Palestine Journal of Mathematics

Vol 13(1)(2024) , 129–141

FOURTH LAPLACE-BELTRAMI OPERATOR OF ROTATIONAL HYPERSURFACES IN E_1^4

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 14J70; Secondary 53A35.

Keywords and phrases: Rotational hypersurface, Fourth Laplace-Beltrami operator, Fundamental forms.

Abstract In the present paper, we obtain the fourth Laplace-Beltrami operator of rotational hypersurfaces about spacelike, timelike, and lightlike axes separately in 4-dimensional Lorentz-Minkowski space and prove theorems about fourth Laplace-Beltrami minimality of them. Also, we construct some examples for these rotational hypersurfaces, obtain their fourth Laplace-Beltrami operators and give their visualizations into 3-spaces.

1 General Information and Basic Concepts

It is known that a rotational hypersurface is defined as a hypersurface rotating a curve around an axis. In this context, if $\alpha : I \subset R \longrightarrow \pi$ is a curve in a plane π of E_1^4 and l is a straight line in E_1^4, then a rotational hypersurface is defined by a hypersurface rotating the profile curve α around the axis l. With the aid of this definition, the differential geometry of rotational (hyper)surfaces in 3 or higher-dimensional Euclidean, Minkowskian, Galilean, and pseudo-Galilean spaces have been studied by mathematicians. For instance, finite type surfaces of revolution in a Euclidean 3-space have been classified in [8] and some properties about surfaces of revolution in four dimensions have been given in [21]. The general rotational surfaces in Minkowski 4-space and the third Laplace-Beltrami operator and the Gauss map of the rotational hypersurface in Euclidean 4-space have been studied in [12] and [13], respectively. In [4], Arslan and his friends have considered generalized rotational surfaces imbedded in a Euclidean space of four dimensions and also they have given some special examples of these surfaces in E^4. In [24], Yoon has studied on rotational surfaces with finite type Gauss maps in Euclidean 4-space. The explicit parameterizations of rotational hypersurfaces in Lorentz-Minkowski space E_1^n have been given and rotational hypersurfaces in E_1^n with constant mean curvature have been obtained in [11]. For more studies about different types of curves and (hyper)surfaces in different spaces, we refer to [1], [2], [5], [6], [7], [9], [10], [14], [15], [16], [18], [19], [20], [22], [23], [25], and etc.

Now, let us recall some fundamental notions for hypersurfaces in Lorentz-Minkowski 4-space.

Let $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ and $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ be three vectors in 4-dimensional Lorentz-Minkowski space E_1^4. Then, the inner product and vector product are defined by

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \tag{1.1}$$

and

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}, \tag{1.2}$$

respectively. Also, the norm of the vector $\overrightarrow{x}$ is $\|\overrightarrow{x}\| = \sqrt{|\langle \overrightarrow{x}, \overrightarrow{x} \rangle|}$.

The Gauss map (i.e., the unit normal vector field), the matrix forms of the first, second and third fundamental forms of a hypersurface

$$\Psi : E^3 \longrightarrow E^4 \tag{1.3}$$
$$(u_1, u_2, u_3) \longrightarrow \Psi(u_1, u_2, u_3) = (\Psi_1(u_1, u_2, u_3), \Psi_2(u_1, u_2, u_3), \Psi_3(u_1, u_2, u_3), \Psi_4(u_1, u_2, u_3))$$

in E_1^4 are

$$N = \frac{\Psi_{u_1} \times \Psi_{u_2} \times \Psi_{u_3}}{\|\Psi_{u_1} \times \Psi_{u_2} \times \Psi_{u_3}\|}, \tag{1.4}$$

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \tag{1.5}$$

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{1.6}$$

and

$$[m_{ij}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \tag{1.7}$$

respectively. Here, we state that $g_{ij} = \langle \Psi_{u_i}, \Psi_{u_j} \rangle$, $h_{ij} = \langle \Psi_{u_i u_j}, N \rangle$, $m_{ij} = \langle N_{u_i}, N_{u_j} \rangle$, $\Psi_{u_i} = \frac{\partial \Psi}{\partial u_i}$, $\Psi_{u_i u_j} = \frac{\partial^2 \Psi}{\partial u_i u_j}$, $N_{u_i} = \frac{\partial N(u_1,u_2,u_3)}{\partial u_i}$, $i, j \in \{1, 2, 3\}$.

If we denote the inverse matrix of $[g_{ij}]$ as $[g_{ij}]^{-1}$, then the shape operator of the hypersurface (1.3) is given by

$$S = [g_{ij}]^{-1}.[h_{ij}]. \tag{1.8}$$

From (1.4)-(1.6) and (1.8), the Gaussian and mean curvatures of the hypersurface (1.3) in E_1^4 are defined by

$$K = \varepsilon \frac{\det[h_{ij}]}{\det[g_{ij}]} \tag{1.9}$$

and

$$3\varepsilon H = tr(S) \tag{1.10}$$

respectively. Here, $\varepsilon = \langle N, N \rangle$. For more details about hypersurfaces in 4-dimensional spaces, one can see [3], [14], [16], and etc.

If we denote fourth fundamental form of the hypersurface (1.3) in E_1^4 by $[n_{ij}]$, then we have [17]

$$\begin{aligned} [h_{ij}] &= [g_{ij}].S, \\ [m_{ij}] &= [h_{ij}].S = [g_{ij}].S.S, \\ [n_{ij}] &= [m_{ij}].S = [h_{ij}].S.S = [g_{ij}].S.S.S. \end{aligned} \tag{1.11}$$

Also, the inverse of an arbitrary matrix

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.12}$$

in E_1^4 is

$$[A^{ij}] = \frac{1}{\det[A_{ij}]} \begin{bmatrix} A_{22}A_{33} - A_{23}A_{32} & A_{13}A_{32} - A_{12}A_{33} & A_{12}A_{23} - A_{13}A_{22} \\ A_{23}A_{31} - A_{21}A_{33} & A_{11}A_{33} - A_{13}A_{31} & A_{13}A_{21} - A_{11}A_{23} \\ A_{21}A_{32} - A_{22}A_{31} & A_{12}A_{31} - A_{11}A_{32} & A_{11}A_{22} - A_{12}A_{21} \end{bmatrix}, \tag{1.13}$$

where

$$\det[A_{ij}] = -A_{13}A_{22}A_{31} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{11}A_{22}A_{33}. \tag{1.14}$$

2 Fourth Laplace-Beltrami Operator of Rotational Hypersurfaces about Spacelike Axis in E_1^4

In this section, firstly we obtain the fourth fundamental form of rotational hypersurfaces about spacelike axis in E_1^4 with the aid of first, second and third fundamental forms and shape operator. After that, we reach the fourth Laplace-Beltrami (LB$^{\mathrm{IV}}$) operator of this hypersurface and give some results and examples for LB$^{\mathrm{IV}}$ operator.

