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ON $(m, n)-$CLOSED $\delta-$PRIMARY IDEALS OF COMMUTATIVE RINGS

Mohammad Hamoda

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Abstract In this paper, we introduce the concept of an $(m, n)-$closed $\delta-$primary ideal of R, where R is a commutative ring with non zero identity, and δ is an expansion function of ideals of R. Several results on $(m, n)-$closed $\delta-$primary ideal of R are proved. We prove that $I = p_1^{k_1}...p_r^{k_r}D$ is an $(m, n)-$closed $\delta-$primary ideal of D if and only if $p_i^{k_i}D$ is an $(m, n)-$closed $\delta-$primary ideal of D for every $i \in \{1, ..., r\}$. Also, we present and prove the relation between $n-$absorbing $\delta-$primary ideals and $(m, n)-$closed $\delta-$primary ideals. Examples of $(m, n)-$closed $\delta-$primary ideals are also studied.

1 Introduction

Recently, extensive researches have been done on prime and primary ideals and submodules. The families of prime and primary ideals (resp. submodules) are very interesting algebraic classes. Ring theorists therefore must often restrict their attention to certain types of these concepts of ideals and submodules or to certain context in which these concepts of ideals and submodules are especially beneficia. The various properties that describe these families and their various generalizations testify to their ubiquitous nature across many branches of mathematics, including number theory, geometry, and topology. New objects related to prime and primary ideals were introduced and studied by Badawi and Badawi et.al. in [4] and [7] respectively. These are the concepts of $2-$absorbing and $2-$absorbing primary ideals of commutative rings. Later, many authors studied on this issue, see for example [6, 8, 9, 10, 12]. The concept of $\delta-$primary ideals in commutative rings was introduced by Zhao in [15]. This concept is considered to unify prime and primary ideals. Many results of prime and primary ideals are extended to these structures, see for examples [11], [14]. In ring theory, it is known that primary ideals are directly closed to prime ideals. These inspired us to define $(m, n)-$closed $\delta-$primary ideals. The innovative idea behind this paper is to construct more accurate results and concepts regarding generalizations of prime ideals. Hence, the motivation of writing this paper lies to create new concepts that can be used in many branches in commutative algebra and its applications. Also, to continue the study of the family of $n-$absorbing ideals and to identify new properties in that subject. The remains of this paper is organized as follows:

Section 2 concerns some basic definitions and results that are used in the sequel of this paper. In section 3, the main results concerning $(m, n)-$closed $\delta-$ primary ideals will be given and then examples have been provided. Section 4 concerns $(m, n)-$closed $\delta-$primary ideals in trivial ring extensions. Section 5 concerns the conclusion.

Throughout this paper, all rings are assumed to be commutative with nonzero identity, all modules are unitary and all ring homomorphisms preserve the identity.

2 Preliminary Notes

Let R be a commutative ring and I be an ideal of R. An ideal I is called proper if $I \neq R$. Let I be a proper ideal of R. Then, the radical of I is defined by $\{x \in R \mid \exists\, n \in \mathbb{N},\ x^n \in I\}$, denoted by $\sqrt{I}$ (note that $\sqrt{R} = R$ and $\sqrt{0}$ is the ideal of all nilpotent elements of R). For the ring R, we shall use $Nil(R)$, $U(R)$, $char(R)$ to denote the set of all nilpotent elements, units, and characteristic of R, respectively.

Definition 2.1. ([4], [7]) A proper ideal I of R is called a $2-$absorbing ideal (respectively, $2-$absorbing primary ideal) of R if whenever $abc \in I$ (respectively, $0 \neq abc \in I$) for some $a, b, c \in R$, implies $ab \in I$ or $ac \in I$ or $bc \in I$ (respectively, $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$).

Another generalization of prime ideals of commutative rings is the concept of $n-$absorbing ideals, defined as follows.

Definition 2.2. [2] Let n be a positive integer. A proper ideal I of R is called an $n-$absorbing ideal (respectively, strongly $n-$absorbing ideal) of R if whenever $x_1...x_{n+1} \in I$ for some $x_1, ..., x_{n+1} \in R$ (respectively, $I_1...I_{n+1} \subseteq I$ for some ideals $I_1, ..., I_{n+1}$ of R), then there are n of the $x_i\,'s$ (respectively, n of the $I_i\,'s$) whose product is in I.

Thus, a $1-$absorbing ideal is just a prime ideal.

Definition 2.3. [3] Let m and n be positive integers. A proper ideal I of R is called a semi$-n-$absorbing ideal of R if whenever $x^{n+1} \in I$ for some $x \in R$ implies $x^n \in I$. More generally, a proper ideal I of R is called an $(m,n)-$closed ideal of R if whenever $x^m \in I$ for some $x \in I$ implies $x^n \in I$.

Definition 2.4. [15] Let $Id(R)$ be the set of all ideals of R. A function $\delta : Id(R) \longrightarrow Id(R)$ is called an expansion function of $Id(R)$ if it has the following two properties: $I \subseteq \delta(I)$ and if $I \subseteq J$ for some ideals, I, J of R, then $\delta(I) \subseteq \delta(J)$. A proper ideal I of R is called a $\delta-$primary ideal of R if whenever $xy \in I$ for some $x, y \in R$ implies $x \in I$ or $y \in \delta(I)$.

Afterwards, Fahid and Zhao introduced the concept of $2-$absorbing $\delta-$primary ideal, which is a generalization of δ primary ideal.

Definition 2.5. [11] A proper ideal I of R is called a $2-$absorbing $\delta-$primary ideal of R if whenever $xyz \in I$ for some $x, y, z \in R$ implies $xy \in I$ or $yz \in \delta(I)$ or $xz \in \delta(I)$. A proper ideal I of R is called a strongly $2-$absorbing $\delta-$primary ideal of R if whenever I_1, I_2, I_3 are ideals of R, $I_1I_2I_3 \subseteq I$, $I_1I_3 \nsubseteq I$ and $I_2I_3 \nsubseteq \delta(I)$, then $I_1I_2 \subseteq \delta(I)$.

The concepts of $n-$absorbing $\delta-$primary ideals and weakly $n-$absorbing $\delta-$primary ideals are generalizations of the concepts of $n-$absorbing primary ideals and weakly $n-$absorbing primary ideals respectively. Recall the following definition:

Definition 2.6. [14] A proper ideal I of R is called an $n-$absorbing $\delta-$primary ideal (respectively, weakly $n-$absorbing $\delta-$primary ideal) of R if whenever $x_1...x_{n+1} \in I$ (respectively, $0 \neq x_1...x_{n+1} \in I$) for some $x_1, ..., x_{n+1} \in R$ implies $x_1...x_n \in I$ or there exists $1 \leq k < n$ such that $x_1...\hat{x}_k...x_{n+1} \in \delta(I)$, where $x_1...\hat{x}_k...x_{n+1}$ denotes the product of $x_1...x_{k-1}x_{k+1}...x_{n+1}$.

