right]_{\max} \le c \le \\ U_{\max}(z_s) + \left[\frac{\left(D^2(U)-\beta\right)(z_s)}{2k^2}\right]_{\min} + \left[\frac{\sqrt{\left[\left(D^2(U)-\beta\right)(z_s)\right]^2+4k^2N^2}}{2k^2}\right]_{\max} \end{array}.$$

□

3 Semi Elliptical Region

Theorem 3.1. *If $|D(U)|^2_{\min} \neq 0$ then the range of complex phase speed $c = c_r + ic_i$ is*

$$\left[c_r - \frac{U_{\min}+U_{\max}}{2}\right]^2 + {c_i}^2 + \frac{J_0}{(1+A_1)^2}c_i^2 \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2,$$

where

$$A_1^2 = \frac{\left|N^2\right|_{\max} - \left|N^2\right|_{\min} + \left|D^2(U)-\beta\right|_{\max}|U_{\max}-U_{\min}|}{|D(U)|^2_{\min}}$$

.

Proof. Equating real parts of (2.3), we get

$$\int_{z_1}^{z_2}|D(\phi)|^2\,dz + k^2\int_{z_1}^{z_2}|\phi|^2\,dz + \int_{z_1}^{z_2}\frac{\left(D^2(U)-\beta\right)(U-c_r)}{|U-c|^2}|\phi|^2\,dz$$
$$-\int_{z_1}^{z_2}\frac{N^2\left[(U-c_r)^2 - c_i^2\right]}{|U-c|^4}|\phi|^2\,dz = 0.$$

Applying triangular inequality, we get

$$\int_{z_1}^{z_2} |D(\phi)|^2 dz + k^2 \int_{z_1}^{z_2} |\phi|^2 dz$$
$$\leq \left| \int_{z_1}^{z_2} \frac{N^2 \left[(U-c_r)^2 - c_i^2 \right]}{|U-c|^4} |\phi|^2 dz - \int_{z_1}^{z_2} \frac{\left(D^2(U) - \beta \right)(U-c_r)}{|U-c|^2} |\phi|^2 dz \right|;$$

$$\int_{z_1}^{z_2} |D(\phi)|^2 dz + k^2 \int_{z_1}^{z_2} |\phi|^2 dz \leq \int_{z_1}^{z_2} \frac{N^2 \left[(U-c_r)^2 - c_i^2 \right]}{|U-c|^4} |\phi|^2 dz + \int_{z_1}^{z_2} \frac{\left(D^2(U) - \beta \right)(U-c_r)}{|U-c|^2} |\phi|^2 dz.$$

Using the inequalities,

$$(U-c_r)^2 \leq c_i^2 + (U-c_r)^2 = |U-c|^2, |U-c_r| \leq |U_{max} - U_{min}| \, and \frac{c_i^2}{|U-c|^2} \leq 1,$$

we have

$$\int_{z_1}^{z_2} |D(\phi)|^2 dz + k^2 \int_{z_1}^{z_2} |\phi|^2 dz$$
$$\leq \left[|N^2|_{\max} - |N^2|_{\min} + \left(D^2(U) - \beta \right)_{\max} |U_{\max} - U_{\min}| \right] \int_{z_1}^{z_2} \frac{|\phi|^2}{|U-c|^2} dz. \tag{3.1}$$

Applying $\phi = (U-c)\varphi$, we get

$$|D(\phi)|^2 \geq |U-c|^2 |D(\varphi)|^2 - 2|U-c| |D(U)| |\varphi| |D(\varphi)| + |D(U)|^2 |\varphi|^2. \tag{3.2}$$

Applying Cauchy-Schwartz inequality , we get

$$\int_{z_1}^{z_2} |U-c| |D(U)| |\varphi| |D(\varphi)| dz \leq \left[\int_{z_1}^{z_2} |D(U)|^2 |\varphi|^2 dz \right]^{\frac{1}{2}} \left[\int_{z_1}^{z_2} |U-c|^2 |D(\varphi)|^2 dz \right]^{\frac{1}{2}}.$$

