Palestine Journal of Mathematics

Vol 13(1)(2024) , 118–128

Anisotropic Buffon's needle problems

Salvatore Vassallo

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 60D05; Secondary 52A22.

Keywords and phrases: Geometric Probability, Stochastic Geometry, Random Sets, Buffon's Needle Problem.

Abstract In this paper, we investigate anisotropic extensions of the classical Buffon's needle problem. In particular, we study the cases where the angle between the needle and a fixed reference direction follows a triangular, a trapezoidal, a wrapped exponential, or a Von Mises distribution law. Within the first two cases, we examine both the oriented and non-oriented needle problems, while within the latter two cases, we study the oriented needle problem exclusively. For the examined distributions, we also determine the minimum and the maximum probability.

1 Introduction and Notations

The classical Buffon's needle problem [10] consists in finding the probability that a "short" needle of length $2l$, dropped onto a ruled paper, crosses one of the lines. This probability depends on the distance d between the lines, or, more precisely, on the ratio $\frac{l}{d}$. A "short" needle has length $2l \leq d$, i.e. it cannot cross two lines simultaneously. The answer to this problem is:

$$p = \frac{4l}{\pi d} \tag{1.1}$$

To obtain this probability, it is possible to evaluate a single straightforward integral. However, Barbier [3] (see also [1]) proposed an alternative proof, which eliminates the need for integrals.

In Barbier's paper [3] there are also extensions of the original Buffon's problem to different bodies such as circles, ellipses, etc..

For a comprehensive historical overview and an extensive bibliography, we recommend consulting to Mathai's book [13], Santaló's book [14], and the paper [8].

1.1 The Buffon's Needle Problem

In Buffon's problem a "needle", which is a segment of length $2l$, is *randomly* dropped onto a set of parallel and equidistant lines in the plane. In this context, "randomly" is defined such that the distance v between the center M of the needle and a line, as well as the angle φ between the needle and the direction of the parallel lines, are independent random variables that follow a specified distribution law (see Figure 1). The distance between the parallel lines is d, with $2l < d$.

To determine the probability of the needle intersecting one of the lines, we assume that point M lies within the strip bounded by the lines $y = kd$ and $y = (k+1)d$ ($k \in \mathbb{Z}$). It is worth noting that, for a given angle $\varphi = \alpha$, the needle does not intersect the lines when point M lies in the strip between the lines $y = kd + l\sin\alpha$ and $y = (k+1)d - l\sin\alpha$ (see Figure 2).

Assuming that the distance v and the angle φ follow independent probability distributions $f(v)$ and $g(\varphi)$ respectively, the probability that the needle does not intersect any line is then given by:

$$q = \frac{\displaystyle\int_I \mathrm{d}\varphi \int_{J'} g(\varphi) f(v)\,\mathrm{d}v}{\displaystyle\int_I \mathrm{d}\varphi \int_J g(\varphi) f(v)\,\mathrm{d}v} \tag{1.2}$$

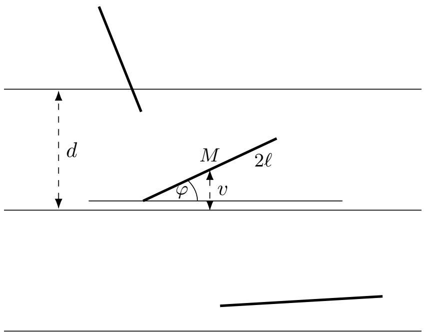


Figure 1. Short needle on parallel lines

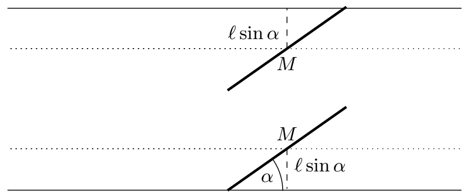


Figure 2. Limit position for the needle in Buffon's problem

where $J = [kd, (k+1)d]$ $J' = [kd + l\sin\varphi, (k+1)d - l\sin\varphi]$ and $I = [0,\pi]$ if the needle is not oriented and $I = [0, 2\pi]$ if the needle is oriented.

If both the distance v and the angle φ are uniformly and independently distributed, then this probability is given by (1.1).

However, some papers consider problems in which either the position of the needle's center or the orientation of the needle does not follow a uniform distribution, (see, for example, [4, 7, 5, 15, 6, 16, 12, 11, 9, 2].)

In this paper, we examine anisotropic Buffon's problem, i.e. when the angle between the needle and the parallel lines is not uniformly distributed. Specifically, we will examine triangular, trapezoidal, wrapped exponential, and Von Mises distributions.

2 The triangular distribution

2.1 The non-oriented needle

In this section, we suppose that the random variable φ follows a *triangular distribution* of support $[0,\pi]$ and mode c. In other words, the density function of φ is given by:

$$g(\varphi) = \begin{cases} \dfrac{2\varphi}{\pi c} & \text{for } 0 \leq \varphi \leq c \\ \dfrac{2(\pi-\varphi)}{\pi(\pi-c)} & \text{for } c < \varphi \leq \pi \end{cases} \tag{2.1}$$

and the random variable v is uniformly distributed in $[kd, (k+1)d]$.

