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BOUNDS FOR SOMBOR EIGENVALUE AND ENERGY OF A GRAPH IN TERMS OF HYPER ZAGREB AND ZAGREB INDICES

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Abstract The Sombor matrix $SOM(G)$ of a graph G is defined as a matrix, where every element of it is defined as $\sum_{i=1}^{n}\sqrt{d_i^2+d_j^2}$ if and only if the vertices v_i and v_j with degrees d_i and d_j are adjacent, otherwise it is zero. Based on this matrix we calculate the Sombor energy and Sombor eigenvalues for some standard graphs. Furthermore, we derive the bounds for the largest eigenvalue and for the partial sum of eigenvalues of Sombor matrix in terms of Zagreb indices, Hyper Zagreb index and Forgotten index.

1 Introduction

Topological indices play a prominent role in chemical graph theory as a molecular descriptors. Recently I. Gutman has introduced a vertex based index called as Sombor index, which is defined for a graph G as $\sum_{i=1}^{n}\sqrt{d_i^2+d_j^2}$ where v_i are the vertices of graph G having degrees d_i, $i=1,2,3,\cdots,n$ [9]. Most recently much of the work is carried out on this index [4,5,14,17,20]. In literature many matrices have been defined on graphs with respect to its adjacency, incidence, distance and many more based on vertex degrees relating to topological indices. Many researchers have defined the energy of graphs in terms of the defined matrices and have studied the bounds for energy [13, 18]. This has motivated us to define a Sombor matrix for a graph, obtain bounds for its eigenvalues and Sombor energy in term of other topological indices. Further we have discussed the Sombor energy for some standard graphs in terms of other indices. Topological indices stated in literature [6–9, 19] which are considered in this paper are

- First Zagreb index $(Zg_1)=\sum_{i=1}^{n}d_i^2=\sum_{edge\ e=ij}(d_i+d_j)$
- Second Zagreb index $Zg_2=\sum_{edge\ e=ij}d_id_j$
- Hyper Zagreb index $HZg=\sum_{edge\ e=ij}(d_i+d_j)^2$
- Forgotten index $FI=\sum_{edge\ e=ij}(d_i^2+d_j^2)$
- Sombor index $SO=\sum_{edge\ e=ij}\sqrt{d_i^2+d_j^2}$.

In this paper we represent a graph in terms of Sombor matrix $SOM(G)$ which is defined as

$$SOM(G)=[dso_{ij}]=\begin{cases}\sqrt{d_i^2+d_j^2} & \text{if } v_i \text{ is adjacent to } v_j\\ 0 & \text{otherwise.}\end{cases}$$

where G is a simple graph with vertices $v_1,v_2,\cdots,v_n$ having degrees $d_1,d_2,\cdots,d_n$. The characteristic polynomial of $SOM(G)$ matrix is

$$P_{SOM}(G) = \mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + \cdots + a_n.$$

As $SOM(G)$ is real and symmetric matrix, its real eigenvalues are arranged as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Lemma 1.1. *Let G be a simple connected graph, with every vertex v_i having the degree d_i, $i = 1, 2, \ldots, n$. Let $SOM(G)$ be the Sombor matrix of G. Then*

$$\sum_{i=1}^{n} \mu_i = 0. \tag{1.1}$$

$$\sum_{i=1}^{n} \mu_i^2 = 2 \sum_{1\leq i<j\leq n} (d_i^2 + d_j^2). \tag{1.2}$$

Lemma 1.2. *Let G be a simple graph of order n, with every vertex v_i having the degree d_i, $i = 1, 2, \ldots, n$. Let $SOM(G)$ be the Sombor matrix of G with $\mu_1, \mu_2, \ldots, \mu_n$ as its eigenvalues. Let $Zg_1(G)$, $Zg_2(G)$, HZg and $FI(G)$ be the first Zagreb index, second Zagreb index hyper Zagreb index and Forgotten index. Then,*

$$\begin{aligned}
\sum_{i=1}^{n} \mu_i^2 &= 2 \sum_{1\leq i<j\leq n} (d_i^2 + d_j^2) = 2FI(G) && (1.3)\\
&= 2\sum_i (d_i^3) = 2\left[\sum_{edge\ e=ij} (d_i + d_j) + \sum_{i=1}^{n} d_i^2(d_i - 1)\right] \\
&= 2\left[Zg_1 + \sum_{i=1}^{n} d_i^2(d_i - 1)\right] && (1.4)\\
\sum_{i=1}^{n} \mu_i^2 &= 2 \sum_{1\leq i<j\leq n} (d_i^2 + d_j^2) = 2 \sum_{1\leq i<j\leq n} ((d_i + d_j)^2 - 2d_i d_j) \\
&= 2HZg - 4Zg_2. && (1.5)
\end{aligned}$$