Let

$$B^2 = \int_{z_1}^{z_2} |D(U)|^2 |\varphi|^2 dz; \tag{3.3}$$

$$C^2 = \int_{z_1}^{z_2} |U-c|^2 \left[|D(\varphi)|^2 + k^2 |\varphi|^2 \right] dz. \tag{3.4}$$

Then

$$\int_{z_1}^{z_2} |U-c| |D(U)| |\varphi| |D(\varphi)| dz \leq BC. \tag{3.5}$$

Using (3.3),(3.4),(3.5) in (3.2), we get

$$\int_{z_1}^{z_2} \left[|D(\phi)|^2 + k^2 |\phi|^2 \right] dz \geq [C-B]^2. \tag{3.6}$$

Substituting (3.6) in (3.1) we get

$$[C-B]^2 \leq \left[|N^2|_{\max} - |N^2|_{\min} + |(D^2(U) - \beta|_{\max} |U_{\max} - U_{\min}| \right] \int_{z_1}^{z_2} |\varphi|^2 dz. \tag{3.7}$$

From (3.3), we have

$$B^2 \geq |D(U)|^2_{\min} \int_{z_1}^{z_2} |\varphi|^2 dz;$$

i.e.,

$$\frac{B^2}{|D(U)|^2_{\min}} \geq \int_{z_1}^{z_2} |\varphi|^2 \, dz. \tag{3.8}$$

Substituting (3.8) in (3.7), we have

$$[C-B]^2 \leq \left[\left|N^2\right|_{\max} - \left|N^2\right|_{\min} + \left|D^2(U) - \beta\right|_{\max} |U_{\max} - U_{\min}|\right] \frac{B^2}{|D(U)|^2_{\min}}.$$

$$B^2 \left[\frac{C}{B} - 1\right]^2 \leq A_1^2 B^2,$$

where

$$A_1^2 = \frac{\left|N^2\right|_{\max} - \left|N^2\right|_{\min} + \left|D^2(U) - \beta\right|_{\max} |U_{\max} - U_{\min}|}{|D(U)|^2_{\min}}.$$

$$\left[\frac{C}{B} - 1\right] \leq A_1;$$

$$i.e., B^2 \geq \frac{C^2}{[1 + A_1]^2}. \tag{3.9}$$

$$\int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \geq \left[\frac{N^2}{(D(U))^2}\right]_{\min} \int_{z_1}^{z_2} |D(U)|^2 |\varphi|^2 \, dz;$$

$$i.e., \int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \geq J_0 B^2, \tag{3.10}$$

where $J_0 = \left[\frac{N^2}{(D(U))^2}\right]_{\min}$, substituting (3.9) in (3.10), we get

$$\int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \geq \frac{J_0 C^2}{[1 + A_1]^2}. \tag{3.11}$$

Using the inequality $|U - c|^2 \geq c_i^2$ in (3.3), we have

$$C^2 \geq c_i^2 \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2\right] dz. \tag{3.12}$$

Substituting (3.12) in (3.11), we have

$$\int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \geq \frac{J_0 c_i^2}{[1 + A_1]^2} \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2\right] dz. \tag{3.13}$$

From [4], we have the equation

$$\left[\left[c_r - \frac{U_{\min} + U_{\max}}{2}\right]^2 + {c_i}^2 - \left[\frac{U_{\max} - U_{\min}}{2}\right]^2\right] \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2\right] dz + \int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \leq 0. \tag{3.14}$$

Substituting (3.13) in (3.14), we get

$$\left[\left[c_r - \frac{U_{\min} + U_{\max}}{2}\right]^2 + {c_i}^2 - \left[\frac{U_{\max} - U_{\min}}{2}\right]^2\right] \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2\right] dz + \frac{J_0 c_i^2}{[1 + A_1]^2} \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2\right] dz \leq 0;$$

i.e.,

$$\left[c_r - \frac{U_{\min} + U_{\max}}{2}\right]^2 + {c_i}^2 + \frac{J_0}{(1 + A_1)^2} c_i^2 \leq \left[\frac{U_{\max} - U_{\min}}{2}\right]^2.$$

□

Theorem 3.2. *If $k \le k_c$, critical wave number $k_c > 0$ then the flow is stable.*

