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COMPACTNESS AND CARDINALITY OF THE SPACE OF CONTINUOUS FUNCTIONS UNDER REGULAR TOPOLOGY

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Abstract In this paper, we investigate the compactness and cardinality of the space $C(X, Y)$ of continuous functions from a topological space X to Y equipped with the regular topology. We prove that different forms of compactness, such as sequential compactness, countable compactness, and pseudocompactness, coincide on a subset of $C(X, Y)$ with regular topology. Moreover, we prove the comparison and coincidence of regular topology with the graph topology on the space $C(X, Y)$. Furthermore, we examine various cardinal invariants, such as density, character, pseudocharacter, etc., on the space $C(X, Y)$ equipped with the regular topology. In addition, we define a type of equivalence between X and Y in terms of $C(X)$ and $C(Y)$ endowed with the regular topology and investigate certain cardinal invariants preserved by this equivalence.

1 Introduction

First of all, for the convenience, we use the following notations throughout the paper. The space X is always a Tychonoff space and Y is a metric space. The spaces $C_r(X, Y)$ and $C_k(X, Y)$ are spaces of continuous functions endowed with the regular and compact-open topology, respectively. The space $C_r(X)$ is space of continuous real-valued function endowed with the regular topology. The space $UC(X, Y)$ is the space of uniformly continuous functions from X to Y, and $C^*(X)$ represents the set of bounded real-valued continuous functions on X. The abbreviations $LSC(X)$ and $NLSC(X)$ represent the lower semi-continuous and normal lower semicontinuous real valued functions on X respectively.

The space $C(X, Y)$ has been endowed with various topologies, including intrinsic topologies like the point-open topology, compact-open topology, and uniform topology. However, stronger topologies than the uniform topology, such as the fine topology (also known as the m-topology) and the graph topology, have also been studied. The fine topology on the space $C(X)$, along with its topological properties, was studied by Hewitt [6]. The basis elements for the fine topology on $C(X, Y)$, where (Y, d) is a metric space, are of the form: $B(f, \epsilon) = \{g \in C(X, Y) \colon d(f(x), g(x)) < \epsilon(x), \forall x \in X\}$, where $f \in C(X, Y)$ and ϵ is a positive unit of the ring $C(X)$. Later, the topological properties associated with the fine topology were also discussed in [14]. In [5], Darani introduces a concept of weakly prime and weakly prime z-filters on a space X, and shows that there exists a one-one correspondence between the weakly prime z-filters on X and weakly prime ideals of $C(X)$. The relation between the class of z-filters on X and the class of primal ideals of $C(X)$ is also discussed.

Iberklied et al. introduced a stronger topology than the fine topology on the space $C(X)$ in [9] and named it the regular topology or the r-topology. This topology is defined in a manner such that the positive unit in the basis elements of the fine topology is replaced by a positive regular element of the ring $C(X)$. That is, the basis elements for the regular topology on the space $C(X)$ are of the form: $R(f, r) = \{g \in C(X) \colon |f(x) - g(x)| < r(x), \forall x \in coz(r)\}$, where $f \in C(X)$, and r is a positive regular element (non-zero divisor) of the ring $C(X)$, and $coz(r) = \{x \in X : r(x) \neq 0\}$. They studied the character of the space $C_r(X)$ and calculated it in terms of D-dominating number of X. Afterwards, Azarpanah investigated compactness, connectedness, and countability of the space $C(X)$ endowed with the regular topology in [3]. Azarpanah proved that the space $C_r(X)$ is connected or locally connected if and only if X is a pseudocompact almost P-space. Moreover, it was also shown that various forms of compactness and countability coincide with the finiteness of the space X.

Later, Jindal et al. explored this regular topology on a more general space $C(X, Y)$, where Y is a metric space with a non-trivial path, in [10]. They used the same idea as before to define the basis element for the regular topology on $C(X, Y)$ as: $R(f, r) = \{g \in C(X, Y) : d(f(x), g(x)) < r(x), \forall x \in coz(r)\}$, where $f \in C(X, Y)$, and r is a positive regular element (non-zero divisor) of the ring $C(X)$. The space $C(X, Y)$ equipped with the regular topology is denoted as $C_r(X, Y)$. Moreover, they studied various topological properties like metrizability, countability and several completeness properties.