3 Properties of $(m, n)-$closed $\delta-$primary ideals

We start by the following definition.

Definition 3.1. Let R be a commutative ring, I a proper ideal of R, δ an expansion function of $Id(R)$ and m and n positive integers.

(1) I is called a semi$-n-$absorbing $\delta-$primary ideal of R if whenever $a^{n+1} \in I$ for some $a \in R$, then $a^n \in \delta(I)$.

(2) I is called an $(m,n)-$closed $\delta-$primary ideal of R if whenever $a^m \in I$ for some $a \in R$, then $a^n \in \delta(I)$.

Clearly, a proper ideal is $(m,n)-$closed $\delta-$primary for $1 \leq m \leq n$; so we usually assume that $1 \leq n < m$. We give our first trivial result.

Theorem 3.2. *Let R be a commutative ring, I a proper ideal of R, δ an expansion function of $Id(R)$ and m and n positive integers. Then,* *(1) I is a semi$-n-$absorbing $\delta-$primary ideal of R if and only if I is an $(n+1,n)-$closed $\delta-$primary ideal of R.*

(2) If I is an $n-$absorbing $\delta-$primary ideal of R, then I is a semi$-n-$absorbing $\delta-$primary ideal of R.

(3) If I is an $(m,n)-$closed $\delta-$primary ideal of R, then I is an $(m,k)-$closed $\delta-$primary ideal of R for every positive integer $k \geq n$.

(4) An $(m,n)-$closed $\delta-$primary ideal of R is $(\acute{m},\acute{n})-$closed $\delta-$primary ideal of R for all positive integers $\acute{m} \leq m$ and $\acute{n} \geq n$.

(5) If $\delta(I)$ is an $(m,n)-$closed ideal of R, then I is an $(m,n)-$closed $\delta-$primary ideal of R.

(6) Any radical ideal I of R, i.e., $\sqrt{I} = I$ is an $(m,n)-$closed $\delta-$primary ideal of R for all positive integers m and n.

(7) An $n-$absorbing ideal of R is an $(m,n)-$closed $\delta-$primary ideal of R for every positive integer m.

Proof. (1)-(6) follows directly from the definitions.
(7) Let I be an $n-$absorbing ideal of R and let $m > n$ be an integer. Assume that $a^m \in I$ for some $a \in R$. Then, $a^n \in I \subseteq \delta(I)$ by ([2], Theorem 2.1 (a)). Thus, I is an $(m,n)-$closed $\delta-$primary ideal of R for $m > n$. Clearly, I is an $(m,n)-$closed $\delta-$primary ideal of R for every integer $1 \leq m \leq n$. Hence, I is an $(m,n)-$closed $\delta-$primary ideal of R for every positive integer m. □

In the following example, we give some expansion functions of ideals of a ring R.

Example 3.3. (1) The identity function δ_I, where $\delta_I(I) = I$ for every $I \in Id(R)$, is an expansion function of ideals of R.

(2) For each ideal I, define $\delta_{\sqrt{I}}(I) = \sqrt{I}$. Then, $\delta_{\sqrt{I}}$ is an expansion function of ideals of R.

Remark 3.4. Let R be a commutative ring, δ_I an expansion function of $Id(R)$ and m and n positive integers, then a proper ideal I of R is an $(m,n)-$closed δ_I-primary ideal of R if and only if I is an $(m,n)-$closed ideal of R.

Example 3.5. It is clear that, any $n-$absorbing ideal of R is an $(n+1,n)-$absorbing ideal of R. However, this need not be true for semi$-n-$absorbing $\delta-$primary ideals. For example, let $R = \mathbb{Z}$ and $I = 12\mathbb{Z}$. Then $\delta_{\sqrt{I}}(I) = \sqrt{I} = 6\mathbb{Z}$. One can easily see that, I is a semi$-1-$absorbing (i.e., $(2,1)-$closed) $\delta_{\sqrt{I}}-$primary ideal of R, but not a semi$-2-$absorbing (i.e, $(3,2)$-closed) $\delta_{\sqrt{I}}-$primary ideal of R.

Theorem 3.6. *Let R be a commutative ring, δ and γ expansion functions of $Id(R)$ with $\delta(I) \subseteq \gamma(I)$, and m and n integers with $1 \leq n < m$. If I is an $(m,n)-$closed $\delta-$primary ideal of R, then I is an $(m,n)-$closed $\gamma-$primary ideal of R.*

Proof. Since $\delta(I) \subseteq \gamma(I)$ and I is an $(m,n)-$closed $\delta-$primary ideal of R, then the claim follows. □

Theorem 3.7. *Let R be a commutative ring, δ an expansion function of $Id(R)$, m and n positive integers, and I an $(m,n)-$closed $\delta-$primary ideal of R with $\sqrt{\delta(I)} = \delta(\sqrt{I})$. Then, $\sqrt{I}$ is an $(m,n)-$closed $\delta-$primary ideal of R.*

Proof. Let $a \in R$ such that $a^m \in \sqrt{I}$. Then, there exists a positive integer k such that $(a^m)^k \in I$. Since I is an $(m,n)-$closed $\delta-$primary ideal of R, we conclude that $(a^n)^k \in \delta(I)$. Hence, $a^n \in \sqrt{\delta(I)} = \delta(\sqrt{I})$. Thus, $\sqrt{I}$ is an $(m,n)-$closed $\delta-$primary ideal of R. □

Example 3.8. Let R be a commutative ring, δ be an expansion function of $Id(R)$, m and n are positive integers, and $J \subseteq I$ be proper ideals of R. Suppose that I is an $(m,n)-$closed $\delta-$primary ideal of R such that $\delta(J) = \delta(I)$. Let K be an ideal of R such that $J \subseteq K \subseteq I$. Then, K is an $(m,n)-$closed $\delta-$primary ideal of R. To show this, let $a^m \in K \subseteq I$ for some $a \in R$. Since $J \subseteq K \subseteq I$ and $\delta(J) = \delta(I)$, we conclude that $\delta(K) = \delta(I)$, and thus $a^n \in \delta(I) = \delta(K)$.