In this case, the numerator in Equation (1.2) is:

$$\int_0^\pi g(\varphi)\,\mathrm{d}\varphi \int_{kd+l\sin\varphi}^{(k+1)d-l\sin\varphi} \frac{1}{d}\,\mathrm{d}v = \int_0^\pi g(\varphi)\left(1 - \frac{2l}{d}\sin\varphi\right)\mathrm{d}\varphi$$

$$= \int_0^\pi g(\varphi)\,\mathrm{d}\varphi - \int_0^c \frac{2l}{d}\frac{2\varphi}{\pi c}\sin\varphi\,\mathrm{d}\varphi - \int_c^\pi \frac{2l}{d}\frac{2\,(\pi-\varphi)}{\pi(\pi-c)c}\sin\varphi\,\mathrm{d}\varphi = 1 - \frac{2l}{d}\,\frac{2\sin c}{c(\pi-c)}$$

So we obtain that the probability that the needle crosses one of the lines if the angle φ follows the distribution law (2.1) is

$$p = \begin{cases} \dfrac{4l}{d}\,\dfrac{\sin c}{c(\pi-c)} & \text{for} \quad 0 < x < \pi \\ \dfrac{4l}{\pi d} & \text{for} \quad x = 0 \text{ and } x = \pi \end{cases} \tag{2.2}$$

It is evident that, for a fixed ratio $\frac{l}{d}$, this probability attains its maximum for $c = \frac{\pi}{2}$, with the maximum value being $\frac{16l}{d\pi^2}$. Conversely, the minimum occurs at $c = 0$ and $c = \pi$, with a value equal to the classical Buffon's probability $\frac{4l}{d\pi}$.

2.2 The oriented needle

Let us assume that the random variable φ follows a triangular distribution of support $[0, 2\pi]$ and mode c, implying the following probability density function:

$$g(\varphi) = \begin{cases} \dfrac{\varphi}{\pi c} & \text{for } 0 \leq \varphi \leq c \\ \dfrac{2\pi - \varphi}{\pi(2\pi - c)} & \text{for } c < \varphi \leq 2\pi \end{cases} \tag{2.3}$$

and the random variable v is uniformly distributed in $[kd, (k+1)d]$. In this case, the numerator in Equation (1.2) is:

$$\int_0^{2\pi} g(\varphi)\, \mathrm{d}\varphi \int_{kd+l\,|\sin\varphi|}^{(k+1)d-l\,|\sin\varphi|} \frac{1}{d}\, \mathrm{d}v = \int_0^{2\pi} g(\varphi) \left(1 - \frac{2l}{d} |\sin\varphi|\right) \mathrm{d}\varphi =$$

$$\int_0^{2\pi} g(\varphi)\, \mathrm{d}\varphi - \int_0^{2\pi} \frac{2l}{d} g(\varphi)\, |\sin\varphi|\, \mathrm{d}\varphi = 1 - \int_0^{2\pi} \frac{2l}{d} g(\varphi)\, |\sin\varphi|\, \mathrm{d}\varphi$$

and the probability that the needle intersects a line is given by:

$$p = \int_0^{2\pi} \frac{2l}{d} g(\varphi)\, |\sin\varphi|\, \mathrm{d}\varphi \tag{2.4}$$

To evaluate this integral we consider separately the cases $c \in [0, \pi]$ and $c \in (\pi, 2\pi]$.

If $0 \leq c \leq \pi$ Equation (2.4) becomes:

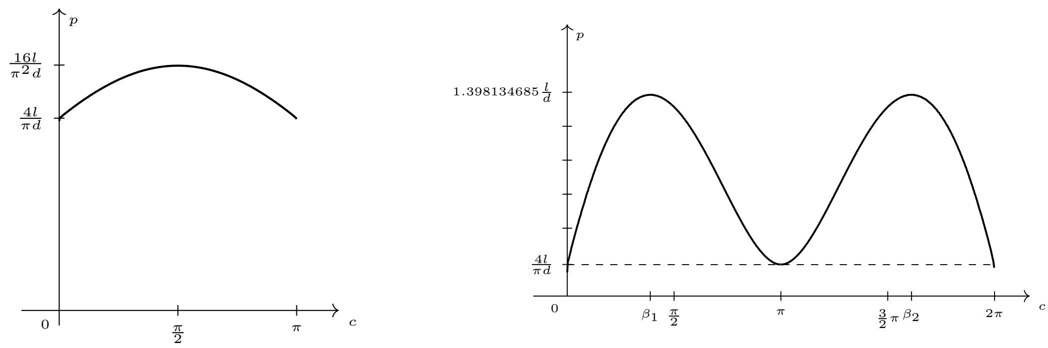
$$\begin{aligned} &\int_0^{2\pi} \frac{2l}{d} g(\varphi)\, |\sin\varphi|\, \mathrm{d}\varphi \\ &= \int_0^{c} \frac{2l}{d} \frac{\varphi}{\pi c} \sin\varphi\, \mathrm{d}\varphi + \int_c^{\pi} \frac{2l}{d} \frac{(2\pi - \varphi)}{\pi(2\pi - c)} \sin\varphi\, \mathrm{d}\varphi + \int_\pi^{2\pi} \frac{2l}{d} \frac{(2\pi - \varphi)}{\pi(2\pi - c)} (-\sin\varphi)\, \mathrm{d}\varphi \\ &= \frac{4l\,(\sin(c) + c)}{(2\,\pi - c)\, dc} \end{aligned}$$