The study of compactness for the space $C(X, Y)$ endowed with Krikorian, fine, and graph topology has been conducted in [7], along with the characterization of compact subsets of the same space. Recently, Aaliya and Mishra examined the study of submetrizability, separation axioms, and various maps corresponding to the regular topology on the space $C(X, Y)$ in [1]. The space $C_r(X, Y)$ has been proven to be submetrizable, and various conditions have been shown to be equivalent to its submetrizability, along with several equivalent conditions for the metrizability of the space $C_r(X, Y)$. Additionally, when Y is considered as a normed linear space, $C_r(X, Y)$ has been proven to be a topological group. Moreover, in [2], Aaliya and Mishra explored the notion of regular topology on a homeomorphism space $H(X)$, for a metric space X and show that it forms a subspace of $C(X, X)$. They study compactness, metrizability and connectedness of the same and prove that the space $H(X)$ forms a topological group under the regular topology.

The paper is organized as follows. In Section 2, we mention some preliminaries that are used throughout the paper. In Section 3, we study the compactness of the space $C(X, Y)$ as a topological property and characterize its compact subsets when endowed with the regular topology. We prove that different forms of compactness coincide on $C_r(X, Y)$ when X is finite and Y is separable and locally compact. We also establish the necessary and sufficient conditions for the compactness of $C_r(X, Y)$. Furthermore, we investigate how different forms of compactness coincide for a subset of the space $C_r(X, Y)$. Notably, we demonstrate that paracompactness is countably additive in $C_r(X)$ under certain conditions. In Section 4, we investigate various cardinal invariants such as character, weight, density, etc., for the space $C_r(X, Y)$. We calculate the density of $C_r(X, Y)$ in terms of the densities of X and Y. In the last Section 5, we define an equivalence between the spaces X and Y in terms of their richer spaces $C_r(X)$ and $C_r(Y)$, respectively, and further investigate the cardinal invariants that remain invariant under this equivalence.

2 Preliminaries

Definition 2.1. [14] A topology τ on $C(X, Y)$ is said to be an ω-type topology if whenever (f_n) is a sequence in $C(X, Y)$ with cluster point f and (x_n) is a sequence in X having no cluster point, then there exists a strictly increasing sequence (n_k) of positive integers such that $f_{n_k}(x_{n_k}) = f(x_{n_k})$ for all $k \in \mathbb{N}$.

Definition 2.2. [7] The subset S (or sequence in) of $C(X, Y)$ said to be compactly supported if there exists a compact subset C of X such that for all $f, g \in S$, $f_{|X \setminus C} = g_{|X \setminus C}$. In more general terms, a subset S of $C(X, Y)$ is called almost compactly supported if it can be written as the union of finitely many compactly supported subsets of $C(X, Y)$.

Definition 2.3. [4]

(i) For a function $f \in C^*(X)$, the lower limit function $I(f)$ and upper limit function of f are respectively defined as

$$I(f)\colon X \to \mathbb{R}, I(f)(x) = \sup_{V \in N_x} \inf_{y \in V} f(y), x \in X$$
$$S(f)\colon X \to \mathbb{R}, S(f)(x) = \inf_{V \in N_x} \sup_{y \in V} f(y), x \in X$$

where N_x is the set of all neighborhoods of the point $x \in X$. It is clear that $I(f)(x) \leq f(x) \leq S(f)(x), x \in X$.

(ii) For lower lower-semicontinuous function f, the normal lower-semicontinuous function as

$$NLSC(X) = \{f \in LSC(X)\colon I(S(f)) = f\}$$

Definition 2.4. [13] A space X is said to be a weak cb-space if each locally bounded, lower semicontinuous function on X is bounded above by a continuous function. Moreover, it is equivalent to say that a space X is weak cb-space if and only if for a given positive normal lower semicontinuous function g on X, there exists $f \in C(X)$ such that $0 < f(x) \leq g(x)$, for each $x \in X$.

3 Compactness of $C_r(X, Y)$

In this section, we investigate various forms of compactness of the space $C_r(X, Y)$.

Theorem 3.1. *For any space X and a metric space (Y, d) with a non-trivial path, the following are equivalent for the space $C_r(X, Y)$*

(i) $C_r(X, Y)$ is separable locally compact.

(ii) $C_r(X, Y)$ is hemicompact.