Theorem 3.9. *Let R be a commutative ring, δ an expansion function of $Id(R)$, and m and n positive integers. If $I_1, ..., I_k$ are radical ideals of R, then $I_1...I_k$ is an $(m,n)-$closed $\delta-$primary ideal of R for all integers $m > 1$ and $n \geq min\{m,k\}$. In particular, $I_1...I_k$ is a semi-$k-$absorbing $\delta-$primary ideal (i.e. $(k+1,k)-$closed $\delta-$primary ideal) of R.*

Proof. Let $a^m \in I_1...I_k$ for some $a \in R$. Then, $a^m \in I_j$ for every $j \in \{1, ..., k\}$, and thus $a \in I_j$ since I_j is a radical ideal of R. Thus, $a^k \in I_1...I_k$. Hence, $a^n \in I_1...I_k \subseteq \delta(I_1...I_k)$ for $n \geq min\{m,k\}$. In particular statement is clear. □

Theorem 3.10. *Let R be a commutative ring, δ an expansion function of $Id(R)$, and m and n positive integers.*

(1) If I is a proper ideal of R with $\delta(\delta(I)) = \delta(I)$, then $\delta(I)$ is an $(m,n)-$closed ideal of R if and only if $\delta(I)$ is an $(m,n)-$closed $\delta-$primary ideal of R.

(2) Suppose that $\delta(0)$ is an $(m,n)-$closed $\delta-$primary ideal of R with $\delta(\delta(0)) = \delta(0)$. Then, $\delta(0)$ is an $(m,n)-$closed ideal of R.

Proof. (1) The necessary part is clear. For the sufficient part, assume that $a^m \in \delta(I)$ for some $a \in R$. Since $\delta(I)$ is an $(m,n)-$closed $\delta-$primary ideal of R, then we have $a^n \in \delta(\delta(I)) = \delta(I)$. Hence, $\delta(I)$ is an $(m,n)-$closed ideal of R.

(2) The proof is similar to that in case (1).

□

We recall the following definition.

Definition 3.11. [14] Let $f : R \longrightarrow A$ be a ring homomorphism and δ, γ expansion functions of $Id(R)$ and $Id(A)$ respectively. Then, f is called a $\delta\gamma-$homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for all ideals I of A.

If γ_r is a radical operation on A and δ_r is a radical operation on R, then any homomorphism from R to A is an example of $\delta_r\gamma_r-$homomorphism. Also, if f is a $\delta\gamma-$epimorphism and I is an ideal of R containing $Ker(f)$, then $\gamma(f(I)) = f(\delta(I))$. In particular, if f is a $\delta\gamma-$ring$-$isomorphism, then $f(\delta(I)) = \gamma(f(I))$ for every ideal I of R.

Theorem 3.12. *Let R and A be commutative rings, m and n positive integers, and $f : R \longrightarrow A$ a $\delta\gamma-$homomorphism, where δ is an expansion function of $Id(R)$ and γ is an expansion function of $Id(A)$.*

(1) If J is an $(m,n)-$closed $\gamma-$primary ideal (respectively, semi$-n-$absorbing $\gamma-$primary ideal) of A, then $f^{-1}(J)$ is an $(m,n)-$closed $\delta-$primary ideal (respectively, semi$-n-$absorbing $\delta-$primary ideal) of R.

(2) If f is surjective and I is an $(m,n)-$closed $\delta-$primary ideal (respectively, semi$-n-$absorbing $\delta-$primary ideal) of R containing $Ker(f)$, then $f(I)$ is an $(m,n)-$closed $\gamma-$primary ideal (respectively, semi$-n-$absorbing $\gamma-$primary ideal) of A.

Proof. (1) Assume that J is an $(m,n)-$closed $\gamma-$primary ideal of A and $a^m \in f^{-1}(J)$ for some $a \in R$. Then, $f(a^m) = [f(a)]^m \in J$. By our assumption, we conclude that $[f(a)]^n \in \gamma(J)$. Thus, $a^n \in f^{-1}(\gamma(J))$. Since $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$, we get $a^n \in \delta(f^{-1}(J))$. Therefore, $f^{-1}(J)$ is an $(m,n)-$closed $\delta-$primary ideal of R. Similarly, it can be verified for semi$-n-$absorbing $\gamma-$primary ideal of A.

(2) Assume that I is an $(m,n)-$closed $\delta-$primary ideal of R and $b^m \in f(I)$ for some $b \in A$. Since f is epimorphism, we have $f(a^m) = b^m$ for some $a \in R$. Thus, $f(a^m) \in f(I)$ and so $a^m \in I$ since $Ker(f) \subseteq I$. As I is an $(m,n)-$closed $\delta-$primary ideal of R, we have $a^n \in \delta(I)$. Then, we have $b^n \in \gamma(f(I))$. Therefore, $f(I)$ is an $(m,n)-$closed $\gamma-$primary ideal of A. Similarly, it can be verified for semi$-n-$absorbing $\delta-$primary ideal of R.

□

Let δ be an expansion function of $Id(R)$ and I a proper ideal of R. The function $\delta_q : R/I \longrightarrow R/I$ defined by $\delta_q(J/I) = \delta(J)/I$ for ideals $I \subseteq J$, becomes an expansion function of R/I. The next corollary is a result of Theorem 3.12 and extends $n-$absorbing ideals in ([2], Corollary 4.3) to $(m,n)-$closed $\delta-$primary ideals, so the proof will be omitted.

Corollary 3.13. *Let R be a commutative ring, $I \subseteq J$ proper ideals of R, δ an expansion function of $Id(R)$ and m and n positive integers. Then, J/I is an $(m,n)-$closed δ_q-primary ideal (respectively, semi$-n-$absorbing δ_q-primary ideal) of R/I if and only if J is an $(m,n)-$closed $\delta-$primary ideal (respectively, semi$-n-$absorbing $\delta-$primary ideal) of R.*

Let R be a commutative ring, m and n positive integers, and δ an expansion function of $Id(R)$. As in [11], a proper ideal I of a commutative ring R is said to be a strongly $n-$absorbing $\delta-$primary ideal of R if whenever $I_1...I_{n+1} \subseteq I$ for some ideals $I_1, ..., I_{n+1}$ of R, then $I_1...I_n \subseteq I$ or there exists $1 \leq k \leq n$ such that $I_1...\widehat{I_k}...I_{n+1} \subseteq \delta(I)$, where $I_1...\widehat{I_k}...I_{n+1}$ denotes the product of $I_1...I_{k-1}I_{k+1}...I_{n+1}$. Clearly, a strongly $n-$absorbing $\delta-$primary ideal of R is an $n-$absorbing $\delta-$primary ideal of R and in [11] the authors conjectured that the two concepts are equivalent. Analogously, we define a proper ideal I of R to be a strongly semi$-n-$absorbing $\delta-$primary ideal of R if whenever $J^{n+1} \subseteq I$ for some ideal J of R implies $J^n \subseteq \delta(I)$, and more generally, a proper ideal I of R is said to be a strongly $(m,n)-$closed $\delta-$primary ideal of R if whenever $J^m \subseteq I$ for some ideal J of R implies $J^n \subseteq I$. It is clear that a strongly $(m,n)-$closed $\delta-$primary ideal of R is also an $(m,n)-$closed $\delta-$primary ideal of R. However, an $(m,n)-$closed $\delta-$primary ideal of R need not be a strongly $(m,n)-$closed $\delta-$primary ideal of R.