Proof. Using the inequality $|U-c|^2 \ge c_i^2$ in (3.4), we get

$$C^2 \ge c_i^2 \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2 \right] dz.$$

The first term inside the integral is positive, hence dropping the term, we get

$$\int_{z_1}^{z_2} |\varphi|^2 \, dz \le \frac{C^2}{k^2 c_i^2} \tag{3.15}$$

Substituting (3.15) in (3.7), we get

$$[C-B]^2 \le \left[|N^2|_{\max} - |N^2|_{\min} + |D^2(U) - \beta|_{\max} |U_{\max} - U_{\min}| \right] \frac{C^2}{k^2 c_i^2};$$

$$i.e., \left[1 - \frac{B}{C}\right]^2 \le \frac{A_2^2}{c_i^2},$$

where

$$A_2^2 = \frac{|N^2|_{\max} - |N^2|_{\min} + |D^2(U) - \beta|_{\max} |U_{\max} - U_{\min}|}{k^2}.$$

Now, we have

$$C^2 \le \frac{B^2}{\left[1 - \frac{A_2}{c_i}\right]^2},$$

Since $C^2 \le \frac{B^2}{\left[1+\frac{A_2}{c_i}\right]^2} \le \frac{B^2}{\left[1-\frac{A_2}{c_i}\right]^2}$, we have

$$C^2 \left[1 + \frac{A_2}{c_i}\right]^2 \le B^2, \tag{3.16}$$

Substituting (3.16) in (3.10), we

$$\int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \ge J_0 C^2 \left[1 + \frac{A_2}{c_i}\right]^2. \tag{3.17}$$

Substituting (3.17) in (3.14), we get

$$\left[c_r - \frac{U_{\max} + U_{\min}}{2}\right]^2 + {c_i}^2 + J_0 \left(c_i^2 + 2A_2 c_i + A_2^2\right) \le \left[\frac{U_{\max} - U_{\min}}{2}\right]^2;$$

$$i.e., \left[c_r - \frac{U_{\max} + U_{\min}}{2}\right]^2 + {c_i}^2 (1 + J_0) + 2A_2 J_0 c_i \le \left[\frac{U_{\max} - U_{\min}}{2}\right]^2 - J_0 A_2^2.$$

From the above equation, we get

$$J_0 A_2^2 \ge \left[\frac{U_{\max} - U_{\min}}{2}\right]^2$$

implies stability;

i.e., $k \le k_c$,

where

$$k_c \le \frac{\sqrt{J_0 \left[|N^2|_{\max} - |N^2|_{\min} + |D^2(U) - \beta|_{\max} |U_{\max} - U_{\min}| \right]}}{\left(\frac{U_{\max} - U_{\min}}{2}\right)}$$

implies stability.

Let us consider the standard examples plane Poiseuille flow and Coutte flow as basic velocity profiles.

Examples:

(i) $U = 1 - z^2$, $z \in [0, 1]$, $N^2 = \beta = z$.
for the above flow, $k \leq 1.732$ implies stability.

(ii) $U = 1 - z^2$, $z \in [0, 1]$, $N^2 = z$, $\beta =$ constant.
for the above flow, $k \leq 2$ implies stability.

(iii) $U = 1 - z^2$, $z \in [0, 1]$, $N^2 =$ constant, $\beta = z$.
for the above flow, $k \leq 1.4142$ implies stability.

(iv) $U = 1 - z^2$, $z \in [0, 1]$, $N^2 = \beta =$ constant.
for the above flow, $k \leq 1.732$ implies stability.

(v) $U = z$, $z \in [0, 1]$, $N^2 = \beta = z$.
for the above flow, $k \leq 2$ implies stability.

(vi) $U = z$, $z \in [0, 1]$, $N^2 = z$, $\beta =$ constant.
for the above flow, $k \leq 2.828$ implies stability.

□

4 Concluding Remarks:

We consider Taylor- Goldstein problem in β - plane for this problem. First we derived upper and lower bound for neutral phase speed. Second, we obtained a semielliptical instability where major axis depends on stratification parameter, curvature and basic velocity profile. Also, we obtained a criterion for stability and illustrated with examples.

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