(iii) $C_r(X, Y)$ is σ-compact.

(iv) X is finite and Y is separable locally compact.

Proof. $(4) \Rightarrow (1), (2), (3)$. Let X be finite. Then, $C_r(X, Y)$ is homeomorphic to Y^n for some positive integer n. Therefore, if Y is separable and locally compact, it implies that $C_r(X, Y)$ is separable and locally compact, which in turn implies that $C_r(X, Y)$ is Lindelöf and locally compact [10]. Hence, $C_r(X, Y)$ is hemicompact, and since every hemicompact space is σ-compact as well, this shows that $(4) \Rightarrow (1), (2), (3)$.

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. Suppose $C_r(X, Y)$ is separable and locally compact. Then, according to [10], $\mathcal{C}_r(X, Y)$ is Lindelöf and locally compact. Consequently, $C_r(X, Y)$ being hemicompact implies that it is σ-compact and thus Lindelöf. Thus, X is finite and Y is separable according to [10]. The finiteness of X implies that $C_r(X, Y)$ is homeomorphic to Y^n for some positive integer n. Therefore, Y^n is separable and locally compact, which implies that Y is separable and locally compact. □

Theorem 3.2. *The space $C_r(X, Y)$ is compact if and only if X is finite and Y is compact.*

Proof. Suppose the space $C_r(X, Y)$ is compact. Then $C_r(X, Y)$ is σ-compact and hence Lindelöf. Therefore, according to [10], X is finite and Y is separable. The finiteness of X implies that $C_r(X, Y)$ is homeomorphic to Y^n for some positive integer n. Thus, Y^n is compact and consequently Y is compact.

Conversely, let X be finite and Y be compact. Then, the finiteness of X implies that $C_r(X, Y)$ is homeomorphic to Y^n for some positive integer n. Moreover, the Tychonoff theorem implies that Y^n is compact. Thus, $C_r(X, Y)$ is compact. □

Corollary 3.3. *If the space $C_r(X, Y)$ is compact then X is compact.*

Example 3.4. Consider $X = \{1, 2, 3\}$ with a topology $\tau = \{X, \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$ and $Y = [0, 1]$. Then X is finite and Y is compact, and hence $C_r(X, [0, 1])$ is compact, and vice versa.

Corollary 3.5. *If the space $C_r(X,Y)$ is compact then Y is second countable.*

Corollary 3.6. *If the space $C_r(X,Y)$ is compact. Then the space $C_r(X,Y)$ can be embedded in $C_p(K(X),K(Y))$ as a closed subspace.*

Proof. The compactness of $C_r(X,Y)$ implies that the space X is finite, which in turn implies that X is compact. Therefore, the regular topology coincides with the compact-open topology on $C(X,Y)$. Hence, the embedding theorem proved in [15] is also applicable to $C_r(X,Y)$. □

Now we characterize various compact subsets of the space $C_r(X,Y)$.

Proposition 3.7 ([16, 11]). *A pseudocompact submetrizable space is metrizable.*

In the upcoming result, we describe the equivalence among various forms of compactness on a subspace of $C_r(X,Y)$.

Theorem 3.8. *For any space X and a metric space (Y,d), the following are equivalent for a subset S of $C_r(X,Y)$:*

(i) S is compact.

(ii) S is sequentially compact.

(iii) S is countably compact.

(iv) S is pseudocompact.

Proof. $(2) \Rightarrow (3) \Rightarrow (4)$ are all immediate. Since $C_d(X,Y) \leq C_r(X,Y)$ and $C_r(X,Y)$ is submetrizable [1], S is submetrizable. Then, by Proposition (3.7), if S is pseudocompact, it is also metrizable. In metrizable spaces, all these types of compactness coincide. Hence, $(1) \Rightarrow (2)$ and $(4) \Rightarrow (1)$. □

To further characterize some other forms of compactness on a subset of $C_r(X,Y)$, we consider a particular class of topology on $C(X,Y)$ called an ω-type topology. As proved in [7], the fine topology is an ω-type topology for a countably paracompact space X and a metric space (Y,d), and every topology finer than an ω-type topology is also an ω-type topology. Since the regular topology is finer than the fine topology on $C(X,Y)$, we conclude that the regular topology is also an ω-type topology for a countably paracompact space X and a metric space (Y,d). Thus, based on the results regarding compactness proved in [7] for an ω-type topology, we can further establish the following important result for the function space $C_r(X,Y)$:

Theorem 3.9. *Let X be a paracompact locally hemicompact k-space, and (Y,d) be a metrizable space. Then the following are equivalent for a subset Q in $C_r(X,Y)$.*

(i) Q is countably compact in $C_r(X,Y)$.