Example 3.14. Let δ_I be the identity expansion function of $Id(R)$, where $R = \mathbb{Z}[X,Y]$. Let $I = (X^2, 2XY, Y^2)$, then $J = \sqrt{I} = (X,Y)$. Assume that $x^m \in I$ for some $x \in R$ and m a positive integer. Then, $x \in \sqrt{I}$, and thus $x = aX + bY$ for some $a, b \in R$. Hence, $x^2 = (aX + bY)^2 = a^2X^2 + 2abXY + b^2Y^2 \in \delta_I(I)$. Thus, I is an $(m,2)-$closed δ_I-primary ideal of R for every positive integer m. One can check that $J^m \subseteq I \; \forall\, m \geq 3$. Now, $XY \notin I$, so $J^2 \nsubseteq \delta_I(I)$ and hence I is not a strongly $(m,2)-$closed δ_I-primary ideal of $R \; \forall\, m \geq 3$.

Theorem 3.15. *Let R be a commutative ring, δ an expansion function of $Id(R)$, m a positive integer, J an $(m,2)-$closed $\delta-$primary ideal of R, and I an ideal of R.*

(1) If $I^m \subseteq J$, then $2I^2 \subseteq \delta(J)$.

(2) If $I^m \subseteq J$ and $2 \in U(R)$, then $I^2 \subseteq \delta(J)$ (i.e., J is a strongly $(m,2)-$closed $\delta-$primary ideal of R).

Proof. (1) Let $a, b \in I$. Then, a^m, b^m, $(a+b)^m \in J$ since $I^m \subseteq J$, and thus a^2, b^2, $(a+b)^2 \in \delta(J)$ since J is an $(m,2)-$closed $\delta-$primary ideal of R. Hence, $2ab = (a+b)^2 - a^2 - b^2 \in \delta(J)$, and therefore, $2I^2 \subseteq \delta(J)$.

(2) Follows directly from (1).

□

Theorem 3.16. *Let R be a commutative ring, m and n positive integers, $S \subseteq R \setminus \{0\}$ a multiplicative set with $I \cap S = \phi$, δ_S an expansion function of $Id(R_S)$ such that $\delta_S(I_S) = (\delta(I))_S$, where δ is an expansion function of $Id(R)$, and I an $(m,n)-$closed $\delta-$primary ideal of R.*

(1) I_S is an $(m,n)-$closed δ_S-primary ideal of R_S. In particular, if I is a semi$-n-$absorbing $\delta-$primary ideal of R, then I_S is a semi$-n-$absorbing δ_S-primary ideal of R_S.

(2) If $n = 2$, $2 \in S$, and $J^m \subseteq I_S$ for some ideal J of R_S, then $J^2 \subseteq I_S$ (i.e., I_S is a strongly $(m,n)-$closed δ_S-primary ideal of R_S).

Proof. (1) Let $a^m \in I_S$ for $a \in R_S$. Then, $a = \frac{r}{s}$ for some $r \in R$ and $s \in S$, and thus $a^m = \frac{r^m}{s^m} = \frac{i}{g}$ for some $i \in I$ and $g \in S$. Hence, $r^m gz = s^m iz \in I$ for some $z \in S$, and thus $(rgz)^m \in I$. Hence, $(rgz)^n \in \delta(I)$ since I is $(m,n)-$closed $\delta-$primary ideal of R, and thus $a^n = \frac{r^n}{s^n} = \frac{r^n g^n z^n}{s^n g^n z^n} \in I_S \subseteq (\delta(I))_S = \delta_S(I_S)$. Hence, I_S is an $(m,n)-$closed δ_S-primary ideal of R_S. In the same way, it is easy to show that I_S is a semi-n-absorbing δ_S-primary ideal of R_S.

(2) Assume that $J^m \subseteq I_S$ for some ideal J of R_S. Then, $2 \in U(R_S)$ since $2 \in S$. Thus, $J^2 \subseteq I_S$ by Theorem 3.15 (2).

□

Recall from [14] that an expansion function δ of $Id(R)$ satisfies the finite intersection property if $\delta(I_1 \cap ... \cap I_n) = \delta(I_1) \cap ... \cap \delta(I_n)$ for some ideals $I_1, ..., I_n$ of the commutative ring R. Note that the radical operation on ideals of a commutative ring is an example of an expansion function satisfying the finite intersection property. The next result generalizes Theorem 3.9.

Theorem 3.17. *Let R be a commutative ring, δ an expansion function of $Id(R)$ satisfying the finite intersection property, $m_1, ..., m_k, n_1, ..., n_k$ positive integers, and $I_1, ..., I_k$ proper ideals of R.*

(1) If I_j is an $(m_j, n_j)-$closed $\delta-$primary ideal of R for all $j \in \{1, ..., k\}$ and $P = \delta(I_j)$ for all $j \in \{1, ..., k\}$, then $I_1 \cap ... \cap I_k$ is an $(m, n)-$closed $\delta-$primary ideal of R for all positive integers $m \leq min\{m_1, ..., m_k\}$ and $n \geq min\{m, max\{n_1, ..., n_k\}\}$.

(2) If I_j is an $(m_j, n_j)-$closed $\delta-$primary ideal of R for all $j \in \{1, ..., k\}$, then $I_1...I_k$ is an $(m, n)-$closed $\delta-$primary ideal of R for all positive integers $m \leq min\{m_1, ..., m_k\}$ and $n \geq min\{n, n_1 + ... + n_k\}$.