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}, \nu \longrightarrow f(\nu), (\nu \in \mathbb{R}-\{0\})$ be a smooth function. If we rotate the profile curve $\alpha(\nu) = (\nu, 0, 0, f(\nu))$ about spacelike axis $(0, 0, 0, 1)$, then the rotational hypersurfaces about spacelike axis in E_1^4 is given by

$$\Psi^S(\nu,\vartheta,\theta) = \begin{bmatrix} \cosh\vartheta\cosh\theta & \sinh\vartheta\cosh\theta & \sinh\theta & 0 \\ \sinh\vartheta & \cosh\vartheta & 0 & 0 \\ \cosh\vartheta\sinh\theta & \sinh\vartheta\sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \nu \\ 0 \\ 0 \\ f(\nu) \end{bmatrix}$$
$$= (\nu\cosh\vartheta\cosh\theta, \nu\sinh\vartheta, \nu\cosh\vartheta\sinh\theta, f(\nu)). \tag{2.1}$$

We must note that, from now on, the superscripts ^{S}, ^{T} and ^{L} denote the rotation axes.

From (1.5), the first fundamental form of rotational hypersurface (2.1), its inverse matrix and determinant are

$$[g_{ij}]^S = \begin{bmatrix} f'^2 - 1 & 0 & 0 \\ 0 & \nu^2 & 0 \\ 0 & 0 & \nu^2\cosh^2\vartheta \end{bmatrix}, \tag{2.2}$$

$$[g^{ij}]^S = \begin{bmatrix} \frac{1}{f'^2-1} & 0 & 0 \\ 0 & \frac{1}{\nu^2} & 0 \\ 0 & 0 & \frac{1}{\nu^2\cosh^2\vartheta} \end{bmatrix} \tag{2.3}$$

and

$$\det[g_{ij}]^S = \nu^4\left(f'^2 - 1\right)\cosh^2\vartheta, \tag{2.4}$$

respectively. Here, if we suppose that $f'^2 - 1 < 0$, then we have $\det[g_{ij}] < 0$ and so, we deal with timelike rotational hypersurface (2.1). Similarly, one can obtain the corresponding results for spacelike rotational hypersurfaces by supposing $f'^2 - 1 > 0$.

From (1.4), the unit normal vector field of rotational hypersurface (2.1) is

$$N^S = -\frac{1}{\sqrt{1-f'^2}}\left(f'\cosh\vartheta\cosh\theta, f'\sinh\vartheta, f'\cosh\vartheta\sinh\theta, 1\right) \tag{2.5}$$

and so, from (2.5), it can be seen that

$$\varepsilon^S = \left\langle N^S, N^S\right\rangle = 1. \tag{2.6}$$

Here, we denote $f = f(\nu)$ and $f' = \frac{df(\nu)}{d\nu}$.

From (1.6) and (1.7), the second and third fundamental forms of the rotational hypersurface (2.1) are obtained by

$$[h_{ij}]^S = \frac{1}{\sqrt{1-f'^2}}\begin{bmatrix} -f'' & 0 & 0 \\ 0 & \nu f' & 0 \\ 0 & 0 & \nu f'\cosh^2\vartheta \end{bmatrix} \tag{2.7}$$

and

$$[m_{ij}]^S = \begin{bmatrix} \frac{-f''^2}{(-1+f'^2)^2} & 0 & 0 \\ 0 & \frac{f'^2}{1-f'^2} & 0 \\ 0 & 0 & \frac{f'^2\cosh^2\vartheta}{1-f'^2} \end{bmatrix}, \tag{2.8}$$

respectively, where $f'' = \frac{d^2f(\nu)}{d\nu^2}$.

Also, using (2.3) and (2.7) in (1.8), the shape operator of the rotational hypersurface (2.1) is given by

$$S^S = \frac{1}{\sqrt{1-f'^2}} \begin{bmatrix} \frac{f''}{1-f'^2} & 0 & 0 \\ 0 & \frac{f'}{\nu} & 0 \\ 0 & 0 & \frac{f'}{\nu} \end{bmatrix}. \tag{2.9}$$

Now, we obtain the LB$^{\text{IV}}$ operator of the rotational hypersurface (2.1) in E_1^4 and prove an important theorem about LB$^{\text{IV}}$-minimality.

For obtaining the LB$^{\text{IV}}$ operator of the rotational hypersurface (2.1) in E_1^4, firstly the fourth fundamental form of this hypersurface must be given. In this context, if (2.8) and (2.9) are used in (1.11), then the fourth fundamental form of the rotational hypersurface (2.1) is obtained by

$$\begin{aligned} [n_{ij}]^S &= \begin{bmatrix} \frac{-f''^2}{(-1+f'^2)^2} & 0 & 0 \\ 0 & \frac{f'^2}{1-f'^2} & 0 \\ 0 & 0 & \frac{f'^2\cosh^2\vartheta}{1-f'^2} \end{bmatrix} \cdot \begin{bmatrix} \frac{f''}{(1-f'^2)^{3/2}} & 0 & 0 \\ 0 & \frac{f'}{\nu\sqrt{1-f'^2}} & 0 \\ 0 & 0 & \frac{f'}{\nu\sqrt{1-f'^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-f''^3}{(1-f'^2)^{7/2}} & 0 & 0 \\ 0 & \frac{f'^3}{\nu(1-f'^2)^{3/2}} & 0 \\ 0 & 0 & \frac{f'^3\cosh^2\vartheta}{\nu(1-f'^2)^{3/2}} \end{bmatrix}. \end{aligned} \tag{2.10}$$

So, from (2.10), we get

$$\det[n_{ij}]^S = -\frac{f'^6 f''^3\cosh^2\vartheta}{\nu^2\left(1-f'^2\right)^{13/2}}. \tag{2.11}$$

Now, we can obtain the LB$^{\text{IV}}$ operator of the rotational hypersurface (2.1) in E_1^4.

The fourth Laplace-Beltrami (LB$^{\text{IV}}$) operator of a smooth function $\varphi = \varphi(\nu^1,\nu^2,\nu^3)|_D$, $(D\subset\mathbb{R}^3)$ of class C^3 with respect to the nondegenerate fourth fundamental form of hypersurface Ψ in E_1^4 is the operator which is defined as follows:

$$\Delta^{IV}\varphi = -\frac{1}{\sqrt{|\det[n_{ij}]|}}\sum_{i,j=1}^{3}\frac{\partial}{\partial\nu^i}\left(\sqrt{|\det[n_{ij}]|}\,n^{ij}\frac{\partial\varphi}{\partial\nu^j}\right), \tag{2.12}$$

where n^{ij} are the components of the inverse matrix $[n_{ij}]^{-1}$.