(ii) Q is compact in $C_r(X,Y)$.

(iii) Q is sequentially compact in $C_r(X,Y)$.

(iv) Q is almost compactly supported and Q is compact in $C_k(X,Y)$.

Now we prove that the paracompactness is additive in $C_r(X)$; before proving the main theorem, we first need to establish the following result :

Theorem 3.10. *For any space X, the space $C_r(X)$ is a Suslin space.*

Proof. Consider an identity map $I\colon C_g(X) \to C_r(X)$. Since an identity map is always a bijection, and since the graph topology is finer than the regular topology, I is continuous. As $C_g(X)$ is a Baire space and also a Polish space [14], we conclude that the space $C_r(X)$ is a continuous image of a Polish space and hence is a Suslin space. □

Theorem 3.11. *If $C_r(X) = \bigcup C_i$, where $i \in \omega$ and C_i are open or closed paracompact subspaces of $C_r(X)$, then $C_r(X)$ is also paracompact.*

Proof. Suppose $C_r(X) = \bigcup Ci$, where $i \in \omega$ and C_i are open or closed paracompact subspaces of $C_r(X)$. From the above result, we have that $C_r(X)$ is a Suslin space. Then C_i, for $i \in \omega$, are also Suslin spaces. This implies that C_i, for $i \in \omega$, are separable and thus have the countable chain condition (ccc). Since a paracompact space with the ccc property is Lindelöf, each C_i is Lindelöf. Recall that Lindelöfness is closed with respect to a countable union. Therefore, $\bigcup C_i$, for $i \in \omega$, is Lindelöf. Thus, $C_r(X)$ is Lindelöf. Additionally, $C_r(X)$ is regular [1], and we know that a regular Lindelöf space is paracompact. Thus, $C_r(X)$ is paracompact. □

Theorem 3.12. *Let X be an arbitrary space. Then $C_r(X)$ is paracompact if and only if it is Lindelöf.*

Proof. Since every Lindelöf space is paracompact, we will prove the converse. Suppose $C_r(X)$ is paracompact. Since $C_r(X)$ is a Suslin space, it is separable and hence has the countable chain condition. Remember that a paracompact space with the countable chain condition is Lindelöf. Hence, $C_r(X)$ is Lindelöf. □

Now we study the comparison and coincidence of the regular topology with that of the graph topology on the space $C_r(X, Y)$.

Theorem 3.13. *For a Tychonoff space X and a metric space (Y, d) with a non-trivial path, we have $C_r(X, Y) \leq C_g(X, Y)$.*

Proof. The basis elements for the regular topology on $C_r(X, Y)$ are of the form $R(f, r) = g \in C(X, Y)\colon d(f(x), g(x)) < r(x), x \in coz(r)$, where r is the regular element of the ring $C(X)$. For the graph topology, the basis elements are of the form

$$B(f, l) = g \in C(X, Y)\colon d(f(x), g(x)) < l(x), x \in X,$$

where l is the lower semi-continuous function of $C(X)$. Since every continuous function is also lower semi-continuous and every regular element is also continuous, we have that the regular topology on $C_r(X, Y)$ is weaker than the graph topology on it. □

Theorem 3.14. *For a Tychonoff space X and a metric space (Y, d) with a non-trivial path, we have $C_r(X, Y) = C_g(X, Y)$ if and only if X is a weak cb-space.*