Proof. (1) Let $a^m \in I_1 \cap ... \cap I_k$ for some $a \in R$, $m \leq min\{m_1, ..., m_k\}$, and $j \in \{1, ..., k\}$. Then $a^m \in I_j$, and thus $a^{m_j} \in I_j$. Thus, $a^{n_j} \in \delta(I_j) = P$ since I_j is an $(m_j, n_j)-$closed $\delta-$primary ideal of R. Thus, $a^n \in \delta(I_1 \cap ... \cap I_k) = \delta(I_1) \cap ... \cap \delta(I_k) = P$ for $n \geq max\{n_1, ..., n_k\}$. Hence, $a^n \in \delta(I_1 \cap ... \cap I_k) = \delta(I_1) \cap ... \cap \delta(I_k) = P$ for $n \geq min\{m, max\{n_1, ..., n_k\}\}$.

(2) Let $a^m \in I_1...I_k$ for some $a \in R$, $m \leq min\{m_1, ..., m_k\}$, and $j \in \{1, ..., k\}$. Then, $a^m \in I_j$, and thus $a^{m_j} \in I_j$. Thus, $a^{n_j} \in \delta(I_j)$ since I_j is an $(m_j, n_j)-$closed $\delta-$primary ideal of R. Hence, $a^{n_1+...+n_k} \in \delta(I_1...I_k)$ and therefore, $a^n \in \delta(I_1...I_k)$ for $n \geq n_1 + ... + n_k$. Thus, $a^n \in \delta(I_1...I_k)$ for $n \geq min\{n, n_1 + ... + n_k\}$.

□

Corollary 3.18. *Let R be a commutative ring, δ an expansion function of $Id(R)$ satisfying the finite intersection property, m and n positive integers, and $I_1, ..., I_k$ proper ideals of R. If $I_1, ..., I_k$ are $(m, n)-$closed $\delta-$primary ideal (respectively, semi$-n-$absorbing ideals) of R, and $P = \delta(I_j)$ for all $j \in \{1, ..., k\}$, then $I_1 \cap ... \cap I_k$ is an $(m, n)-$closed $\delta-$primary ideal (respectively, semi$-n-$absorbing ideal) of R.*

Proof. It is clear. □

Theorem 3.19. *Let D be an integral domain, δ an expansion function of $Id(D)$, m and n integers with $1 \leq n < m$, and $I = p^j D$, where p is a prime element of D and j is a positive integer. Suppose that the following two statements hold.*
(i) $j = mx + r$, where x and r are positive integers with $x \geq 0$, $1 \leq r \leq n$, $x(mod\, n) + r \leq n$; and
(ii) $x \neq 0 \implies m = n + y$ for some integer y with $1 \leq y \leq n - 1$.
Then I is an $(m, n)-$closed $\delta-$primary ideal of D.

Proof. Assume that $j = mx + r$, where x and r are integers with $x \geq 0$, $1 \leq r \leq n$, $x(mod\, n) + r \leq n$, and if $x \neq 0$, then $m = n + y$ for some integer y such that $1 \leq y \leq n - 1$. Suppose that $a^m \in I$ for some $a \in D$. We have two cases.
Case(i): $x = 0$. Then, $j = r$, and thus $1 \leq j \leq n$. Then, $p|a$, and hence $p^j|a^j$. Hence, $p^j|a^n$ since $n \geq j$, and thus $a^n \in I \subseteq \delta(I)$.
Case(ii): $x \neq 0$. We show that $p^j|a^n$, and hence $a^n \in I \subseteq \delta(I)$. Then, $p|a$ and $p^j|a^m$ since $a^m \in I$. If $p^j|a$, then $a^n \in I$. So assume that $p^j \nmid a$. Let i be the largest positive integer such that $p^i|a$. Thus, $p^{mi}|a^m$ and mi is the largest positive integer such that $p^{mi}|a^m$. Hence, $mi > j$, so $0 > j - mi = (mx + r) - mi = m(x - i) + r$. Since $1 \leq r \leq n$, we have $i > x$. Thus, $i = x + z$ for some integer $z \geq 1$. Then, $j = mx + r$ and $m = n + y$ give $\frac{j}{n} = \frac{mx+r}{n} = \frac{(n+y)x+r}{n} = \frac{nx+yx+r}{n} = x + \frac{cx+r}{n} \leq x + 1$ since $xc + r = x(m \bmod n) + r \leq n$. Since $z \geq 1$, we have $i = x + z \geq x + 1 \geq \frac{j}{n}$, and hence $ni \geq j$. Thus, $p^{ni}|a^n$ since $p^i|a$, and thus $p^j|a^n$ since $ni \geq j$. Thus, $a^n \in I \subseteq \delta(I)$. Thus, I is an $(m, n)-$closed $\delta-$primary ideal of D. □

Corollary 3.20. *Let D be an integral domain, δ an expansion function of $Id(D)$, n a positive integer and $I = p^j D$, where p is a prime element of D and j is a positive integer. If $j = (n+1)x + r$, where x and r are positive integers with $x \geq 0$, $1 \leq r < n$, and $x + r \leq n$, then I is a semi$-n-$absorbing $\delta-$primary ideal (i.e. $(n+1, n)-$closed $\delta-$primary ideal) of R.*

Proof. The proof is clear by Theorem 3.19 since an ideal I of R is a semi $-n-$absorbing $\delta-$primary ideal if and only if I is $(n+1, n)-$closed $\delta-$primary ideal. □

Example 3.21. Led D be an integral domain, δ an expansion of $Id(D)$, and $I = p^k D$, where p is a prime ideal of D and k is a positive integer. Then by Corollary 3.20, I is a semi$-2-$absorbing $\delta-$primary ideal (i.e., $(3,2)-$closed $\delta-$primary ideal) of R if and only if $k \in \{1, 2, 4\}$.

Let D be an integral domain. It is well known that if $p_1, ..., p_r$ are non associate prime elements of D, then $p_1^{k_1} D \cap ... \cap\ p_r^{k_r} D = p_1^{k_1} ... p_r^{k_r} D$ for all positive integers $k_1, ..., k_r$. Note that $p_1^{k_1} ... p_r^{k_r} D$ is an $n-$absorbing ideal if and only if $n \geq\ k_1 + ... + k_r$ ([2], Theorem 2.1. (d)).

Theorem 3.22. *Let D be an integral domain, δ an expansion function of $Id(D)$ satisfying the finite intersection property, m and n integers with $1 \leq\ n < m$, and $I = p_1^{k_1} ... p_r^{k_r} D$, where $p_1, ..., p_r$ are non associate prime elements of D and $k_1, ..., k_r$ are positive integers. Then, the following statements are equivalent.*

(1) I is an $(m, n)-$closed $\delta-$primary ideal of D.