Lemma 1.3. *Let G be a graph with n vertices and m edges, with adjacency eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Consider another graph H with n vertices having vertex degrees as $d_1, d_2, \ldots, d_n$ and let the Sombor eigenvalues of H be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then*

$$\begin{aligned}
\left(\sum_{i=1}^{n} \lambda_i \mu_i\right)^2 &\leq \sqrt{4m\left[Zg_1 + \sum_{i=1}^{n} d_i^2(d_i - 1)\right]} && (1.6)\\
&\leq \sqrt{4m\left[HZg - 2Zg_2\right]} && (1.7)\\
&\leq 2\sqrt{mFI} && (1.8)
\end{aligned}$$

Proof. By using Cauchy-Schwartz inequality [2] and Lemma 1.2 we have,

$$\begin{aligned}
\sum_{i=1}^{n} (\lambda_i \mu_i)^2 &\leq \left(\sum_{i=1}^{n} \lambda^2\right)\left(\sum_{i=1}^{n} \mu^2\right) \\
&= 2m(2[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]) \\
\sum_{i=1}^{n} \lambda_i \mu_i &\leq \sqrt{4m[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]}.
\end{aligned}$$

Similarly we get the other two results.
The equality holds if $\lambda_i = \mu_i$ for all i. □

2 Bounds for spectra of $SOM(G)$

If G is a simple graph having n vertices and e edges with adjacency eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then for $1 \leq k \leq n$, [1]

$$\sqrt{\frac{(n-k)2e}{nk}} \geq \lambda_k \geq -\sqrt{\frac{(k-1)2e}{n(n-k+1)}} \tag{2.1}$$

For adjacency matrix we have $\sum_{i=1}^{n} \lambda_i^2 = 2e$. For Sombor matrix, the terms $2[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]$, $2\left[HZg_2 - 2Zg_2\right]$ and $2FI$ plays the same role. So the direct outcome of Eq.(2.1) will be the results in Lemma (2.1).

Lemma 2.1. *For a graph G, with Sombor eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and for $1 \leq k \leq n$*

$$\begin{aligned}
\sqrt{\frac{(n-k)2\left[Zg_1 + \sum_{i=1}^{n} d_1^2(d_i-1)\right]}{nk}} \quad &\geq \mu_k \geq \quad -\sqrt{\frac{(k-1)2\left[Zg_1 + \sum_{i=1}^{n} d_1^2(d_i-1)\right]}{n(n-k+1)}} \\
\sqrt{\frac{(n-k)2\left[HZg_2 - 2Zg_2\right]}{nk}} \quad &\geq \mu_k \geq \quad -\sqrt{\frac{(k-1)2\left[HZg_2 - 2Zg_2\right]}{n(n-k+1)}} \\
\sqrt{\frac{(n-k)2FI}{nk}} \quad &\geq \mu_k \geq \quad -\sqrt{\frac{(k-1)2FI}{n(n-k+1)}}.
\end{aligned} \tag{2.2}$$

For complete graph and complete bipartite graphs direct Sombor eigenvalues can be obtained. But for a $r-$regular graph and for a path graph P_n we can obtain the bounds for Sombor eigenvalues using Lemma (2.1) by substituting their corresponding FI.

For $r-$regular graphs: $r\sqrt{\frac{2(n-k)r}{k}} \geq \mu_k \geq -r\sqrt{\frac{2(k-1)r}{(n-k+1)}}$

For a P_n: $2\sqrt{\frac{(n-k)(4n-7)}{nk}} \geq \mu_k \geq -2\sqrt{\frac{(k-1)(4n-7)}{n(n-k+1)}}$.