Proof. First, suppose that $C_r(X, Y) = C_g(X, Y)$. Let $\eta \in NLC^+(X)$ and $h\colon [0, 1] \to Y$ be a continuous function such that $h(z) \neq h(0)$ for all $z \neq 0$. Define $f(x) = h(0)$ for all $x \in X$. Then, $f \in C(X, Y)$, and if $\lambda = \min\{\eta, d(h(0), h(1))/2\}$, then $\lambda \in NLC^+(X)$. Since $C_r(X, Y) = C_g(X, Y)$, there exists an $r \in r^+(X)$ such that $B_r(f, r) \subseteq B_g(f, \lambda)$. We claim that $r(x) \leq \lambda(x)$, otherwise, $\lambda(x_0) < r(x_0)$ for some $x_0 \in X$. Let $O(x_0)$ be some open neighborhood of x_0 such that $\lambda(x_0) < r(x)$ for every $x \in O(x_0)$. Since $\{z \in [0, 1]\colon d(h(0), h(z)) \geq \lambda(x_0)\}$ is a non-empty compact subset of $[0, 1]$, it has a minimum $b > 0$. Note that $d(h(0), h(z)) \leq \lambda(x_0)$ for all $z \in [0, b]$ open, and $d(h(0), h(b)) = \lambda(x_0)$.

Since X is a Tychonoff space, there is a continuous function $H\colon X \to [0, b]$ such that $H(x_0) = b$ and $H(x) = 0$ for all $x \notin O(x_0)$. Define the function $G\colon X \to Y$ as $G(z) = h(H(z))$ for all $z \in X$. Then, G is a continuous function that is different from f. Since for $x \in O(x_0)$, $d(f(x), G(x)) = d(h(0), h(H(x))) \leq \lambda(x_0) < r(x)$, and for $x \notin O(x_0)$, $d(f(x), G(x)) = d(h(0), h(0)) = 0 < r(x)$, $G \in B_r(f, r) \Rightarrow G \in B_g(f, \lambda)$, which is a contradiction since $d(f(x_0), G(x_0)) = d(h(x_0), h(b)) = \lambda(x_0)$. Consequently, we have found that $r \in r^+(X)$ with $r \leq \lambda \leq \eta$. Thus, X is a weak *cb*-space.

Conversely, suppose that X is a weak *cb*-space. To prove that $C_r(X, Y) = C_g(X, Y)$, it is sufficient to prove that $C_g(X, Y) \leq C_r(X, Y)$. Let $B_r(f, r)$ be an open set in $C_r(X, Y)$ and let $g \in B_r(f, r)$. Thus, we have $d(f(x), g(x)) < r(x)$ for all $x \in coz(r)$, where $r \in r^+(X)$. Since X is a weak *cb*-space, that is, for all $l \in NLC^+(X)$ there exists $\varphi \in U^+(X)$ such that $\varphi(x) \leq l(x)$ for all $x \in X$. Then, for some $l \in LSC^+(X)$, there exists $\varphi \in U^+(X)$ such that $\varphi(x) \leq l(x)$ for all $x \in X$. Since $U^+(X) \subseteq r^+(X)$, we can write, for some $l \in LSC^+(X)$, $r(x) \leq l(x)$ for $r \in r^+(X)$. Therefore, we have $d(f(x), g(x)) < l(x)$. Hence, $g \in B_g(f, l)$ and so $C_r(X, Y) = C_g(X, Y)$. □

Example 3.15. Consider $X = [0, 1]$ and $Y = \mathbb{R}$. Since $[0, 1]$ is Tychonoff and compact and hence pseudocompact. Remind that a pseudocompact Tychonoff space is weak cb-space. Therefore, we have $C_r([0, 1]) = C_g([0, 1])$

4 Cardinal functions on $C_r(X, Y)$

To study the cardinal invariants of the space $C_r(X,Y)$, we need to make certain assumptions that will hold throughout, namely, the space X is a Tychonoff space that is also pseudocompact and almost a P-space, while (Y,d) is a metric space with a non-trivial path. Consequently, the space $C_r(X,Y)$ is always metrizable with the supremum metric on it [10]. Moreover, for metrizable spaces, the weight, density, Lindelof number, and cellularity coincide. Hence, it is sufficient to focus on the density when considering the space $C_r(X,Y)$. Throughout, the symbols $w(X), d(X), l(X)$, and $c(X)$ represent the weight, density, Lindelof number, and cellularity of the space X respectively.