(2) $p_i^{k_i} D$ is an $(m, n)-$closed $\delta-$primary ideal of D for every $i \in \{1, ..., r\}$.

Proof. (1) $\Longrightarrow$ (2) Let $I_i = p_i^{k_i} D$. Assume that $a^m \in\ I_i$ for some $a \in\ D$. Let $b = \frac{a(p_1^{k_1} ... p_r^{k_r})}{p_i^{k_i}} \in\ D$. Then, $b^m \in\ I$, and hence $b^n \in \delta(I)$ since I is an $(m, n)-$closed $\delta-$primary ideal of D. By construction, $b^n \in\ I \subseteq \delta(I)$ if and only if $a^n \in\ I_i \subseteq \delta(I_i)$. Thus, I_i is an $(m, n)-$closed $\delta-$primary ideal of D for every $i \in \{1, ..., r\}$.
(2) $\Longrightarrow$ (1) This is clear by Corollary 3.18 since $p_1^{k_1} D \cap ... \cap\ p_r^{k_r} D = p_1^{k_1} ... p_r^{k_r} D$. □

Let D be an integral domain, δ an expansion function of $Id(R)$ satisfying the finite intersection property, and I a proper ideal of D. For fixed positive integers $m, k_1, ..., k_r$, we next determine the smallest positive integer n such that $I = p_1^{k_1} ... p_r^{k_r} D$ is an $(m, n)-$closed $\delta-$primary ideal of D, where $p_1, ..., p_r$ are non associate prime elements of D. Note that $m \geq\ n$ since every proper ideal is $(n, n)-$closed $\delta-$primary and I is $(\acute{m}, n)-$closed $\delta-$primary for every positive integer $\acute{m} \leq\ m$. Also, for fixed positive integers $n, k_1, ..., k_r$, we determine the largest positive integer m such that $I = p_1^{k_1} ... p_r^{k_r} D$ is an $(m, n)-$closed $\delta-$primary ideal of D, where $p_1, ..., p_r$ are non associate prime elements of D. Note that $n \geq\ m$ since every proper ideal is $(m, m)-$closed $\delta-$primary and I is $(m, \acute{n})-$closed $\delta-$primary for every positive integer $\acute{n} \leq\ n$.

Theorem 3.23. *Let D be an integral domain, δ an expansion function of $Id(D)$ satisfying the finite intersection property, and $I = p_1^{k_1} ... p_r^{k_r} D$, where $p_1, ..., p_r$ are non associate prime elements of D and $k_1, ..., k_r$ are positive integers.*

(1) Let m be a positive integer. If n_i is the smallest positive integer such that $p_i^{k_i} D$ is an $(m, n_i)-$closed $\delta-$primary ideal of D for every $i \in \{1, ..., r\}$, then $n = max\{n_1, ..., n_r\}$ is the smallest positive integer such that I is an $(m, n)-$closed $\delta-$primary ideal of D.

(2) Let n be a positive integer. If m_i is the smallest positive integer such that $p_i^{k_i} D$ is an $(m_i, n)-$closed $\delta-$primary ideal of D for every $i \in \{1, ..., r\}$, then $m = min\{m_1, ..., m_r\}$ is the largest positive integer such that I is an $(m, n)-$closed $\delta-$primary ideal of D.

Proof. This follows since I is an $(m, n)-$closed $\delta-$primary ideal of D if and only if every $p_i^{k_i} D$ is an $(m, n)-$closed $\delta-$primary ideal of D by Theorem 3.22. □

Recall that an ideal of $R_1 \times\ R_2$ has the form $I_1 \times\ I_2$ for some ideals I_1 of R_1 and I_2 of R_2, where R_1 and R_2 are commutative rings. Let $R = R_1 \times ... \times\ R_k$, where R_i is a commutative ring with nonzero identity and δ_i be an expansion function of $Id(R_i)$ for each $i \in \{1, 2, ..., k\}$. Let $\delta_\times$ be a function of $Id(R)$, which is defined by $\delta_\times(I_1 \times\ I_2 \times ... \times\ I_k) = \delta_1(I_1) \times \delta_2(I_2) \times ... \times \delta_k(I_k)$ for each ideal I_i of R_i were $i \in \{1, 2, ..., k\}$. Then $\delta_\times$ is an expansion function of $Id(R)$. Note that every ideal of R is of the form $I_1 \times\ I_2 \times\ ... \times\ I_k$, where each ideal I_i is an ideal of R_i, $i \in \{1, 2, ..., k\}$.

Theorem 3.24. *Let $R = R_1 \times R_2$, where where R_1 and R_2 are commutative rings, δ_i be an expansion function of $Id(R_i)$ for each $i \in \{1,2\}$, J a proper ideal of R, and m and n positive integers. Then the following statements are equivalent.*

(1) J is an $(m,n)-$closed $\delta_\times-$primary ideal of R.

(2) $J = I_1 \times R_2$, or $J = R_1 \times I_2$ or $J = I_1 \times I_2$ for $(m,n)-$closed δ_1-primary ideal I_1 of R_1 and $(m,n)-$closed δ_2-primary ideal I_2 of R_2.

Proof. Follows directly from the definitions. □

4 $(m,n)-$closed $\delta-$primary ideals in trivial ring extensions of commutative rings

Let R be a commutative ring, δ an expansion function of $Id(R)$, M an $R-$module and m and n positive integers. In this suction, we study $(m,n)-$closed $\delta-$primary ideals in trivial ring extension of R by M (or the idealization of M over R) that is denoted by $R(+)M$. As in [13], $R(+)M = \{(a,b) : a \in R,\ b \in M\}$ is a commutative ring with identity $(1,0)$ under addition defined by $(a,b)+(c,d) = (a+c, b+d)$ and multiplication defined by $(a,b)(c,d) = (ac, ad+bc)$ for each $a,c \in R$ and $b,d \in M$. Note that $(\{0\}(+)M)^2 = \{0\}$, so $\{0\}(+)M \subseteq Nil(R(+)M)$. We define a function $\delta_{(+)} : Id(R(+)M) \longrightarrow Id(R(+)M)$ such that $\delta_{(+)}(I(+)N) = \delta(I)(+)M$ for every ideal I of R and every submodule N of M. Then, $\delta_{(+)}$ is an expansion function of ideals of $R(+)M$.

Theorem 4.1. *Let R be a commutative ring, δ an expansion function of $Id(R)$, n a positive integer, I a proper ideal of R, and M an $R-$module. Then,*

(1) I is an $n-$absorbing $\delta-$primary ideal of R if and only if $I(+)M$ is an $n-$absorbing $\delta_{(+)}-$primary ideal of $R(+)M$.