If $\pi < c \leq 2\pi$ Equation (2.4) becomes:

$$\begin{aligned} &\int_0^{2\pi} \frac{2l}{d} g(\varphi)\, |\sin\varphi|\, \mathrm{d}\varphi \\ &= \int_0^{\pi} \frac{2l}{d} \frac{\varphi}{\pi c} \sin\varphi\, \mathrm{d}\varphi + \int_\pi^{c} \frac{2l}{d} \frac{\varphi}{\pi c} (-\sin\varphi)\, \mathrm{d}\varphi + \int_c^{2\pi} \frac{2l}{d} \frac{(2\pi - \varphi)}{\pi(2\pi - c)} (-\sin\varphi)\, \mathrm{d}\varphi \\ &= \frac{4l\,(2\,\pi - c - \sin(c))}{(2\,\pi - c)\, dc} \end{aligned}$$

The probability that the needle intersects one line is:

$$p = \begin{cases} \dfrac{4l}{d}\, \dfrac{|\sin(c)| + \min\{c, 2\pi - c\}}{(2\,\pi - c)\, c} & \text{for } 0 < c < 2\pi \\ \dfrac{4l}{\pi d} & \text{for } c = 0 \text{ and } c = 2\pi \end{cases} \tag{2.5}$$



(a) Non-oriented needle

(b) Oriented needle

Figure 3. Triangular distribution

The function $p = p(c)$ is symmetric about π. By conducting computer simulations using a mathematical software such as *Maxima* or *Mathematica*, it is observed that the function $p = p(c)$ achieves its minimum value at $c = 0$, $c = \pi$, and $c = 2\pi$ where $p = \frac{4l}{\pi d}$, i.e. the classical Buffon's probability. Conversely, the function attains its maximum at $c = \beta_1 \approx 1.22134468$ and $c = \beta_2 = 2\pi - \beta_1$, with $p(\beta_1) = p(\beta_2) \approx 1.398134685\frac{l}{d}$ as depicted in Figure 2.2.

3 The trapezoidal distribution

3.1 The non-oriented needle

Now, we suppose that the random variable φ follows a *trapezoidal distribution* of support $[0, \pi]$, lower mode c_1 and upper mode c_2, i.e. the density function of φ is:

$$g(\varphi) = \begin{cases} \dfrac{2}{\pi + c_2 - c_1}\dfrac{\varphi}{c_1} & \text{for } 0 \le \varphi \le c_1 \\ \dfrac{2}{\pi + c_2 - c_1} & \text{for } c_1 \le \varphi \le c_2 \\ \dfrac{2}{\pi + c_2 - c_1}\dfrac{\pi - \varphi}{\pi - c_2} & \text{for } c_2 < \varphi \le \pi \end{cases} \tag{3.1}$$

and the random variable v is uniformly distributed in $[kd, (k+1)d]$

In this case the numerator in Equation (1.2) is:

$$\int_0^\pi g(\varphi)\,\mathrm{d}\varphi \int_{kd+l\sin\varphi}^{(k+1)d-l\sin\varphi} \frac{1}{d}\,\mathrm{d}v = \int_0^\pi g(\varphi)\left(1 - \frac{2l}{d}\sin\varphi\right)\mathrm{d}\varphi =$$

$$\int_0^\pi g(\varphi)\,\mathrm{d}\varphi - \frac{2l}{d}\frac{2}{\pi + c_2 - c_1}\left[\int_0^{c_1} \frac{\varphi}{c_1}\sin\varphi\,\mathrm{d}\varphi - \int_{c_1}^{c_2} \sin\varphi\,\mathrm{d}\varphi - \int_{c_2}^{\pi} \frac{\pi\varphi}{\pi - c_2}\sin\varphi\,\mathrm{d}\varphi\right] =$$

$$1 - \frac{4l}{d}\frac{\pi\sin c_1 + c_1\sin c_2 - c_2\sin c_1}{c_1(\pi - c_2)(\pi + c_2 - c_1)}$$