Theorem 4.1. *The space $C_r(X,Y)$ has the character $\chi(C_r(X,Y)) = \omega$.*

Theorem 4.2. *For a finite space X and a separable space (Y,d),*

$$w(C_r(X,Y)) = \omega \text{ and } d(C_r(X,Y)) = \omega$$

Example 4.3. For a real line $\mathbb{R}$, let $\beta\mathbb{R}$ denotes its Stone-Cech compactification. Let $X = \beta\mathbb{R} - \mathbb{R}$, the X is an almost P-space [12] and since $\mathbb{R}$ is locally compact, so it is open in $\beta\mathbb{R}$, and $\beta\mathbb{R} - \mathbb{R}$ is therefore compact, thus pseudocompact. Then $C_r(\beta\mathbb{R} - \mathbb{R})$ is metrizable and hence has countable weight, density and cellularity.

Theorem 4.4. *[1] The space $C_r(X,Y)$ is countably tight.*

Before moving ahead to calculate the density of $C_r(X,Y)$ in terms of the density of X and Y, we will require the following results:

Definition 4.5. For a metric space (X,d) and $\epsilon > 0$, a non-empty subset E of X is called ϵ-uniformly discrete if for any $a,b \in E$ such that $a \neq b$, we have $d(a,b) \geq \epsilon$.

Let ξ_ϵ be the family of ϵ-uniformly discrete subsets of X, and let ξ be the family of all uniformly discrete subsets of X. Let $\xi_\epsilon^{\max}$ be the subfamily of ξ_ϵ containing all the elements which are maximal with respect to the set-theoretic inclusion. Then V in $\xi_\epsilon^{\max}$, in addition to being ϵ-uniformly discrete, is ϵ-dense, meaning that for each $x \in X$, there exists $v \in V$ such that $d(x,v) < \epsilon$.

Definition 4.6. A metric space (X,d) is called generalized totally bounded (GTB) if for every $\epsilon > 0$, there exists an ϵ-dense subset G of X with $|G| < d(X)$.

Proposition 4.7. *[14] A metric space Y is generalized totally bounded if and only if every uniformly discrete subset V of Y satisfies $|V| < d(Y)$.*

Definition 4.8. A topological space X is said to be generalized compact (GK) if every open cover $\Im$ of X has a subcover $\wp$ such that $|\wp| < d(X)$.

Theorem 4.9. *[14] A metrizable space attains its extent if and only if it is not GK.*

Lemma 4.10. *[14] Let C be a closed and discrete subset of a metrizable space X, let $0 < \epsilon < 1$, and U be an ϵ-uniformly discrete subset of a pathwise connected metric space (Y,ρ). Then there exists an ϵ-uniformly discrete subset of $C(X,Y)$ of cardinality $|U|^{|C|}$.*

Theorem 4.11. *Let X be a metrizable space, and (Y,d) be a pathwise connected metric space with $d(X) = \upsilon$ and $d(Y) = \gamma$. If X is not GK and Y is not GTB, then $d(C_r(X,Y)) = \gamma^\upsilon$.*

Proof. Consider a dense subset D of X such that $|D| = \upsilon$. Since $d(Y) = \gamma$ and (Y,d) is a metric space, we have $|Y| = \gamma^{\aleph_0}$. Hence, $|Y^D| = |Y|^{|D|} < \gamma^\upsilon$. Define a map $\psi\colon C_r(X,Y) \to Y^D$ as $\psi(f) = f_{|D}$. Since D is dense, ψ is one-to-one. Therefore, $|C_r(X,Y)| \leq |Y^D| \leq \gamma^\upsilon$. Hence, $d(C_r(X,Y)) \leq |C_r(X,Y)| \leq \gamma^\upsilon$.

Now we show that $d(C_r(X,Y)) \leq |C_r(X,Y)| \geq \gamma^\upsilon$. Since X is not GK, then by the Proposition (4.9), there exists a closed and discrete subset D of X such that $|D| = \upsilon$. Since Y is not GTB, then by the Proposition (4.7), there exists a uniformly discrete subset U of Y with $|Y| = \gamma$. Therefore, by using the Lemma (4.10), we have that there exists a uniformly discrete subset V of $C_r(X,Y)$ with $|V| = \gamma^\upsilon$. Thus, $d(C_r(X,Y)) \geq \gamma^\upsilon$. □

Theorem 4.12. *For a space X and a normed linear space $(Y, ||.||)$, we have*

$$\chi(f, C_r(X,Y)) = \chi(C_r(X,Y)) = \pi_\chi(C_r(X,Y)) = \pi_\chi(f, C_r(X,Y))$$

and

$$\psi(f, C_r(X,Y)) = \psi(C_r(X,Y)) = \triangle(C_r(X,Y))$$

where χ, ψ, and $\triangle$ signify character, pseudocharacter, and diagonal degree respectively.