Theorem 2.1. Let G be a simple graph of order n. Let μ_1 be the largest eigenvalue of $SOM(G)$ and $SO(G)$ be the Sombor index of G. Then

$$\mu_1 \geq \frac{2}{n} SO(G). \tag{2.3}$$

Proof. Let G be a simple graph having n vertices v_i with degree d_i respectively. By the definition of $SOM(G)$ it is observed that the sum of all the entries of $SOM(G)$ is $\sum_{i \neq j} dso_{ij} = \sum_{i \neq j} \sqrt{d_i^2 + d_j^2}$. Let $y = [1, 1, \ldots, 1]$ be the all one vector. Then by Rayleigh principle we have,

$$\begin{aligned}
\mu_1 \quad &\geq \quad \frac{ySOMy^T}{xx^T} = \frac{1}{n}\sum_{i \neq j} \sqrt{d_i^2 + d_j^2} \\
&= \quad \frac{1}{n} 2 \sum_{i<j} \sqrt{d_i^2 + d_j^2} \\
&\geq \quad \frac{2}{n} SO(G).
\end{aligned}$$

If G is a simple $r-$regular graph, then $SO(G) = \frac{nr^2}{\sqrt{2}}$, $\mu_1 = r^2\sqrt{2}$.
Hence equality holds true for regular graph. □

Theorem 2.2. Let G be a graph having n vertices and m edges, with the vertex degrees $d_1, d_2, \ldots, d_n$ and the Sombor eigenvalues be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then

$$\mu_1 \leq \sqrt{\frac{2p}{p-1}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]} + \frac{1}{p-1}\sum_{i=2}^{p} \mu_{n-p+i}. \tag{2.4}$$

Proof. Let $\mu_1, \mu_2, \ldots, \mu_{n-p+1}, \mu_{n-p+2}, \ldots, \mu_n$ be the Sombor eigenvalues of G. Let $H = K_p \cup \overline{K_{n-p}}$. Then the number of edges of H are $m = \dfrac{p(p-1)}{2}$ and the adjacency eigenvalues (λ_i) of H are

$$\begin{pmatrix} p-1 & 0 & -1 \\ 1 & n-p & p-1 \end{pmatrix}.$$

Using Lemma 1.3 we get

$$\sum_{i=1}^{n} (\lambda_i \mu_i) \leq \sqrt{4m[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]}$$

$$(p-1)\mu_1 + (0)\sum_{i=2}^{n-p-1} \mu_i - \sum_{i=n-p+2}^{n} \mu_i \leq \sqrt{4\frac{p(p-1)}{2}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]}$$

$$(p-1)\mu_1 \leq \sqrt{2p(p-1)[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]} + \sum_{i=n-p+2}^{n} \mu_i$$

$$\mu_1 \leq \sqrt{\frac{2p}{(p-1)}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]} + \frac{1}{p-1}\sum_{i=2}^{p} \mu_{n-p+i}.$$

Using the results (1.7),(1.8) we get

$$\mu_1 \leq \sqrt{\frac{2p}{p-1}[2HZg - 4ZG_2]} + \frac{1}{p-1}\sum_{i=2}^{p} \mu_{n-p+i}.$$

$$\mu_1 \leq \sqrt{\frac{2p}{p-1}[FI]} + \frac{1}{p-1}\sum_{i=2}^{p} \mu_{n-p+i}.$$

□

Corollary 2.3. Let G be a graph having n vertices and m edges, with vertex degrees $d_1, d_2, \ldots, d_n$. Then,

$$\mu_1 \leq \sqrt{\frac{2(n-1)}{n}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]}. \tag{2.5}$$

Proof. Substituting $p = n$ in (2.4) and using Eq.(1.1) we get,

$$\begin{aligned}
\mu_1 &\leq \sqrt{\frac{2n}{(n-1)}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]} + \frac{1}{n-1}\sum_{i=2}^{n} \mu_i \\
\mu_1 &\leq \sqrt{\frac{2n}{(n-1)}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]} + \frac{1}{n-1}(-\mu_1) \\
\mu_1 &\leq \sqrt{\frac{2(n-1)}{n}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]}.
\end{aligned}$$

Similarly we get $\mu_1 \leq \sqrt{\frac{2(n-1)}{n}[HZg(G) - 4ZG_2]}$ and $\mu_1 \leq \sqrt{\frac{2(n-1)}{n}[FI]}$.

□

Remark 2.2. Equality in (2.5) holds for complete graphs.
In a complete graph $Zg_1(G) = nr^2 = n(n-1)^2$ and $\sum_{i=1}^{n} d_i^2(d_i - 1) = n(n-2)(n-1)^2$, substituting in (2.5) we get $\mu_1 = \sqrt{2}(n-1)^2$.
As the Sombor index for complete graph is $SO(K_n) = \frac{n(n-1)^2\sqrt{2}}{2}$ we can express the largest Sombor eigenvalue of complete graph in terms of Sombor index as $\mu_1 = (n-1)^2\sqrt{2} = \frac{2(SO)}{n}$.