The probability that the needle crosses one of the lines, if the angle φ follows the distribution law (3.1), is:

$$p=\begin{cases}\dfrac{4l}{d}\,\dfrac{\pi\sin c_1+c_1\sin c_2-c_2\sin c_1}{c_1(\pi-c_2)(\pi+c_2-c_1)} & \text{for } 0<c_1\le c_2<\pi\\ \dfrac{4l}{d}\,\dfrac{\pi+\sin(c_2)-c_2}{(\pi-c_2)(\pi+c_2)} & \text{for } c_1=0,\ 0\le c_2<\pi\\ \dfrac{4l}{d}\,\dfrac{c_1+\sin(c_1)}{(2\pi-c_1)\,c_1} & \text{for } c_2=\pi,\ 0<c_1\le\pi\\ \dfrac{4l}{\pi d} & \text{for } c_1=0 \text{ and } c_2=\pi\end{cases}\tag{3.2}$$

This probability reduces to the triangular case with mode c_2 if $c_1=c_2$ or $c_1=\pi-c_2$, and to the classical case if $c_1=c_2=0$, $c_1=c_2=\pi$, or $c_1=0$ and $c_2=\pi$.

3.2 The oriented needle

Now we suppose that the random variable φ follows a *trapezoidal distribution* of support $[0,2\pi]$, lower mode c_1 and upper mode c_2 i.e. its density is:

$$g(\varphi)=\begin{cases}\dfrac{2}{2\pi+c_2-c_1}\,\dfrac{\varphi}{c_1} & \text{for } 0\le\varphi\le c_1\\ \dfrac{2}{2\pi+c_2-c_1} & \text{for } c_1\le\varphi\le c_2\\ \dfrac{2}{2\pi+c_2-c_1}\,\dfrac{2\pi-\varphi}{2\pi-c_2} & \text{for } c_2<\varphi\le 2\pi\end{cases}\tag{3.3}$$

and the random variable v is uniformly distributed in $[kd,(k+1)d]$.

As above, we obtain the probability

$$p=\frac{2l}{d}\int_0^{2\pi} g(\varphi)\,|\sin\varphi|\,\mathrm{d}\varphi\tag{3.4}$$

To evaluate the integral in Equation (3.4) we have to consider three cases: (i) $0<c_1<c_2<\pi$; (ii) $0<c_1<\pi<c_2<2\pi$, and (iii) $0<c_1<\pi<c_2<2\pi$.

In the case (i) we obtain:

$$\int_0^{2\pi} g(\varphi)\,|\sin\varphi|\,\mathrm{d}\varphi=$$

$$\frac{2l}{d}\,\frac{2}{2\pi+c_2-c_1}\left[\int_0^{c_1}\frac{\varphi}{c_1}\sin(\varphi)\,\mathrm{d}\varphi+\int_{c_1}^{c_2}\sin(\varphi)\,\mathrm{d}\varphi+\int_{c_2}^{\pi}\frac{2\pi-\varphi}{2\pi-c_2}\sin(\varphi)\,\mathrm{d}\varphi+\right.$$

$$\left.\int_{\pi}^{2\pi}\frac{2\pi-\varphi}{2\pi-c_2}\sin(\varphi)\,\mathrm{d}\varphi\right]=$$

$$\frac{4l}{d}\,\frac{2\pi c_1+2\pi\sin(c_1)+\sin(c_2)\,c_1-\sin(c_1)\,c_2}{(2\pi-c_2)(2\pi+c_2-c_1)\,c_1}$$

In the case (ii) we obtain:

$$\int_0^{2\pi} g(\varphi)\,|\sin\varphi|\,\mathrm{d}\varphi =$$

$$\frac{2l}{d}\frac{2}{2\pi+c_2-c_1}\left[\int_0^{c_1}\frac{\varphi}{c_1}\sin(\varphi)\,\mathrm{d}\varphi+\int_{c_1}^{\pi}\sin(\varphi)\,\mathrm{d}\varphi-\int_{\pi}^{c_2}\sin(\varphi)\,\mathrm{d}\varphi-\right.$$

$$\left.\int_{c_2}^{2\pi}\frac{2\pi-\varphi}{2\pi-c_2}\sin(\varphi)\,\mathrm{d}\varphi\right]=$$

$$\frac{4l}{d}\,\frac{4\,\pi\,c_1+2\,\pi\,\sin(c_1)-\sin(c_2)\,c_1-2\,c_2c_1-\sin(c_1)\,c_2}{(2\,\pi-c_2)\,(2\,\pi+c_2-c_1)\,c_1}$$