So, using (1.13), (1.14) and (2.12), the LB$^{\text{IV}}$ operator of a smooth function $\varphi=\varphi(\nu,\vartheta,\theta)$ can be written as

$$\Delta^{IV}\varphi = -\frac{1}{\sqrt{|\det[n_{ij}]|}}\left\{\begin{array}{l} \frac{\partial}{\partial\nu}\left(\frac{(-n_{23}n_{32}+n_{22}n_{33})\varphi_\nu+(n_{13}n_{23}-n_{12}n_{33})\varphi_\vartheta+(n_{12}n_{23}-n_{13}n_{22})\varphi_\theta}{\sqrt{|\det[n_{ij}]|}}\right) \\ +\frac{\partial}{\partial\vartheta}\left(\frac{(-n_{21}n_{33}+n_{23}n_{31})\varphi_\nu+(n_{11}n_{33}-n_{13}n_{31})\varphi_\vartheta+(n_{13}n_{21}-n_{11}n_{23})\varphi_\theta}{\sqrt{|\det[n_{ij}]|}}\right) \\ +\frac{\partial}{\partial\theta}\left(\frac{(-n_{22}n_{31}+n_{21}n_{32})\varphi_\nu+(n_{12}n_{31}-n_{11}n_{32})\varphi_\vartheta+(n_{11}n_{22}-n_{12}n_{21})\varphi_\theta}{\sqrt{|\det[n_{ij}]|}}\right) \end{array}\right\}, \tag{2.13}$$

where

$$\det[n_{ij}] = -n_{13}n_{22}n_{31}+n_{12}n_{23}n_{31}+n_{13}n_{21}n_{32}-n_{11}n_{23}n_{32}-n_{12}n_{21}n_{33}+n_{11}n_{22}n_{33}.$$

Let us denote the LB$^{\text{IV}}$ operator of a rotational hypersurface Ψ in E_1^4 as $\Delta^{IV}\Psi$. Then, with the aid of (2.13), we can write

$$\begin{aligned} \Delta^{IV}\Psi &= ((\Delta^{IV}\Psi)_1,(\Delta^{IV}\Psi)_2,(\Delta^{IV}\Psi)_3,(\Delta^{IV}\Psi)_4) \\ &= -\frac{1}{\sqrt{|\det[n_{ij}]|}}\left(\begin{array}{l} (\mathfrak{U}_1)_\nu+(\mathfrak{V}_1)_\vartheta+(\mathfrak{W}_1)_\theta,(\mathfrak{U}_2)_\nu+(\mathfrak{V}_2)_\vartheta+(\mathfrak{W}_2)_\theta, \\ (\mathfrak{U}_3)_\nu+(\mathfrak{V}_3)_\vartheta+(\mathfrak{W}_3)_\theta,(\mathfrak{U}_4)_\nu+(\mathfrak{V}_4)_\vartheta+(\mathfrak{W}_4)_\theta \end{array}\right), \end{aligned} \tag{2.14}$$

where

$$\left.\begin{array}{l}
\mathfrak{U}_i = \frac{1}{\sqrt{|\det[n_{ij}]|}}\left((-n_{23}n_{32}+n_{22}n_{33})(\Psi_i)_\nu+(n_{13}n_{23}-n_{12}n_{33})(\Psi_i)_\vartheta+(n_{12}n_{23}-n_{13}n_{22})(\Psi_i)_\theta\right), \\
\mathfrak{V}_i = \frac{1}{\sqrt{|\det[n_{ij}]|}}\left((-n_{21}n_{33}+n_{23}n_{31})(\Psi_i)_\nu+(n_{11}n_{33}-n_{13}n_{31})(\Psi_i)_\vartheta+(n_{13}n_{21}-n_{11}n_{23})(\Psi_i)_\theta\right), \\
\mathfrak{W}_i = \frac{1}{\sqrt{|\det[n_{ij}]|}}\left((-n_{22}n_{31}+n_{21}n_{32})(\Psi_i)_\nu+(n_{12}n_{31}-n_{11}n_{32})(\Psi_i)_\vartheta+(n_{11}n_{22}-n_{12}n_{21})(\Psi_i)_\theta\right).
\end{array}\right\}. \tag{2.15}$$

Using (2.1) and (2.10) in (2.15), we have

$$\left.\begin{array}{l}
\mathfrak{U}_1^S = -\frac{f'^3(1-f'^2)^{1/4}\cosh^2\vartheta\cosh\theta}{\nu\sqrt{-f''^3}},\ \mathfrak{U}_2^S = -\frac{f'^3(1-f'^2)^{1/4}\cosh\vartheta\sinh\vartheta}{\nu\sqrt{-f''^3}} \\
\mathfrak{U}_3^S = -\frac{f'^3(1-f'^2)^{1/4}\cosh^2\vartheta\sinh\theta}{\nu\sqrt{-f''^3}},\ \mathfrak{U}_4^S = -\frac{f'^4(1-f'^2)^{1/4}\cosh\vartheta}{\nu\sqrt{-f''^3}};
\end{array}\right\} \tag{2.16}$$

$$\left.\begin{array}{l}
\mathfrak{V}_1^S = -\frac{\nu\sqrt{-f''^3}\cosh\vartheta\sinh\vartheta\cosh\theta}{(1-f'^2)^{7/4}},\ \ \mathfrak{V}_2^S = -\frac{\nu\sqrt{-f''^3}\cosh^2\vartheta}{(1-f'^2)^{7/4}} \\
\mathfrak{V}_3^S = -\frac{\nu\sqrt{-f''^3}\cosh\vartheta\sinh\vartheta\sinh\theta}{(1-f'^2)^{7/4}},\ \mathfrak{V}_4^S = 0
\end{array}\right\} \tag{2.17}$$

and

$$\left.\begin{array}{l}
\mathfrak{W}_1^S = -\frac{\nu\sqrt{-f''^3}\sinh\theta}{(1-f'^2)^{7/4}},\ \mathfrak{W}_2^S = 0 \\
\mathfrak{W}_3^S = -\frac{\nu\sqrt{-f''^3}\cosh\theta}{(1-f'^2)^{7/4}},\ \mathfrak{W}_4^S = 0.
\end{array}\right\} \tag{2.18}$$

Thus, if we write (2.11), (2.16), (2.17) and (2.18) in equation (2.14), we can obtain the following theorem:

Theorem 2.1. *The LB^{IV} operator of the rotational hypersurface (2.1) about spacelike axis in E_1^4 is*

$$\Delta^{IV}\Psi^S = \frac{(1-f'^2)^{3/2}}{2\nu f'^3 f''^4}(-A\cosh\vartheta\cosh\theta, -A\sinh\vartheta, -A\cosh\vartheta\sinh\theta, B(1-f'^2)f'^3),$$

where

$$A = \nu f'^2 f''^2\left(-6+13f'^2-7f'^4\right)+4\nu^3 f''^4+f'^3\left(1-f'^2\right)^2\left(2f''+3\nu f'''\right)$$

and

$$B = \nu f''^2(8-9f'^2) - f'\left(1-f'^2\right)\left(2f''+3\nu f'''\right).$$

Using Theorem 2.1, we have

Theorem 2.2. *The rotational hypersurface (2.1) about spacelike axis in E_1^4 cannot be LB^{IV}-minimal.*