Proof. It has been proven that for a space X and a normed linear space $(Y, ||.||)$, the space $C_r(X,Y)$ is a topological group [1]. Also, for a topological group X, the following hold: $\chi(x, X) = \chi(X) = \pi_\chi(X) = \pi_\chi(x, X)$ and $\psi(x, X) = \psi(X) = \triangle(X)$. □

Example 4.13. Lets consider X as $(0, 1)$ and Y as a real line $\mathbb{R}$ with Euclidean norm, then $C_r(X,Y)$ is a topological group [1, Theorem 3.5] and thus we have for some $f \in C((0,1))$, $\chi(f, C((0,1))) = \chi(C((0,1))) = \pi_\chi(C((0,1))) = \pi_\chi(f, C((0,1)))$ and $\psi(f, C((0,1))) = \psi(C((0,1))) = \triangle(C((0,1)))$.

Theorem 4.14. *For every X, $w(C_r(X)) = \chi(C_r(X)) \cdot d(C_r(X))$.*

Proof. Since $w(Y) \geq \chi(Y) \cdot d(Y)$ for any Y, it is sufficient to show that $w(C_r(X)) \leq \chi(C_r(X)) \cdot d(C_r(X))$. Let $\Im$ be a base at 0_X in $C_r(X)$ such that $|\Im| = \chi(C_r(X))$, where 0_X is the constant element in $C_r(X)$. Also, let S be a dense subset of $C_r(X)$ such that $|S| = d(C_r(X))$. We may assume that each $B \in \mathcal{I}$ looks like $B = B(0_X, r)$ for some $r \in r^+(X)$. Define $\Re = B(f, r) \colon f \in S, B \in \Im$. Clearly, $|\Re| = \chi(C_r(X)) \cdot d(C_r(X))$. Now we show that $\Re$ is a base for $C_r(X)$. Let $g \in C(X)$ and $s \in r^+(X)$. As S is dense in $C_r(X)$, there is an $h \in S \cap B(g, s)$. Then there exists $\varphi \in r^+(X)$ such that $h \in B(h, \varphi) \subseteq B(h, s)$. Also, there is a $B \in \Im$, so that $B(0_X, r) \subseteq B(0_X, \varphi)$. To see that $B(h, r) \subseteq B(h, \varphi)$, let $t \in B(h, r)$, then $t - h \in B(0_X, r) \subseteq B(0_X, \varphi)$. Thus $t \in B(h, \varphi)$. Hence it follows that $\Re$ is a base for $C_r(X)$. □

Theorem 4.15. *Let X be a metrizable space and (Y, ρ) be a metric space with $d(X) = \upsilon$ and $d(Y) = \zeta$. Then $d(C(X,Y)) \leq |C(X,Y)| \leq \zeta^\upsilon$.*

Proof. Fix a dense subset E of X with $|E| = \upsilon$. Since $d(Y) = \zeta$, by a well-known property of metrizable spaces, we have $|Y| \leq \zeta^{\aleph_0}$. Therefore, $|Y^E| = |Y|^{|E|} \leq (\zeta^{\aleph_0})^\upsilon = \zeta^\upsilon$. For every $f \in C(X,Y)$, consider $\varphi(f) = f|_E$. Then φ is one-to-one, because two continuous functions which coincide on a dense subset of the domain must coincide everywhere. Therefore, $|C(X,Y)| \leq |Y^E| \leq \zeta^\upsilon$. □

Theorem 4.16. *Let (X, d) be a non-empty finite metric space and (Y, ρ) be a non-trivial pathwise connected metric space, and let $d(Y) = \zeta$. Then $d(UC(X,Y)) = d(C(X,Y)) = \zeta$.*

Proof. Consider a fixed dense subset D of Y with $|D| = \zeta$. Let F be the set of all functions from X to D. Then $|F| = \zeta^{|X|}$. Every element of F is trivially continuous and uniformly continuous. Since F is dense in $C(X,Y)$, we have that $d(UC(X,Y)) \leq d(C(X,Y)) \leq \zeta$. On the other hand, the set of all constant functions from X to Y is a subset of $UC(X,Y)$ homeomorphic to Y and isometric to $(Y, \min(\rho, 1))$. Thus, $d(UC(X,Y)) = d(C(X,Y)) = \zeta$. □

5 $t_r - equivalence$

To characterize the properties of X using its richer space $C(X)$ and to investigate the properties of X and Y by defining a homeomorphism between their richer spaces $C(X)$ and $C(Y)$ is of great interest in the theory of function spaces. In this section, we define a form of equivalence between the spaces X and Y and determine the properties preserved by that equivalence.