In the case (iii) we finally obtain:

$$\int_0^{2\pi} g(\varphi)\,|\sin\varphi|\,\mathrm{d}\varphi =$$

$$\frac{2l}{d}\frac{2}{2\pi+c_2-c_1}\left[\int_0^{c_1}\frac{\varphi}{c_1}\sin(\varphi)\,\mathrm{d}\varphi+\int_{c_1}^{\pi}\sin(\varphi)\,\mathrm{d}\varphi-\int_{\pi}^{c_2}\sin(\varphi)\,\mathrm{d}\varphi-\right.$$

$$\left.\int_{c_2}^{2\pi}\frac{2\pi-\varphi}{2\pi-c_2}\sin(\varphi)\,\mathrm{d}\varphi\right]=$$

$$\frac{4l}{d}\,\frac{4\,\pi^2-2\,\pi\,c_2-2\,\sin(c_1)\,\pi+\sin(c_1)\,c_2-c_1\sin(c_2)}{(2\,\pi-c_2)\,(2\,\pi+c_2-c_1)\,c_1}$$

So, the probability that the needle intersects one line is:

$$p=\begin{cases}\dfrac{4l}{d}\,\dfrac{2\,\pi\,c_1+2\,\pi\,\sin(c_1)+\sin(c_2)\,c_1-\sin(c_1)\,c_2}{(2\,\pi-c_2)\,(2\,\pi+c_2-c_1)\,c_1} & \text{for } 0<c_1\le c_2\le\pi\\[2ex] \dfrac{4l}{d}\,\dfrac{4\,\pi\,c_1+2\,\pi\,\sin(c_1)-\sin(c_2)\,c_1-2\,c_1c_2-\sin(c_1)\,c_2}{(2\,\pi-c_2)\,(2\,\pi+c_2-c_1)\,c_1} & \text{for } 0<c_1\le\pi\le c_2<2\pi\\[2ex] \dfrac{4l}{d}\,\dfrac{4\,\pi^2-2\,\pi\,c_2-2\pi\,\sin(c_1)+\sin(c_1)\,c_2-c_1\sin(c_2)}{(2\,\pi-c_2)\,(2\,\pi+c_2-c_1)\,c_1} & \text{for } \pi\le c_1\le c_2<2\pi\\[2ex] \dfrac{4l}{d}\,\dfrac{2(2\,\pi-c_2)+\min\{c_2,2\pi-c_2\}+|\sin(c_2)|}{(2\,\pi-c_2)\,(2\,\pi+c_2)} & \text{for } c_1=0\le c_2\le 2\pi\\[2ex] \dfrac{4l}{d}\,\dfrac{2c_1+|\sin(c_1)|+\min\{c_1,2\pi-c_1\}}{(4\,\pi-c_1)\,c_1} & \text{for } 0<c_1\le c_2=2\pi\\[2ex] \dfrac{4l}{\pi d} & \text{for } c_1=0 \text{ and } c_2=2\pi\end{cases}\tag{3.5}$$

This probability reduces to the triangular case with mode c_2 if $c_1 = c_2$ and if $c_1 = 2\pi - c_2$, and it reduces to the classical Buffon's probability for $c_1 = 0,\ c_2 = 2\pi$, $c_1 = c_2 = 0$, or $c_1 = c_2 = 2\pi$.

4 The wrapped exponential distribution

A wrapped exponential distribution is a wrapped probability distribution that results from the "wrapping" of the exponential distribution around the unit circle. Its support is the interval

$[0, 2\pi]$ and its density function is expressed as:

$$g(\varphi) = \frac{\lambda e^{-\lambda\varphi}}{1 - e^{-2\pi\lambda}} \tag{4.1}$$

where $\lambda > 0$. If the angle φ follows a wrapped exponential distribution, it is necessary to use an oriented needle. We consider $\varphi \in [0, 2\pi)$ and, as before, we assume that the random variable v is uniformly distributed in $[kd, (k+1)d]$. Consequently, the numerator in Equation (1.2) takes the form:

$$\begin{gathered} \int_0^{2\pi} d\varphi \int\limits_{kd+l\,|\sin\varphi|}^{(k+1)d-l\,|\sin\varphi|} g(\varphi) f(v)\, dv = \\ 1 - \frac{2l}{d} \int_0^{2\pi} |\sin(\varphi)| \frac{\lambda e^{-\lambda\varphi}}{1 - e^{-2\pi\lambda}}\, d\varphi \end{gathered} \tag{4.2}$$

Therefore, the probability is given by:

$$p = \frac{2l}{d} \int_0^{2\pi} |\sin(\varphi)| \frac{\lambda e^{-\lambda\varphi}}{1 - e^{-2\pi\lambda}}\, d\varphi = \frac{2l}{d} \frac{\lambda \left(e^{\lambda\pi} + 1\right)}{\left(e^{\lambda\pi} - 1\right)\left(\lambda^2 + 1\right)} \tag{4.3}$$

As $\lambda \to 0$, the probability converges to the classical Buffon's one since the wrapped exponential distribution tends to the uniform distribution.