Proof. A hypersurface is *LB*$^{\text{IV}}$*-minimal,* if all components of *LB*$^{\text{IV}}$ operator vanishes. According to this definition, when we state the *LB*$^{\text{IV}}$ operator of the rotational hypersurface (2.1) as

$$\Delta^{IV}\Psi^S = ((\Delta^{IV}\Psi^S)_1, (\Delta^{IV}\Psi^S)_2, (\Delta^{IV}\Psi^S)_3, (\Delta^{IV}\Psi^S)_4),$$

it must be

$$(\Delta^{IV}\Psi^S)_1 = (\Delta^{IV}\Psi^S)_2 = (\Delta^{IV}\Psi^S)_3 = (\Delta^{IV}\Psi^S)_4 = 0. \tag{2.19}$$

From Theorem 2.1, for the equation (2.19) satisfies, we must have

$$\nu f'^2 f''^2\left(-6+13f'^2-7f'^4\right)+4\nu^3 f''^4+f'^3\left(1-f'^2\right)^2\left(2f''+3\nu f^3\right)=0 \tag{2.20}$$

and

$$\nu f''^2(8-9f'^2) - f'\left(1-f'^2\right)\left(2f''+3\nu f^3\right)=0 \tag{2.21}$$

simultaneously. From (2.21), we get

$$\frac{\nu f''^2(8-9f'^2)}{(1-f'^2)} = f'\left(2f''+3\nu f^3\right)$$

and using the last equation in (2.20), we reach that

$$f'^2(1-f'^2)^2+2\nu^2 f''^2=0.$$

Since $f''\neq 0\neq f'\neq 1$ for our rotational hypersurface (2.1), the last expression is not possible for any f. Thus, the rotational hypersurface (2.1) is never LB$^{\text{IV}}$-minimal and the proof completes. □

Example 2.3. Taking the profile curve of the rotational hypersurface (2.1) in E_1^4 as $\left(\nu,0,0,\nu^2\right)$, we have

$$\Psi^S(\nu,\vartheta,\theta) = \left(\nu\cosh\vartheta\cosh\theta, \nu\sinh\vartheta, \nu\cosh\vartheta\sinh\theta, \nu^2\right). \tag{2.22}$$

and the LB$^{\text{IV}}$ operator of this hypersurface is obtained by

$$\Delta^{IV}\Psi^S = \frac{1}{4}\begin{pmatrix} \nu(1-4\nu^2)^{3/2}(20\nu^2-9)\cosh\vartheta\cosh\theta, \nu(1-4\nu^2)^{3/2}(20\nu^2-9)\sinh\vartheta, \\ \nu(1-4\nu^2)^{3/2}(20\nu^2-9)\cosh\vartheta\sinh\theta, (1-4\nu^2)^{5/2}(3-14\nu^2) \end{pmatrix}.$$

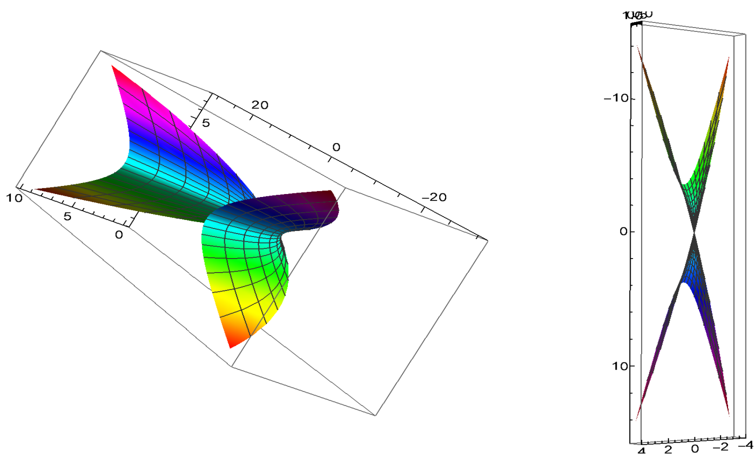

Figure 1. Projections of the rotational hypersurface (2.22) for $\theta=2$ into $x_2x_3x_4$-space (left) and $x_1x_2x_4$-space (right)

3 Fourth Laplace-Beltrami Operator of Rotational Hypersurfaces about Timelike Axis in E_1^4

Let $g: I\subset\mathbb{R}\longrightarrow\mathbb{R}$, $\nu\longrightarrow g(\nu)$, $(\nu\in\mathbb{R}-\{0\})$ be a smooth function. If we rotate the profile curve $\beta(\nu)=(g(\nu),0,0,\nu)$ about timelike axis $(1,0,0,0)$, then the rotational hypersurfaces about timelike axis in E_1^4 is given by

$$\Psi^T(\nu,\vartheta,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\vartheta\sin\theta & -\cos\vartheta\sin\theta \\ 0 & 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\theta & \sin\vartheta\cos\theta & \cos\vartheta\cos\theta \end{bmatrix}.\begin{bmatrix} g(\nu) \\ 0 \\ 0 \\ \nu \end{bmatrix} \tag{3.1}$$
$$= \left(g(\nu), -\nu\cos\vartheta\sin\theta, -\nu\sin\vartheta, \nu\cos\vartheta\cos\theta\right).$$

From (1.5), the first fundamental form of rotational hypersurface (3.1), its inverse matrix and determinant are

$$[g_{ij}]^T = \begin{bmatrix} 1-g'^2 & 0 & 0 \\ 0 & \nu^2 & 0 \\ 0 & 0 & \nu^2\cos^2\vartheta \end{bmatrix}, \tag{3.2}$$

$$[g^{ij}]^T = \begin{bmatrix} \frac{1}{1-g'^2} & 0 & 0 \\ 0 & \frac{1}{\nu^2} & 0 \\ 0 & 0 & \frac{1}{\nu^2\cos^2\vartheta} \end{bmatrix} \tag{3.3}$$

and

$$\det[g_{ij}]^T = -\nu^4\left(g'^2-1\right)\cos^2\vartheta, \tag{3.4}$$

respectively. Here, if we suppose that $g'^2-1>0$, then we have $\det[g_{ij}]<0$ and so, we deal with timelike rotational hypersurface (3.1). Similarly, one can obtain the corresponding results for spacelike rotational hypersurfaces by supposing $g'^2-1<0$.

From (1.4), the unit normal vector field of rotational hypersurface (3.1) is

$$N^T = \frac{1}{\sqrt{g'^2-1}}\left(1, -g'\cos\vartheta\sin\theta, -g'\sin\vartheta, g'\cos\vartheta\cos\theta\right) \tag{3.5}$$

and so, from (3.5), it can be seen that

$$\varepsilon^T = \left\langle N^T, N^T\right\rangle = 1. \tag{3.6}$$

Here, we denote $g=g(\nu)$ and $g'=\frac{dg(\nu)}{d\nu}$.