Definition 5.1. Topological spaces X and Y are said to be t_r-equivalent if the spaces $C_r(X)$ and $C_r(Y)$ are homeomorphic. We write it as $X \sim^{t_r} Y$.

Definition 5.2. A property P is said to be t_r-invariant if it remains preserved during t_r-equivalence.

Theorem 5.3. *If X is a metrizable, non-GK space with $d(X) = \nu$, and if the space X is t_r-equivalent to Y, then the density is t_r-invariant.*

Proof. Given that $X \sim^{t_r} Y$ and $d(X) = \nu$, let $d(Y) = \lambda$. This implies that $C_r(X) \cong C_r(Y)$. From Theorem (4.11), we have $d(C_r(X)) = d(\mathbb{R})^{d(X)} = \aleph_0^{\nu}$. Since $C_r(X) \cong C_r(Y)$, then $d(C_r(X)) = d(C_r(Y))$. This means $\aleph_0^{\nu} = \aleph_0^{\lambda}$, which implies $\nu = \lambda$, where $\nu, \lambda \geq \aleph_0$. Thus, $d(X) = d(Y)$. Hence, density remains invariant under t_r-equivalence. □

Theorem 5.4. *If X is a pseudocompact, almost P-space and $X \sim^{t_r} Y$, then weight is t_r-invariant.*

Proof. Given that $X \sim^{t_r} Y \Rightarrow C_r(X) \cong C_r(Y)$. As we have already proved in the Theorem (4.14) that for every X, $w(C_r(X)) = \chi(C_r(X)) \cdot d(C_r(X))$. The space X being an almost P-space implies that $\chi(C_r(X)) = dn(X)$, [9, Corolary 1.12]. Furthermore, X being pseudocompact implies that $dn(X) = \aleph_0$, [8, Theorem 3.3]. Therefore, we have $w(C_r(X) = d(C_r(X)) = \aleph_0^{d(X)}$. Since $C_r(X) \cong C_r(Y)$, then $d(C_r(X)) = w(C_r(X)) = w(C_r(Y)) = d(C_r(Y))$. As we proved above that the density remains invariant under t_r-equivalence under certain conditions, the same can be proved for weight as well. □

By allowing X to be an almost P-space, the space $C_r(X)$ turns out to be equal to $C_f(X)$. Assuming the same, then from [8, Theorem 3.2 and 3.4], we have the following equalities, and now we prove that all these below are t_r-invariant.

Theorem 5.5. *Let X be a Tychonoff space, then*

$$u(C_r(X)) = \chi(C_r(X)) = \pi_\chi(C_r(X)) = t(C_r(X)) = dn(X)$$

and

$$\begin{aligned} ue(C_r(X)) = c(C_r(X)) = d(C_r(X)) = L(C_r(X)) = e(C_r(X)) = s(C_r(X)) \\ = \pi(C_r(X)) = nw(C_r(X)) = w(C_r(X)) = d(C_r(X)) \geq dn(X) \end{aligned}$$

and all these cardinal invariants are t_r-invariant.

Proof. For the first equality, we prove that the dominating number is preserved under the t_r-equivalence. Suppose $X \sim^{t_r} Y \Rightarrow C_r(X) \cong C_r(Y)$. Considering X to be an almost P-space, we have $\chi(C_r(X)) = dn(X)$ [9, Corollary 1.12]. The homeomorphism between $C_r(X)$ and $C_r(Y)$ implies that $dn(X) = \chi(C_r(X)) = \chi(C_r(Y)) = dn(Y) \Rightarrow dn(X) = dn(Y)$. Thus, χ, π_χ, t, u are all t_r-invariant.

For the second equality, since we have shown earlier that the weight also remains preserved under t_r-equivalence when X is an almost P-space, all other cardinal functions in the second equality are also t_r-invariant. □

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