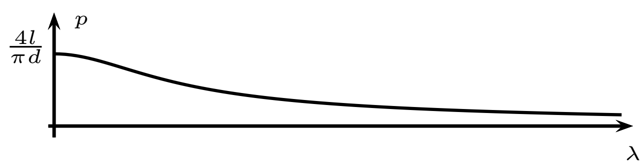


Figure 4. Wrapped exponential distribution

5 The Von Mises distribution

The *Von Mises distribution*, also called the circular normal or Tikhonov distribution, is a continuous probability distribution with support $[0, 2\pi)$ (or any interval of length 2π). Its density function is defined as:

$$g(\varphi) = \frac{e^{\kappa \cos(\varphi - \mu)}}{2\pi I_0(\kappa)} \tag{5.1}$$

where $\mu \in \mathbb{R}$ represents the mean direction of the distribution, $\kappa > 0$ is a shape parameter called the "concentration", and $I_0(\kappa)$ denotes the modified Bessel function with order zero defined for all κ as:

$$I_0(\kappa) = \sum_0^{\infty} \frac{\kappa^{2i}}{2^{2i}(i!)^2}$$

When $\kappa = 0$, the distribution becomes uniform. For large values of κ, it approaches a normal distribution in φ with mean μ and variance $\frac{1}{\kappa}$.

If the angle φ follows a Von Mises distribution, it is necessary to use an oriented needle, and we will take $\varphi \in [0, 2\pi)$ and, as above, we will suppose that the random variable v is uniformly distributed in $[kd, (k+1)d]$.

Consequently, the numerator in Equation (1.2) becomes:

$$\int_0^{2\pi} d\varphi \int\limits_{kd+l\,|\sin\varphi|}^{(k+1)d-l\,|\sin\varphi|} g(\varphi) f(v)\,dv = \tag{5.2}$$
$$1 - \frac{l}{d}\,\frac{1}{\pi I_0(\kappa)} \int_0^{2\pi} |\sin(\varphi)|\, e^{\kappa\cos(\varphi-\mu)}\, d\varphi$$

Thus the probability is given by:

$$p = \frac{l}{d}\,\frac{1}{\pi I_0(\kappa)} \int_0^{2\pi} |\sin(\varphi)|\, e^{\kappa\cos(\varphi-\mu)}\, d\varphi \tag{5.3}$$

It is possible to evaluate the probability in a closed form only for $\mu = 0$ and $\mu = \pi$ and we show the graphics for some other case.

When $\kappa = 0$ the Von Mises distribution reduces to the uniform distribution, yielding the classical Buffon's probability.

In the case $\mu = \pi$, the distribution is symmetric about π and we have:

$$\begin{aligned} p = &\frac{l}{d}\,\frac{1}{\pi I_0(\kappa)} \int_0^{2\pi} |\sin(\varphi)|\, e^{-\kappa\cos(\varphi)}\, d\varphi = \\ &\frac{l}{d}\,\frac{1}{\pi I_0(\kappa)} \left[\int_0^{\pi} \sin(\varphi) e^{-\kappa\cos(\varphi)}\, d\varphi - \int_\pi^{2\pi} \sin(\varphi) e^{-\kappa\cos(\varphi)}\, d\varphi\right] = \\ &\frac{2l}{\pi d}\,\frac{e^{\kappa} - e^{-\kappa}}{\kappa I_0(\kappa)} \end{aligned} \tag{5.4}$$

For $\mu = 0$, we obtain the same probability as above.

In Figure 5 we show some graphics for different values of μ.

When considering the probability (5.3) for a fixed κ as a function of the mean direction μ we can obtain some more information.

Let:

$$h(\mu) = \int_0^{2\pi} |\sin(\varphi)|\, e^{\kappa\cos(\varphi-\mu)}\, d\varphi \tag{5.5}$$

First of all let us note that $h(\mu)$ is symmetric about π, indeed:

$$h(2\pi-\mu) = \int_0^{2\pi} |\sin(\varphi)|\, e^{\kappa\cos(\varphi+\mu)}\, d\varphi =$$
$$\text{(with the substitution } \psi = 2\pi-\varphi) \int_0^{2\pi} |\sin(\psi)|\, e^{\kappa\cos(\psi-\mu)}\, d\psi = h(\mu)$$

The derivative of $h(\mu)$ is:

$$h'(\mu) = \int_0^{2\pi} |\sin(\varphi)|\, \kappa\sin(\varphi-\mu) e^{\kappa\cos(\varphi-\mu)}\, d\varphi$$