From (1.6) and (1.7), the second and third fundamental forms of the rotational hypersurface (3.1) are obtained by

$$[h_{ij}]^T = -\frac{1}{\sqrt{g'^2-1}}\begin{bmatrix} g'' & 0 & 0 \\ 0 & \nu g' & 0 \\ 0 & 0 & \nu g'\cos^2\vartheta \end{bmatrix} \tag{3.7}$$

and

$$[m_{ij}]^T = \begin{bmatrix} \frac{-g''^2}{(g'^2-1)^2} & 0 & 0 \\ 0 & \frac{g'^2}{g'^2-1} & 0 \\ 0 & 0 & \frac{g'^2\cos^2\vartheta}{g'^2-1} \end{bmatrix}, \tag{3.8}$$

respectively, where $g''=\frac{d^2g(\nu)}{d\nu^2}$.

Also, using (3.3) and (3.7) in (1.8), the shape operator of the rotational hypersurface (3.1) is given by

$$S^T = \frac{1}{\sqrt{g'^2-1}}\begin{bmatrix} \frac{g''}{g'^2-1} & 0 & 0 \\ 0 & -\frac{g'}{\nu} & 0 \\ 0 & 0 & -\frac{g'}{\nu} \end{bmatrix}. \tag{3.9}$$

Now, we obtain the LB$^{\rm IV}$ operator of the rotational hypersurface (3.1) in E_1^4 and prove an important theorem about LB$^{\rm IV}$-minimality.

For obtaining the LB$^{\rm IV}$ operator of the rotational hypersurface (3.1) in E_1^4, firstly the fourth fundamental form of this hypersurface must be given. In this context, if (3.8) and (3.9) are used in (1.11), then the fourth fundamental form of the rotational hypersurface (3.1) is obtained by

$$\begin{aligned}[n_{ij}]^T &= \begin{bmatrix} \frac{-g''^2}{(g'^2-1)^2} & 0 & 0 \\ 0 & \frac{g'^2}{g'^2-1} & 0 \\ 0 & 0 & \frac{g'^2\cos^2\vartheta}{g'^2-1} \end{bmatrix}.\begin{bmatrix} \frac{g''}{(g'^2-1)^{3/2}} & 0 & 0 \\ 0 & -\frac{g'}{\nu\sqrt{g'^2-1}} & 0 \\ 0 & 0 & -\frac{g'}{\nu\sqrt{g'^2-1}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-g''^3}{(g'^2-1)^{7/2}} & 0 & 0 \\ 0 & \frac{-g'^3}{\nu(g'^2-1)^{3/2}} & 0 \\ 0 & 0 & \frac{-g'^3\cos^2\vartheta}{\nu(g'^2-1)^{3/2}} \end{bmatrix}. \end{aligned} \tag{3.10}$$

So, from (3.10), we get

$$\det[n_{ij}]^T = -\frac{g'^6g''^3\cos^2\vartheta}{\nu^2\left(g'^2-1\right)^{13/2}}. \tag{3.11}$$

Using (3.1) and (3.10) in equation (2.15), we have

$$\left.\begin{array}{l}\mathfrak{U}_1^T = \frac{g'^4(g'^2-1)^{1/4}\cos\vartheta}{\nu\sqrt{-g''^3}},\ \mathfrak{U}_2^T = -\frac{g'^3(g'^2-1)^{1/4}\cos^2\vartheta\sin\theta}{\nu\sqrt{-g''^3}} \\ \mathfrak{U}_3^T = -\frac{g'^3(g'^2-1)^{1/4}\cos\vartheta\sin\vartheta}{\nu\sqrt{-g''^3}},\ \mathfrak{U}_4^T = \frac{g'^3(g'^2-1)^{1/4}\cos^2\vartheta\cos\theta}{\nu\sqrt{-g''^3}};\end{array}\right\} \quad (3.12)$$

$$\left.\begin{array}{l}\mathfrak{V}_1^T = 0,\ \mathfrak{V}_2^T = \frac{\nu\sqrt{-g''^3}\cos\vartheta\sin\vartheta\sin\theta}{(g'^2-1)^{7/4}} \\ \mathfrak{V}_3^T = -\frac{\nu\sqrt{-g''^3}\cos^2\vartheta}{(g'^2-1)^{7/4}},\ \mathfrak{V}_4^T = -\frac{\nu\sqrt{-g''^3}\cos\vartheta\sin\vartheta\cos\theta}{(g'^2-1)^{7/4}}\end{array}\right\} \quad (3.13)$$

and

$$\left.\begin{array}{l}\mathfrak{W}_1^T = 0,\ \mathfrak{W}_2^T = -\frac{\nu\sqrt{-g''^3}\cos\theta}{(g'^2-1)^{7/4}}, \\ \mathfrak{W}_3^T = 0,\ \mathfrak{W}_4^T = -\frac{\nu\sqrt{-g''^3}\cos\vartheta\sin\theta}{(g'^2-1)^{7/4}}.\end{array}\right\} \quad (3.14)$$

Thus, writing the equations (3.11), (3.12), (3.13) and (3.14) in (2.14), we can give the following theorem:

Theorem 3.1. *LB^{IV} operator of the rotational hypersurface (3.1) about timelike axis in E_1^4 is*

$$\Delta^{IV}\Psi^T = \frac{(g'^2-1)^{3/2}}{2\nu g'^3 g''^4}\left(-Cg'^3(g'^2-1), D\cos\vartheta\sin\theta, D\sin\vartheta, -D\cos\vartheta\cos\theta\right),$$

where

$$C = \nu g''^2(8-9g'^2) + g'\left(g'^2-1\right)\left(2g''+3\nu g^3\right)$$

and

$$D = \nu g'^2 g''^2\left(-6+13g'^2-7g'^4\right) + 4\nu^3 g''^4 + g'^3\left(g'^2-1\right)^2\left(2g''+3\nu g^3\right).$$

Using similar method with proof of Theorem 2.2, one can give the following theorem:

Theorem 3.2. *The rotational hypersurface (3.1) about timelike axis in E_1^4 cannot be LB^{IV}-minimal.*

Example 3.3. Taking the profile curve of the rotational hypersurface (3.1) in E_1^4 as $(\sin\nu, 0, 0, \nu)$, we have

$$\Psi^T(\nu,\vartheta,\theta) = (\sin\nu, -\nu\cos\vartheta\sin\theta, -\nu\sin\vartheta, \nu\cos\vartheta\cos\theta) \quad (3.15)$$

and the LB^{IV} operator of this hypersurface is obtained by

$$\Delta^{IV}\Psi^T = \frac{1}{8\nu}\begin{pmatrix} 4\sin^3\nu(5\nu-3\nu\cos(2\nu)+\sin(2\nu)), \\ \tan^3\nu\cos\vartheta\sin\theta(6\nu-16\nu^3+4\nu\cos(2\nu)-2\nu\cos(4\nu)+2\sin(2\nu)+\sin(4\nu)), \\ \sin\vartheta\tan^3\nu(-6\nu+16\nu^3-4\nu\cos(2\nu)+2\nu\cos(4\nu)-2\sin(2\nu)-\sin(4\nu)), \\ \tan^3\nu\cos\vartheta\cos\theta(-6\nu+16\nu^3-4\nu\cos(2\nu)+2\nu\cos(4\nu)-2\sin(2\nu)-\sin(4\nu)). \end{pmatrix}$$