Remark 2.3. For a $r-$regular graph $FI = nr^3$, hence we get $\mu_1 < r\sqrt{2r(n-1)}$.
For a cycle with $n, n \geq 3$ vertices, $\mu_1 < 4\sqrt{n-1}$.
For a Path P_n, $FI = 2(4n-7)$ hence, $\mu_1 < 2\sqrt{\frac{(n-1)(4n-7}{n}}$.
Star graph with n vertices has $FI = (n^2+1)n$
hence, $\mu_1 < \sqrt{\frac{2(n-1)^2(n^2-2n+2)}{n}}$.

Corollary 2.4. The largest eigenvalue of $SOM(G)$ is bounded by

$$\frac{2}{n}SO(G) \leq \mu_1 \leq \sqrt{\frac{2(n-1)}{n}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]}. \tag{2.6}$$

Proof. Using Eq.(2.3) and Eq.(2.5) we get the bounds for the largest Sombor eigenvalue of graph G. □

Remark 2.4. This Equality holds good for complete graphs.

Theorem 2.5. Let G be a graph with n vertices and m edges, with vertex degrees $d_1, d_2, \ldots, d_n$ and Sombor eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then,

$$\sum_{i=1}^{k} \mu_i \leq \sqrt{\frac{2k(p-1)}{p}[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]} \qquad 1 \leq k \leq n. \tag{2.7}$$

$$\sum_{i=1}^{k} \mu_i \leq \sqrt{\frac{2k(p-1)}{p}[HZg(G) - Zg_2]} \qquad 1 \leq k \leq n. \tag{2.8}$$

Proof. Let $\mu_1, \mu_2, \ldots, \mu_k, \mu_{k+1}, \ldots, \mu_n$ be the Sombor eigenvalues of G. Consider H to be the union of k copies of complete graph K_p, that is $H = \cup_k K_p$ where $kp = n$. Then the adjacency eigenvalues (λ_i) of H are

$$\begin{pmatrix} p-1 & -1 \\ k & n-k \end{pmatrix}.$$

As the number of vertices of H are $n = pk$ its edges will be $\dfrac{kp(p-1)}{2}$. Using Lemma (1.3),

$$\begin{aligned}
(p-1)\sum_{i=1}^{k}\mu_i - \sum_{i=k+1}^{n}\mu_i &\leq \sqrt{\frac{4kp(p-1)}{p}[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}\\
p\sum_{i=1}^{k}\mu_i - \sum_{i=1}^{n}\mu_i &\leq \sqrt{2kp(p-1)[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}\\
p\sum_{i=1}^{k}\mu_i &\leq \sqrt{2kp(p-1)[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}\\
\sum_{i=1}^{k}\mu_i &\leq \sqrt{\frac{2k(p-1)}{p}[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}.
\end{aligned}$$

Thus we have obtained the bound for the sum of k, Sombor eigenvalues of a graph G. If $k = 1$, then Eq.(2.7) gets reduced to Eq.(2.5). Similarly using lemma (1.3), Eq.(2.8) can be obtained. □

Theorem 2.6. Let G be a graph on n vertices and m edges, with vertex degrees $d_1, d_2, \ldots, d_n$ and Sombor eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then,

$$\sum_{i=1}^{k}(\mu_i - \mu_{n-k+i}) \leq \sqrt{4k[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}. \tag{2.9}$$

$$\sum_{i=1}^{k}(\mu_i - \mu_{n-k+i}) \leq \sqrt{4k[HZg(G)-2Zg_2]}. \tag{2.10}$$

Proof. Let $\mu_1, \mu_2, \ldots, \mu_k, \mu_{k+1}, \ldots, \mu_{n-k}, \mu_{n-k+1}, \ldots, \mu_n$ be the Sombor eigenvalues of G. Consider H to be the union of k copies of complete bipartite graph $K_{p,q}$, that is $H = \cup_k K_{p,q}$ where $kp = n$.Then the number of edges of H are kpq and the adjacency eigenvalues (λ_i) of H are

$$\begin{pmatrix} \sqrt{pq} & 0 & -\sqrt{pq} \\ k & n-2k & k \end{pmatrix}.$$

Using Lemma 1.3 we get,

$$\begin{aligned}
\sqrt{pq}\sum_{i=1}^{k}\mu_i + 0\sum_{i=k+1}^{n-k}\mu_i - \sqrt{pq}\sum_{i=k+1}^{n}\mu_i &\leq \sqrt{4kpq[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}\\
\sqrt{pq}\sum_{i=1}^{k}\mu_i - \sum_{i=1}^{k}\mu_{n-k+i} &\leq \sqrt{4kpq[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}\\
\sum_{i=1}^{k}(\mu_i - \mu_{n-k+i}) &\leq \sqrt{4k[Zg_1(G)+\sum_{i=1}^{n}d_i^2(d_i-1)]}.
\end{aligned}$$