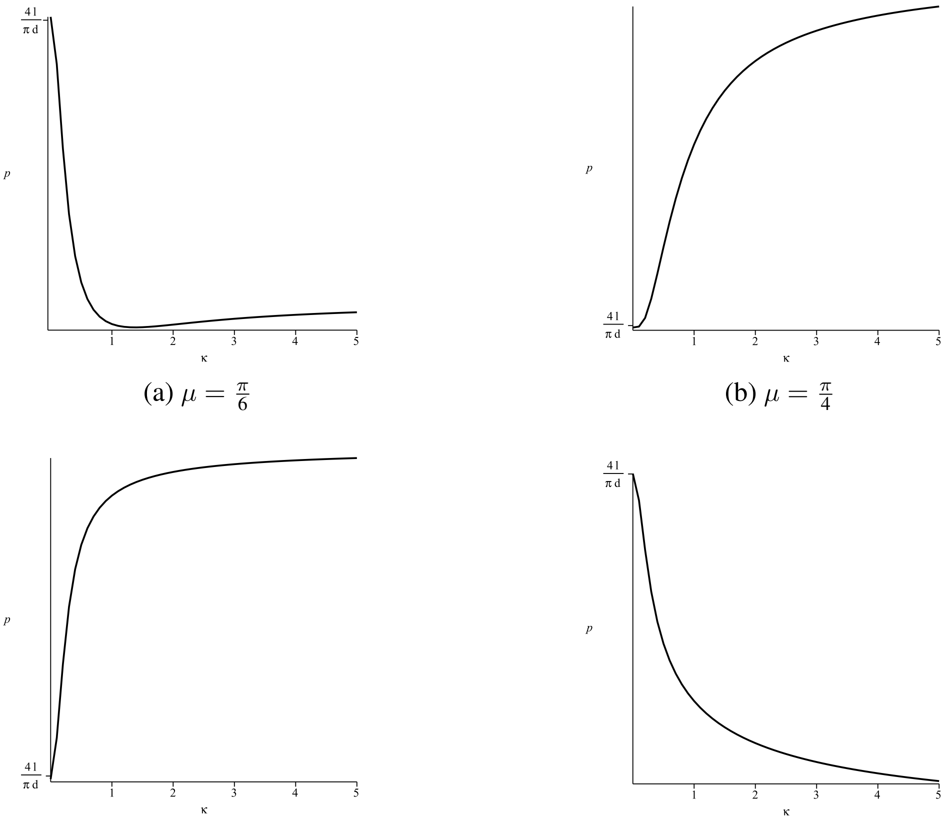
Applying integration by parts, we obtain:

$$h'(\mu) = \int_0^{\pi} \cos(\varphi)\, e^{\kappa\cos(\varphi-\mu)}\, d\varphi - \int_\pi^{2\pi} \cos(\varphi)\, e^{\kappa\cos(\varphi-\mu)}\, d\varphi$$

It is easy to check, integrating by parts again, that $h'\left(\frac{\pi}{2}\right) = 0$ and, by symmetry,

$$\int_0^{\pi} \cos(\varphi)\, e^{\kappa\cos(\varphi)}\, d\varphi = \int_\pi^{2\pi} \cos(\varphi)\, e^{\kappa\cos(\varphi)}\, d\varphi$$

and so also $h'0) = h'(\pi) = 0$.



(a) $\mu = \frac{\pi}{6}$ (b) $\mu = \frac{\pi}{4}$

(c) $\mu = \frac{\pi}{2}$ (d) $\mu = \pi$

Figure 5. Some cases of Von Mises distribution in function of κ

The second derivative of the function $h(\mu)$ is:

$$h''(\mu) = \int_0^{\pi} \kappa \cos(\varphi) \sin(\varphi - \mu) \, \mathrm{e}^{\kappa \cos(\varphi - \mu)} \mathrm{d}\varphi - \int_{\pi}^{2\pi} \kappa \cos(\varphi) \sin(\varphi - \mu) \, \mathrm{e}^{\kappa \cos(\varphi - \mu)} \mathrm{d}\varphi$$

By integration by parts we obtain:

$$h''(\mu) = 2\,\mathrm{e}^{-\cos(\mu)\kappa} + 2\,\mathrm{e}^{\cos(\mu)\kappa} - \int_0^{2\pi} |\sin(\varphi)| \, \mathrm{e}^{\kappa \cos(\varphi - \mu)} \mathrm{d}\varphi$$

As

$$\int_0^{2\pi} |\sin(\varphi)| \, \mathrm{e}^{\kappa \cos(\varphi)} \mathrm{d}\varphi = \frac{2\left(\mathrm{e}^{\kappa} - \mathrm{e}^{-\kappa}\right)}{\kappa}$$

we obtain

$$h''(0) = h''(\pi) = \frac{2(\kappa \mathrm{e}^{\kappa} + \kappa \mathrm{e}^{-\kappa} - \mathrm{e}^{\kappa} + \mathrm{e}^{-\kappa})}{k} > 0$$

Since

$$\int_0^{2\pi} |\sin(\varphi)| \, \mathrm{e}^{\kappa \cos\left(\varphi - \frac{\pi}{2}\right)} \mathrm{d}\varphi = \int_0^{\pi} \sin(\varphi) \, \mathrm{e}^{\kappa \sin(\varphi)} \mathrm{d}\varphi - \int_{\pi}^{2\pi} \sin(\varphi) \, \mathrm{e}^{\kappa \sin(\varphi)} \mathrm{d}\varphi =$$