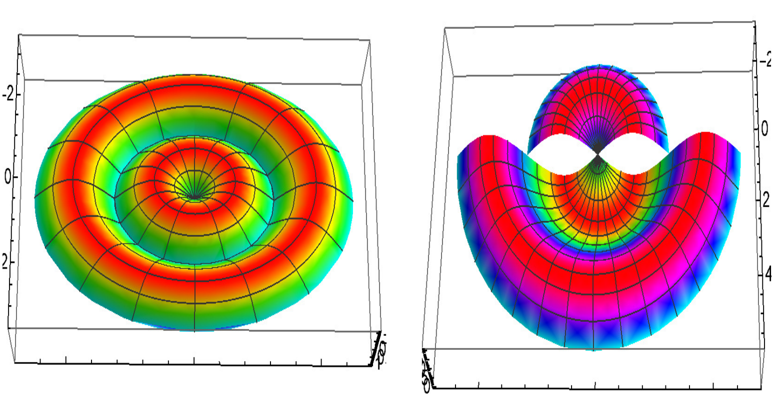

Figure 2. Projections of the rotational hypersurface (3.15) for $\theta = \pi/3$ into $x_1x_3x_4$-space (left) and $x_1x_2x_3$-space (right)

4 Fourth Laplace-Beltrami Operator of Rotational Hypersurfaces about Lightlike Axis in E_1^4

Let $h : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\nu \longrightarrow h(\nu)$, $(\nu \in \mathbb{R} - \{0\})$ be a smooth function. If we rotate the profile curve $\gamma(\nu) = (\nu, h(\nu), 0, 0)$ about lightlike axis $(1, 1, 0, 0)$, then the rotational hypersurfaces about lightlike axis in E_1^4 is given by

$$\Psi^L(\nu,\vartheta,\theta) = \begin{bmatrix} \frac{\vartheta^2+\theta^2}{2}+1 & -\frac{\vartheta^2+\theta^2}{2} & \vartheta & \theta \\ \frac{\vartheta^2+\theta^2}{2} & 1-\frac{\vartheta^2+\theta^2}{2} & \vartheta & \theta \\ \vartheta & -\vartheta & 1 & 0 \\ \theta & -\theta & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \nu \\ h(\nu) \\ 0 \\ 0 \end{bmatrix} \tag{4.1}$$

$$= \left(\frac{1}{2}\left(\left(\vartheta^2+\theta^2+2\right)\nu - \left(\vartheta^2+\theta^2\right)h(\nu)\right), \frac{1}{2}\left(\left(\vartheta^2+\theta^2\right)\nu + \left(2-\vartheta^2-\theta^2\right)h(\nu)\right), \nu\vartheta - h(\nu)\vartheta, \nu\theta - h(\nu)\theta\right)$$

From (1.5), the first fundamental form of rotational hypersurface (4.1), its inverse matrix and determinant are

$$[g_{ij}]^L = \begin{bmatrix} h'^2-1 & 0 & 0 \\ 0 & (\nu-h)^2 & 0 \\ 0 & 0 & (\nu-h)^2 \end{bmatrix}, \tag{4.2}$$

$$[g^{ij}]^L = \begin{bmatrix} \frac{1}{h'^2-1} & 0 & 0 \\ 0 & \frac{1}{(\nu-h)^2} & 0 \\ 0 & 0 & \frac{1}{(\nu-h)^2} \end{bmatrix} \tag{4.3}$$

and

$$\det[g_{ij}]^L = (\nu-h)^4\left(h'^2-1\right), \tag{4.4}$$

respectively. Here, if we suppose that $h'^2 - 1 < 0$, then we have $\det[g_{ij}] < 0$ and so, we deal with timelike rotational hypersurface (4.1). Similarly, one can obtain the corresponding results for spacelike rotational hypersurfaces by supposing $h'^2 - 1 > 0$.

From (1.4), the unit normal vector field of rotational hypersurface (4.1) is

$$N^L = \frac{1}{2\sqrt{1-h'^2}}\left(\vartheta^2+\theta^2-\left(\vartheta^2+\theta^2+2\right)h', \vartheta^2+\theta^2-2-\left(\vartheta^2+\theta^2\right)h', 2\vartheta\left(1-h'\right), 2\theta\left(1-h'\right)\right) \tag{4.5}$$

and so, from (3.5), it can be seen that

$$\varepsilon^L = \left\langle N^L, N^L \right\rangle = 1 \tag{4.6}$$

Here, we denote $h = h(\nu)$ and $h' = \frac{dh(\nu)}{d\nu}$.

From (1.6) and (1.7), the second and third fundamental forms of the rotational hypersurface (4.1) are obtained by

$$[h_{ij}]^L = \frac{1}{\sqrt{1-h'^2}} \begin{bmatrix} -h'' & 0 & 0 \\ 0 & (\nu-h)(h'-1) & 0 \\ 0 & 0 & (\nu-h)(h'-1) \end{bmatrix} \tag{4.7}$$

and

$$[m_{ij}]^L = \begin{bmatrix} \frac{-h''^2}{(-1+h'^2)^2} & 0 & 0 \\ 0 & \frac{1-h'}{1+h'} & 0 \\ 0 & 0 & \frac{1-h'}{1+h'} \end{bmatrix}, \tag{4.8}$$

respectively, where $h'' = \frac{d^2h(\nu)}{d\nu^2}$.

Also, using (4.3) and (4.7) in (1.8), the shape operator of the rotational hypersurface (4.1) is given by

$$S^L = \frac{1}{\sqrt{1-h'^2}} \begin{bmatrix} \frac{h''}{(1-h'^2)} & 0 & 0 \\ 0 & \frac{h'-1}{(\nu-h)} & 0 \\ 0 & 0 & \frac{h'-1}{(\nu-h)} \end{bmatrix}. \tag{4.9}$$

Now, we obtain the LB^{IV} operator of the rotational hypersurface (4.1) in E_1^4 and prove an important theorem about LB^{IV}-minimality.