Similarly using lemma (1.3), Eq.(2.10) can be obtained. □

3 Bounds for Sombor Energy $SOME(G)$ of a graph

Let G be a graph with n vertices and m edges. Then the Sombor energy of graph G can be defined as the absolute sum of Sombor eigenvalues of G.

$$SOME(G) = \sum_{i=1}^{n} |\mu_i|.$$

analogous to the energy defined for the graph with respect to its adjacency matrix [10]. As $2m$ is equal to the sum of the squares of adjacency eigenvalues of G. It is analogy to take $2m = 2[Zg_1 + \sum_{i=1}^{n} d_i(d_i - 1)^2] = 2[HZg - 2Zg_2] = 2FI$, lemma(1.2). Hence the direct consequence for the bounds of Sombor energy are as follows

$$\sqrt{2[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]} \leq SOME(G) \leq \sqrt{2n[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)]}$$

$$\sqrt{2[HZg(G) - 2Zg_2]} \leq SOME(G) \leq \sqrt{2n[HZg(G) - 2Zg_2]}$$

$$\sqrt{2[FI]} \leq SOME(G) \leq \sqrt{2n[FI]}$$

$$SOME(G) \geq \sqrt{2[Zg_1(G) + \sum_{i=1}^{n} d_i^2(d_i - 1)] + n(n-1)|det(SOM(G))|^{2/n}}$$

Theorem 3.1. The Sombor energy of a complete graph is equal to its Sombor index

$$SOME(K_n) = (n-1)^2\sqrt{2} = SO(K_n)$$

Proof. As the Sombor matrix for a complete graph can be expressed in terms of its adjacency matrix as $SOM(K_n) = (n-1)\sqrt{2}A(K_n)$, its Sombor spectra is

$$\begin{pmatrix} (n-1)\sqrt{2} & -(n-1)^2\sqrt{2} \\ n-1 & 1 \end{pmatrix}$$

Hence Sombor energy $SOME(K_n) = (n-1)^2\sqrt{2}$ which is the sum of absolute Sombor eigenvalues. □

Theorem 3.2. If $K_{p,q}$ is a complete bipartite graph then its Sombor energy is

$$SOME(K_{p,q}) = 2\frac{SO(K_{p,q})}{\sqrt{pq}}$$

Proof. Sombor matrix of a complete bipartite graph can be expressed in terms of adjacency matrix as $SOM(K_{p,q}) = \sqrt{p^2 + q^2}A(K_{p,q})$, so its Sombor spectra is

$$\begin{pmatrix} -\sqrt{pq(p^2+q^2)} & 0 & \sqrt{pq(p^2+q^2)} \\ 1 & p+q-2 & 1 \end{pmatrix}$$

Hence Sombor energy for complete bipartite graph is $SOME(K_{p,q}) = 2\sqrt{pq(p^2+q^2)}$. As Sombor index of $K_{p,q}$ is $SOME(K_{p,q}) = pq\sqrt{p^2+q^2}$ by direct substitution we get the required result. □

Theorem 3.3. Let G be a r regular with n vertices. Then

$$SOME(G) \geq 2\sqrt{2}r^2. \tag{3.1}$$

Proof. Let G be a r regular graph having n vertices and $r^2\sqrt{2}, r\sqrt{2}\lambda_2, r\sqrt{2}\lambda_3, \ldots, r\sqrt{2}\lambda_n$ be its

Sombor eigenvalues in terms of its adjacency eigenvalues. Then

$$\begin{aligned} SOME(G) &= |r^2\sqrt{2}| + \sum_{i=2}^{n} |r\sqrt{2}\lambda_i| \\ &\geq r^2\sqrt{2} + \left|\sum_{i=2}^{n} r\sqrt{2}(-r)\right| \\ &\geq r^2\sqrt{2} + \left|-r^2\sqrt{2}\right| \\ &\geq 2\sqrt{2}r^2. \end{aligned}$$

□

4 Conclusion remarks

We have obtained the bounds for the largest eigenvalues of a Sombor matrix and the sum of eigenvalues of a Sombor matrix for any general graph. These bounds are expressed in relation to Zagreb indices, Hyper Zagreb index, and Forgotten index. We have proved that these bounds are only equal for complete graphs, but for certain standard graphs, we have established specific inequalities for upper bounds. In our future work, we aim to characterize extremal graphs based on the indices.

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