(with the substitution $\varphi = \pi + \psi$ in the second integral)

$$\int_0^{\pi} \sin(\varphi) \left(\mathrm{e}^{\kappa \sin(\varphi)} + \mathrm{e}^{-\kappa \sin(\varphi)}\right) \mathrm{d}\varphi > \int_0^{\pi} 2 \sin(\varphi) \, \mathrm{d}\varphi = 4$$

we have

$$h''\left(\frac{\pi}{2}\right) = 4 - \int_0^{2\pi} |\sin(\varphi)|\, \mathrm{e}^{\kappa\cos\left(\varphi - \frac{\pi}{2}\right)} \mathrm{d}\varphi < 0$$

So, as expected, the points $\mu = 0$ and $\mu = \pi$ are of minimum for the probability, with

$$p = \frac{2l}{\pi d}\frac{\mathrm{e}^{\kappa} - \mathrm{e}^{-\kappa}}{\kappa I_0(\kappa)},$$

as the points $\mu = \frac{\pi}{2}$ and $\mu = \frac{3\pi}{2}$ are of maximum for p. The behavior of the function is shown in Figure 6.

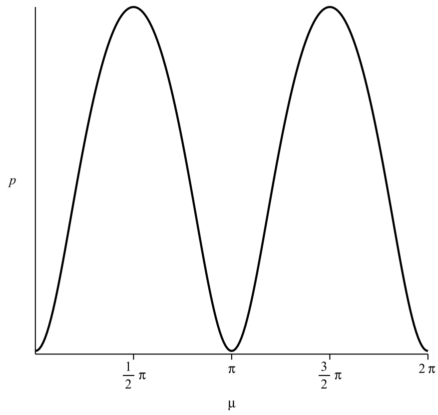


Figure 6. Von Mises distribution as a function of μ

References

[1] M. Aigner, and G. Ziegler, *Proofs from The Book*, Springer-Verlag, Berlin, (1999).

[2] F. Altomare, *Sur quelques problèmes de probabilités du type aiguille de Buffon*, Publications de l'Institut de Statistique de l'Université de Paris, **30**, 1–14, (1985).

[3] É. Barbier, *Note sur le problème de l'aiguille et le jeu du joint couvert*, Journal De Mathématiques Pures Et Appliquées, 273–286, (1860).

[4] G. Caristi, E. Saitta, and M. Stoka, *A Laplace type problem with non uniform distribution*, Applied Mathematical Sciences, **10**, 1595–1602 (2016).

[5] V. Conserva, and S. Rizzo, *Non uniform geometric probability for regular tilings*, Rendiconti Del Seminario Matematico Di Messina, Serie II, **XV**, Suppl. n.1, (1991).

[6] V. Conserva, and S. Rizzo, *Quelques extensions anisotropiques du problème de Buffon-Laplace*, Publ. Inst. Stat. Univ. Paris, **XXXIII**, (1), 91-105 (1988).

[7] V. Conserva, and S. Rizzo, *Some problems of geometric probability in a weak magnetic field*, Rend. Circ. Mat. Palermo (2) Supplemento, **65**, Part I, 79–88, (2000).

[8] M. Hykšová, A., Kalousová, and I. Saxl, *Early History of Geometric Probability and Stereology*, Image Analysis And Stereology, **31**, no.1, 1–16, (2012).

[9] M. Klamkin, *On the Uniqueness of the Distribution Function for the Buffon Needle Problem*, American Mathematical Monthly, **60**, 677-680 (1953)

[10] G. L. Leclerc, Comte de Buffon, *Essai d'arithmétique morale, Appendix to "Histoire naturelle générale et particulière", Vol. 4* (1777).

[11] F. Marriott, *337. Note: Buffon's Problems for Non-Random Distribution*, Biometrics, **28**, 621–624, (1972).

[12] F. Marriott, *302. Note: Buffon's Problem for Non-Random Directions*, Biometrics, **27**, 233–235, (1971).

[13] A. Mathai, *An introduction to geometrical probability*, Gordon and Breach Science Publishers, Amsterdam, (1999).

[14] L. Santaló, *Integral geometry and Geometric Probability*, Addison-Wesley,(1976).

[15] C. Tanasi, *L'extension anisotropique du problème de Buffon-Laplace*, Rend. Sem. Matem. Univ. Politecn. Torino, **41**, n.1, 115–130, (1983).

[16] S. Vassallo, *Some problems of non uniform geometric probability relative to plane lattices*, Rend. Circ. Mat. Palermo (2) Supplemento, **40**, 270-280 (1991).

Author information

Salvatore Vassallo, Dipartimento di Matematica per le Scienze economiche, finanziarie ed attuariali
Università Cattolica del S. Cuore
Milano, Italy.
E-mail: salvatore.vassallo@unicatt.it

Received: 2022-09-08
Accepted: 2023-10-02