For obtaining the LB^{IV} operator of the rotational hypersurface (4.1) in E_1^4, firstly the fourth fundamental form of this hypersurface must be given. In this context, if (4.8) and (4.9) are used in (1.11), then the fourth fundamental form of the rotational hypersurface (4.1) is obtained by

$$[n_{ij}]^L = \begin{bmatrix} \frac{-h''^2}{(-1+h'^2)^2} & 0 & 0 \\ 0 & \frac{1-h'}{1+h'} & 0 \\ 0 & 0 & \frac{1-h'}{1+h'} \end{bmatrix} \cdot \begin{bmatrix} \frac{h''}{(1-h'^2)^{3/2}} & 0 & 0 \\ 0 & \frac{h'-1}{(\nu-h)\sqrt{1-h'^2}} & 0 \\ 0 & 0 & \frac{h'-1}{(\nu-h)\sqrt{1-h'^2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-h''^3}{(1-h'^2)^{7/2}} & 0 & 0 \\ 0 & \frac{(-1+h')\sqrt{1-h'^2}}{(\nu-h)(1+h')^2} & 0 \\ 0 & 0 & \frac{(-1+h')\sqrt{1-h'^2}}{(\nu-h)(1+h')^2} \end{bmatrix}. \tag{4.10}$$

So, from (4.10), we get

$$\det[n_{ij}]^L = \frac{\sqrt{1-h'^2}h''^3}{(\nu-h)^2(1+h')^7(-1+h')}. \tag{4.11}$$

Using (4.1) and (4.10) in equation (2.15), we have

$$\left.\begin{aligned} &\mathfrak{U}_1^L = \frac{\left(-1+h'\right)^{7/2}\sqrt{1+h'}\left(-2-\vartheta^2-\theta^2+\left(\vartheta^2+\theta^2\right)h'\right)}{2(\nu-h)\sqrt{h''^3\sqrt{1-h'^2}}},\ \mathfrak{U}_2^L = \frac{\left(-1+h'\right)^{7/2}\sqrt{1+h'}\left(-\vartheta^2-\theta^2+\left(-2+\vartheta^2+\theta^2\right)h'\right)}{2(\nu-h)\sqrt{h''^3\sqrt{1-h'^2}}} \\ &\mathfrak{U}_3^L = \frac{\vartheta\left(-1+h'\right)^{9/2}\sqrt{1+h'}}{(\nu-h)\sqrt{h''^3\sqrt{1-h'^2}}},\ \mathfrak{U}_4^L = \frac{\theta\left(-1+h'\right)^{9/2}\sqrt{1+h'}}{(\nu-h)\sqrt{h''^3\sqrt{1-h'^2}}}, \end{aligned}\right\} \tag{4.12}$$

$$\left.\begin{aligned} &\mathfrak{V}_1^L = \frac{\vartheta(\nu-h)h''^{3/2}}{(-1+h')^{3/2}(1+h')^{3/2}(1-h'^2)^{1/4}},\ \mathfrak{V}_2^L = \frac{\vartheta(\nu-h)h''^{3/2}}{(-1+h')^{3/2}(1+h')^{3/2}(1-h'^2)^{1/4}} \\ &\mathfrak{V}_3^L = \frac{(\nu-h)h''^{3/2}}{(-1+h')^{3/2}(1+h')^{3/2}(1-h'^2)^{1/4}},\ \mathfrak{V}_4^L = 0 \end{aligned}\right\} \tag{4.13}$$

and

$$\left.\begin{aligned} &\mathfrak{W}_1^L = \frac{\theta(\nu-h)h''^{3/2}}{(-1+h')^{3/2}(1+h')^{3/2}(1-h'^2)^{1/4}},\ \mathfrak{W}_2^L = \frac{\theta(\nu-h)h''^{3/2}}{(-1+h')^{3/2}(1+h')^{3/2}(1-h'^2)^{1/4}}, \\ &\mathfrak{W}_3^L = 0,\ \mathfrak{W}_4^L = \frac{(\nu-h)h''^{3/2}}{(-1+h')^{3/2}(1+h')^{3/2}(1-h'^2)^{1/4}}. \end{aligned}\right\} \tag{4.14}$$

Thus, writing the equations (4.11), (4.12), (4.13) and (4.14) in (2.14), we can give the following theorem:

Theorem 4.1. *LB^{IV} operator of the rotational hypersurface (4.1) about lightlike axis in E_1^4 is*

$$\Delta^{IV}\Psi^L = ((\Delta^{IV}\Psi^L)_1, (\Delta^{IV}\Psi^L)_2, (\Delta^{IV}\Psi^L)_3, (\Delta^{IV}\Psi^L)_4),$$

where

$$(\Delta^{IV}\Psi^L)_1 = \frac{(1+h')^2 \left(\begin{array}{l} 2\left(-1+h'\right)^6\left(1+h'\right)^2\left(-2-\vartheta^2-\theta^2+\left(\vartheta^2+\theta^2\right)h'\right)h'' \\ +\left(\nu-h\right)\left(-1+h'\right)^4\left(1+h'\right)h''^2\left(\begin{array}{c} -4\left(3+2\vartheta^2+2\theta^2\right) \\ -\left(14+\vartheta^2+\theta^2\right)h' \\ +9\left(\vartheta^2+\theta^2\right)h'^2 \end{array}\right) \\ -3\left(\nu-h\right)\left(-1+h'\right)^5\left(1+h'\right)^2h'''\left(\begin{array}{c} -2-\vartheta^2-\theta^2 \\ +\left(\vartheta^2+\theta^2\right)h' \end{array}\right) \\ +8\left(\nu-h\right)^3h''^4 \end{array}\right)}{4\left(h-\nu\right)\left(-1+h'\right)h''^4\sqrt{1-h'^2}}$$

$$(\Delta^{IV}\Psi^L)_2 = \frac{(1+h')^2\begin{pmatrix} 2(-1+h')^6(1+h')^2\left(-\vartheta^2-\theta^2+\left(-2+\vartheta^2+\theta^2\right)h'\right)h'' \\ +(\nu-h)(-1+h')^4(1+h')h''^2\begin{pmatrix} 4\left(1-2\vartheta^2-2\theta^2\right) \\ -\left(12+\vartheta^2+\theta^2\right)h' \\ +9\left(-2+\vartheta^2+\theta^2\right)h'^2\end{pmatrix} \\ -3(\nu-h)(-1+h')^5(1+h')^2h'''\begin{pmatrix} -\vartheta^2-\theta^2 \\ +\left(-2+\vartheta^2+\theta^2\right)h'\end{pmatrix} \\ +8(\nu-h)^3h''^4\end{pmatrix}}{4(h-\nu)(-1+h')h''^4\sqrt{1-h'^2}},$$

$$(\Delta^{IV}\Psi^L)_3 = \frac{\vartheta(-1+h')^4(1+h')^3\begin{pmatrix} -2(-1+h')^2(1+h')h'' \\ -(\nu-h)(8+9h')h''^2 \\ +3(\nu-h)\left(-1+h'^2\right)h'''\end{pmatrix}}{2h''^4(\nu-h)\sqrt{1-h'^2}},$$

$$(\Delta^{IV}\Psi^L)_4 = \frac{\theta(-1+h')^4(1+h')^3\begin{pmatrix} -2(-1+h')^2(1+h')h'' \\ -(\nu-h)(8+9h')h''^2 \\ +3(\nu-h)\left(-1+h'^2\right)h'''\end{pmatrix}}{2h''^4(\nu-h)\sqrt{1-h'^2}}.$$