(2) I is a strongly $n-$absorbing $\delta-$primary ideal of R if and only if $I(+)M$ is a strongly $n-$absorbing $\delta_{(+)}-$primary ideal of $R(+)M$.

(3) I is an $(m,n)-$closed $\delta-$primary ideal of R if and only if $I(+)M$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of $R(+)M$.

Proof. It is clear. □

Lemma 4.2. *Let D be an integral domain with quotient field K, M a $K-$vector space, F a $K-$subspase of M, δ an expansion function of $Id(D)$, and n a positive integer. Then, $\{0\}(+)F$ is a strongly $2-$absorbing $\delta_{(+)}-$primary ideal of $D(+)M$, and hence $\{0\}(+)F$ is a strongly $n-$absorbing $\delta_{(+)}-$primary ideal of $D(+)M$.*

Proof. First of all, we show that $\{0\}(+)F$ is a $2-$absorbing $\delta_{(+)}-$primary ideal of $D(+)M$. Let $(r_i, e_i) \in D(+)M$, where $1 \leq i \leq 3$. Suppose that $(r_1,e_1)(r_2,e_2)(r_3,e_3) \in \{0\}(+)F$. Since D is an integral domain, we may assume that $r_3 = 0$. Suppose that $r_1r_2 = 0$. Then, $(r_1,e_1)(r_3,e_3) \in \{0\}(+)F$ or $(r_2,e_2)(r_3,e_3) \in \delta_{(+)}(\{0\}(+)F)$. Suppose that $r_1r_2 \neq 0$. Then, $(r_1,e_1)(r_2,e_2)(r_3,e_3) = (0, r_1r_2e_3)$. Since F is a $K-$subspace of M, we conclude that $r_2^{-1}r_1^{-1}(r_1r_2e_3) = e_3 \in F$. Thus, $(r_3,e_3) = (0,e_3) \in \delta_{(+)}(\{0\}(+)F)$, and hence $(r_1,e_1)(r_3,e_3) \in \delta_{(+)}(\{0\}(+)F)$. Thus, $\{0\}(+)F$ is a $2-$absorbing $\delta_{(+)}-$primary ideal of $D(+)M$. Hence, $\{0\}(+)F$ is a strongly $2-$absorbing $\delta_{(+)}-$primary ideal of $D(+)M$ by ([4], Theorem 2.13). □

Theorem 4.3. *Let D be an integral domain with quotient field K, δ an expansion function of $Id(D)$, M a $K-$vector space, and N a $D-$submodule of M. If N is a $K-$subspace of M, then $\{0\}(+)N$ is an $n-$absorbing $\delta_{(+)}-$primary ideal of $D(+)M$ for some integer $n \geq 2$.*

Proof. Follows directly from Lemma 4.2. □

Theorem 4.4. *Let R be a commutative ring, δ an expansion function of $Id(R)$, M be an $A-$module, and $A = R(+)M$. Suppose that $I(+)N$ is a proper ideal of A, where I is a proper ideal of R and N is a submodule of M with $IM \subseteq N$. If I is an $(m,n)-$closed $\delta-$primary ideal of R for some positive integers $n < m$, then $I(+)N$ is an $(m,n+1)-$closed $\delta_{(+)}-$primary ideal of A.*

Proof. Assume that I is an $(m,n)-$closed $\delta-$primary ideal of R for some positive integers $n < m$. Let $a = (x,y) \in A$ and suppose that $a^m = (x^m, mx^{m-1}y) \in I(+)N$. Since I is an $(m,n)-$closed $\delta-$primary ideal of R, so $(x^{n+1},(n+1)x^n y) = a^{n+1} \in \delta_{(+)}(I(+)N)$. Thus, $I(+)N$ is an $(m,n+1)-$closed $\delta_{(+)}-$primary ideal of A. □

Theorem 4.5. *Let R be a commutative ring, δ an expansion function of $Id(R)$, M be an $A-$module, and $A = R(+)M$. Suppose that $I(+)N$ is a proper ideal of A, where I is a proper ideal of R and N is a submodule of M with $IM \subseteq N$. Let m and n be integers with $1 \leq n < m$. Then the following statements are equivalent.*

(1) $I(+)N$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A.

(2) I is an $(m,n)-$closed $\delta-$primary ideal of R and whenever $x^m \in I$ for some $x \in R$ implies $nx^{n-1}M \subseteq N$.

Proof. (1) $\Longrightarrow$ (2) Assume that $I(+)N$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A. Then, it is clear that I is an $(m,n)-$closed $\delta-$primary ideal of R. Suppose that $x^m \in I$ for some $x \in R$ and let $y \in M$ with $a = (x,y)$. Since I is an $(m,n)-$closed $\delta-$primary ideal of R, so $x^n \in \delta(I)$. As $n \leq m-1$, so $x^{m-1} \in I$. Since $IM \subseteq N$ and $x^{m-1} \in I$, we conclude that $a^m = (x^m, mx^{m-1}y) \in (I(+)N)$. Since $I(+)N$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A, we conclude that $a^n = (x^n, nx^{n-1}y) \in \delta_{(+)}(A)$ and therefore $nx^{n-1}M \subseteq N$.
(2) $\Longrightarrow$ (1) Assume that I is an $(m,n)-$closed $\delta-$primary ideal of R and whenever $x^m \in I$ for some $x \in R$ implies $nx^{n-1}M \subseteq N$. Let $a = (x,y) \in A$ for some $x \in R$ and some $y \in M$, and suppose that $a^m = (x^m, mx^{m-1}y) \in I(+)N$. Since $x^m \in I$ and I is an $(m,n)-$closed ideal of R, we have $x^n \in R$ and $nx^{n-1}y \in N$. Thus, $(x^n, nx^{n-1}y) \in \delta_{(+)}(I(+)N)$. Hence, $I(+)N$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A. □

Theorem 4.6. *Let R be a commutative ring, δ an expansion function of $Id(R)$, M be an $A-$module, m and n integers with $1 \leq n < m$, I a proper ideal of R, and $A = R(+)M$. Suppose that $char(R)$ divides n. Then the following statements are equivalent.*

(1) $I(+)N$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A for every submodule N of M where $IM \subseteq N$.

(2) I is an $(m,n)-$closed $\delta-$primary ideal of R.

Proof. (1) $\Longrightarrow$ (2) It is clear by Theorem 4.5.
(2) $\Longrightarrow$ (1) Let N be a submodule of M with $IM \subseteq N$. Since $char(R)$ divides n, so whenever $x^m \in I$ for some $x \in R$ implies $nx^{n-1}M = 0_M \subseteq N$, where 0_M is the additive identity of M. Thus, $I(+)N$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A by Theorem 4.5. □

We recall the following definition.

Definition 4.7. [5] Let p be a prime element of an integral domain D. Suppose that $p^t | d$ for some $d \in D$ and a positive integer t but $p^{t+1} \nmid d$. Then we write $p^t \| d$.

Theorem 4.8. *Let D be an integral domain, $A = D(+)D$, δ an expansion function of $Id(D)$, m and n integers with $1 \leq n < m$, and $I = p^k D$, where p is a prime element of D and k is a positive integer. Assume that $m < k$ and $char(D) \neq n$. Let $x = \lceil \frac{k}{m} \rceil$ and $y = \lceil \frac{k}{x} \rceil$. Then the following statements are equivalent.*

(1) $I(+)p^i D$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A for some integer $i \geq 1$.

(2) One of the following three cases must hold:
(a) $y < n < m$ and $i \leq k$.
(b) $y = n$, $p \nmid n.1_D(in\ D)$, where 1_D is the identity of D and $i \leq x(n-1) < k$.
(c) $y = n$, $p^t \| n.1_D(in\ D)$, and $i \leq min\{x(n-1)+t, k\}$.

Proof. (1) $\Longrightarrow$ (2) Assume that $I(+)p^i D$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A for some integer $i \geq 1$. Since $I(+)p^i p$ is an ideal of D, we have $I \subseteq p^i D$. Thus, $i \leq k$. Now $x = \lceil \frac{k}{m} \rceil$ is the smallest positive integer where $(p^x)^m \in I$. Also, $y = \lceil \frac{k}{x} \rceil$ is the smallest positive integer where $(p^x)^y \in I$. Since $I(+)p^i D$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A and $1 \leq n < m$,

we have $y \leq n < m$. Since $I(+)p^iD$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A, we have I is an $(m,n)-$closed $\delta-$primary ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$ by Theorem 4.5. Hence, since $(p^x)^m \in I$, we have $n(p^x)^{n-1} \in p^iD$ by Theorem 4.5. If $y < n < m$, then $y < n-1$ and thus $n(p^x)^{n-1} \in p^kD = I$ (note that $(p^x)^y \in I$ and $i \leq k$). Assume that $n = y$ and $p \nmid n.1_D(in\ D)$. Since y is the smallest positive integer where $(p^x)^y \in I$ and $p \nmid n.1_D$, we have $x(n-1) < k$ and $n(p^x)^{n-1} \in p^iD$ if and only if $i \leq x(n-1) < k$. Assume that $y = n$ and $p^t \| n.1_D(in\ D)$. Since $i \leq x(n-1) < k$, we have $n(p^x)^{n-1} \in p^iD$ if and only if $i \leq min\{x(n-1)+t, k\}$.
(2) $\Longrightarrow$ (1) In view of proof (1) $\Longrightarrow$ (2) above, one can easily see that if (a), (b) or (c) holds, then I is an $(m,n)-$closed $\delta-$primary ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$. Thus, $I(+)p^iD$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A by Theorem 4.5. □

Theorem 4.9. *Let D be an integral domain, $A = D(+)D$, δ an expansion function of $Id(D)$, m and n integers with $1 \leq n < m$, and $I = p^kD$, where p is a prime element of D and k is a positive integer. Assume that $m > k$ and $char(D) \neq n$. If one of the following three cases hold:*
(a) $k < n < m$ and $i \leq k$ for some integer $i \geq 1$.
(b) $n = k$, and $1 \leq i < k$.
(c) $n = i = k$, and $p|k.1_D(in\ D)$, where 1_D is the identity of D.
Then, $I(+)p^iD$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A.

Proof. If (a), (b) or (c) holds, then one can easily verify that I is an $(m,n)-$closed $\delta-$primary ideal of D and whenever $x^m \in I$ for some $x \in D$ implies $nx^{n-1}D \subseteq p^iD$. Hence, $I(+)p^iD$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of A by Theorem 4.5. □

Theorem 4.10. *Let D be an integral domain with quotient field K, M a $K-$vector space, and δ an expansion function of $Id(D)$. Then the following statements are equivalent.*

(1) Every proper ideal of D is an $(m,n)-$closed $\delta-$primary ideal of D for some integers $1 \leq n < m$.

(2) Every proper ideal of $D(+)M$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of $D(+)M$ for some integers $1 \leq n < m$.

Proof. (1) $\Longrightarrow$ (2) Assume that every proper ideal of D is an $(m,n)-$closed $\delta-$primary ideal of D for some integers $1 \leq n < m$, and let I be an ideal of D. Since M is a divisor $D-$module, we have $I = J(+)M$ for some proper ideal J of D or $I = \{0\}(+)N$ for some $D-$submodule N of M by ([1], Corollary 3.4). If $I = J(+)M$, then it is clear that I is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of $D(+)M$ since J is an $(m,n)-$closed $\delta-$primary ideal of D for some integers $1 \leq n < m$. If $I = \{0\}(+)N$, then I is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of $D(+)M$ for every integer $m \geq 3$ since D is an integral domain. Therefore, every proper ideal of $D(+)M$ is an $(m,n)-$closed $\delta_{(+)}-$primary ideal of $D(+)M$ for some integers $1 \leq n < m$.
(2) $\Longrightarrow$ (1) It is clear. □

5 Conclusion

Here, we represented a new form of the theory of absorbing ideals of commutative rings. We discussed and proved new theorems in this area. We proved that if $S \subseteq R\backslash\{0\}$ is a multiplicative set with $I \cap S = \phi$, δ_S is an expansion function of $Id(R_S)$ such that $\delta_S(I_S) = (\delta(I))_S$, where δ is an expansion function of $Id(R)$, and I is an $(m,n)-$closed $\delta-$primary ideal of R, then I_S is an $(m,n)-$closed δ_S-primary ideal of R_S. Also, we considered the trivial ring extensions of commutative rings and studied the properties of $(m,n)-$closed $\delta-$primary ideals in these rings. We can generalize the concept of $(m,n)-$closed $\delta-$primary ideals to the concept of weakly $(m,n)-$closed $\delta-$ primary ideals in the next work.

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Author information

Mohammad Hamoda, Department of Mathematics, Faculty of Applied Science, Al-Aqsa University, 4051, Palestine.
E-mail: `ma.hmodeh@alaqsa.edu.ps`

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