Theorem 4.2. *The rotational hypersurface (4.1) about lightlike axis in E_1^4 cannot be LB^{IV}-minimal.*

Proof. Let us suppose that the rotational hypersurface (4.1) about lightlike axis in E_1^4 is LB$^{\text{IV}}$-minimal. So we have $(\Delta^{IV}\Psi^L)_1 = (\Delta^{IV}\Psi^L)_2 = (\Delta^{IV}\Psi^L)_3 = (\Delta^{IV}\Psi^L)_4 = 0$. From $(\Delta^{IV}\Psi^L)_1 = 0$, we have

$$\begin{pmatrix} 2(-1+h')^6(1+h')^2\left(-2-\vartheta^2-\theta^2+\left(\vartheta^2+\theta^2\right)h'\right)h'' \\ +(\nu-h)(-1+h')^4(1+h')\begin{pmatrix} -4\left(3+2\vartheta^2+2\theta^2\right) \\ -\left(14+\vartheta^2+\theta^2\right)h' \\ +9\left(\vartheta^2+\theta^2\right)h'^2\end{pmatrix}h''^2 \\ +8(\nu-h)^3h''^4-3(\nu-h)(-1+h')^5(1+h')^2\begin{pmatrix} -2-\vartheta^2-\theta^2 \\ +\left(\vartheta^2+\theta^2\right)h'\end{pmatrix}h''' \end{pmatrix} = 0. \quad (4.15)$$

The equation (4.15) can be written by

$$\vartheta^2 A + \theta^2 A + B = 0, \quad (4.16)$$

where

$$A = (-1+h')^5(1+h')\left(2(-1+h')^2(1+h')h'' + (\nu-h)(8+9h')h''^2 - 3(\nu-h)(-1+h'^2)h'''\right) \quad (4.17)$$

and

$$B = -4(-1+h')^6(1+h')^2h'' - 2(\nu-h)(-1+h')^4(6+13h'+7h'^2)h''^2 + 8(\nu-h)^3h''^4 + 6(\nu-h)(-1+h')^5(1+h')^2h'''. \quad (4.18)$$

Since $\{\vartheta^2, \theta^2, 1\}$ are linear independent, it must be $A = 0$ ve $B = 0$ in (4.16). Since $h' \neq \pm 1$ in (4.17), for $A = 0$, it must be

$$h''' = \frac{2(-1+h')^2(1+h')h'' + (\nu-h)(8+9h')h''^2}{3(\nu-h)(-1+h'^2)}. \quad (4.19)$$

If we use the equation (4.19) in the expression of $B = 0$, i.e.

$$-4(-1+h')^6(1+h')^2h'' - 2(\nu-h)(-1+h')^4(6+13h'+7h'^2)h''^2 + 8(\nu-h)^3h''^4 + 6(\nu-h)(-1+h')^5(1+h')^2h''' = 0; \quad (4.20)$$

then we have,

$$4(\nu-h)h''^2((-1+h')^4(1+h')^2+2(\nu-h)^2h''^2)=0. \tag{4.21}$$

Since the function h cannot be linear, (4.21) doesn't hold and so it is a contradiction. The reason of this contradiction is our assumption of $(\Delta^{IV}\Psi^L)_1=0$. So, $(\Delta^{IV}\Psi^L)_1$ cannot be zero and this completes the proof. □

Example 4.3. Taking the profile curve of the rotational hypersurface (4.1) in E_1^4 as $\left(\nu,\nu^2+\nu,0,0\right)$, we have

$$\Psi^L(\nu,\vartheta,\theta)=\left(\begin{array}{c}\left(\frac{\vartheta^2+\theta^2}{2}+1\right)\nu-\frac{\vartheta^2+\theta^2}{2}(\nu^2+\nu),\ \frac{\vartheta^2+\theta^2}{2}\nu+\left(1-\frac{\vartheta^2+\theta^2}{2}\right)(\nu^2+\nu),\\ \nu\vartheta-(\nu^2+\nu)\vartheta,\nu\theta-(\nu^2+\nu)\theta\end{array}\right) \tag{4.22}$$

and the LB$^{\mathrm{IV}}$ operator of this hypersurface is obtained by

$$\Delta^{IV}\Psi^L=\frac{1}{\nu^{11}}\left(\begin{array}{c}-2(-\nu^9(1+\nu))^{\frac{3}{2}}(-9+20\nu^3(\vartheta^2+\theta^2)\\ +2\nu(-11+9\vartheta^2+9\theta^2)+2\nu^2(-6+19\vartheta^2+19\theta^2),\\ -2(-\nu^9(1+\nu))^{\frac{3}{2}}(-9+20\nu^3(-2+\vartheta^2+\theta^2)\\ +2\nu(-29+9\vartheta^2+9\theta^2)+2\nu^2(-44+19\vartheta^2+19\theta^2),\\ 8(1+\nu)^2\sqrt{-\nu^{29}(1+\nu)}(9+10\nu)\vartheta,\\ 8(1+\nu)^2\sqrt{-\nu^{29}(1+\nu)}(9+10\nu)\theta.\end{array}\right)$$

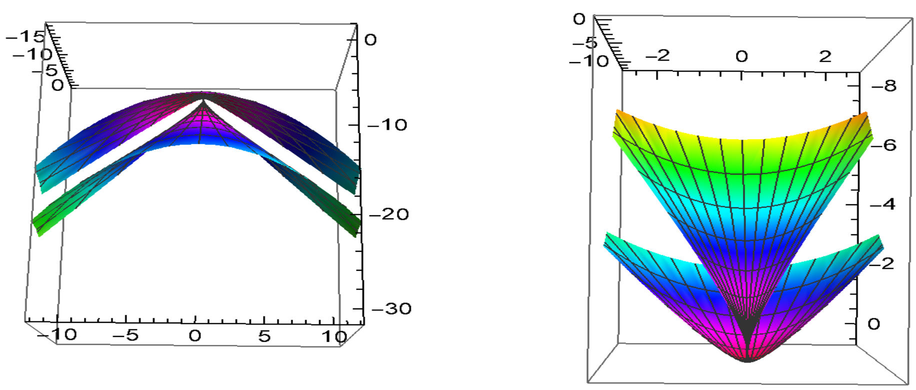

Figure 3. Projections of the rotational hypersurface (4.22) for $\theta=2$ into $x_2x_3x_4$-space (left) and $x_1x_2x_3$-space (right)

Here, we must note that the first and second fundamental forms, which are given for the rotational hypersurfaces in this study, can be obtained by taking a = b = 0 in [16] and also, they can be found in [3], too.

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Received: 2022-09-09
Accepted: 2023-10-05