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On a Neumann problem driven by $p(x)$-Laplacian-like operators in variable-exponent Sobolev spaces

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Abstract *In this work, we consider a Neumann boundary value problem for equations involving the $p(x)$-Laplacian-like operators with two real parameters and two Carathéodory functions that satisfy only the growth condition. We apply the topological degree for a class of demicontinuous operators of generalized (S_+) type and the theory of variable-exponent Sobolev spaces to etablish the existence of weak solutions for the considered problem.*

1 Introduction and motivation

Partial differential equations with nonlinearities and nonconstant exponents has been received considerable attention in recent years. Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents. Modeling with classic Lebesgue and Sobolev spaces has been demonstrated to be limited for a number of materials with inhomogeneities. In the subject of fluid mechanics, for example, great emphasis has been paid to the study of electrorological fluids, which have the ability to modify their mechanical properties when exposed to an electric field. Rajagopal and M. Ruzicka recently developed a very interesting model for these fluids in [31] (see also [33]), taking into account the delicate interaction between the electric field $E(x)$ and the moving liquid. This type of problem's energy is provided by $\int_\Omega |\nabla u|^{p(x)}dx$. This type of energy can also be found in elasticity problems [36]. Other applications relate to image processing [1, 9], elasticity [37], the flow in porous media [5, 25], and problems in the calculus of variations involving variational integrals with nonstandard growth [2, 7, 10, 26, 34].

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$. In this paper we deal with the question of the existence of a weak solutions for a class of $p(x)$-Laplacian-like Neumann boundary value problem, arising from capillarity phenomena, of the following form:

$$\begin{cases} -\Delta^l_{p(x)}u + \delta(x)|u|^{\alpha(x)-2}u = \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta^l_{p(x)}u := \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)$$

is the $p(x)$-Laplacian-like operators $p(\cdot), \alpha(\cdot) \in C_+(\overline{\Omega})$ with $p(\cdot)$ is log-Hölder continuous func-

tion (in a sense to be precised in Section 2), $\delta \in L^{\infty}(\Omega)$, μ and λ are two real parameters, η is the outer unit normal to $\partial\Omega$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth.

The motivation for this research originated from the application of similar problems in physics to model the behavior of electrorheological fluids (see [31, 33]), specifically the phenomenon of capillarity, which depends solid-liquid interfacial characteristics as surface tension, contact angle, and solid surface geometry. Recently problems like (1.1) has received more and more attention, such as [4, 6, 15, 16, 17, 18, 19, 20, 21].

Problems related to (1.1) have been studied by many scholars, for example, W. Ni et al. [29, 30] study the following equations

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The operator $-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)$ *is most often denoted by the specified mean curvature operator.*

For $\alpha(x) = p(x)$, $\mu \geq 0$, $\lambda > 0$, $\delta \in L^{\infty}(\Omega)$ with $\mathrm{ess}\inf_{\Omega} \delta > 0$ and f independent of ∇u, Afrouzi et al. [3] established some new sufficient conditions underwhich the problem (1.1) possesses infinitely many weak solutions. Their discussion is based on a fully variational method and the main tool is a general critical point theorem.

In the present paper, we will generalize these works, by proving, under a suitable growth conditions on g and f, the existence of a weak solutions for the problem (1.1) by using another approach based on the topological degree for a class of demicontinuous operators of generalized (S_+) type of [8] and the theory of the variable-exponent Sobolev spaces. To the best of our knowledge, this is the first paper that discusses a Neumann boundary value problem driven by $p(x)$-Laplacian-like operators depending on two real parameters via topological degree methods.

The remainder of the article is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problem. In Section 3, we introduce some classes of operators of generalized (S_+) type, as well as the Berkovits topological degrees. Finaly, in Section 4, we give our basic assumptions, some technical lemmas, and we will state and prove the main result of the paper.

2 Preliminaries

In the analysis of problem (1.1), we will use the theory of the generalized Lebesgue-Sobolev space $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [24, 28, 12, 13] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{p(x),\ x \in \overline{\Omega}\Big\} \ \text{ and } \ p^- := \min\Big\{p(x),\ x \in \overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \ \text{ is measurable such that } \int_{\Omega} |u(x)|^{p(x)} dx < +\infty\Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\},$$

where

$$\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx,\ \forall\, u \in L^{p(x)}(\Omega).$$

Proposition 2.1. *[24, Theorem 1.3 and Theorem 1.4] Let (u_n) and $u \in L^{p(x)}(\Omega)$, then*

$$|u|_{p(x)} < 1\big(resp. = 1; > 1\big) \Leftrightarrow \rho_{p(x)}(u) < 1\big(resp. = 1; > 1\big), \tag{2.1}$$

$$|u|_{p(x)} > 1 \ \Rightarrow \ |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow \ |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \ \Leftrightarrow \ \lim_{n\to\infty} \rho_{p(x)}\big(u_n - u\big) = 0. \tag{2.4}$$

Remark 2.2. According to (2.2) and (2.3), we have

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.3. *[28, Theorem 2.5 and Corollary 2.7] The space $\big(L^{p(x)}(\Omega), |\cdot|_{p(x)}\big)$ is a separable and reflexive Banach space.*

Proposition 2.4. *[28, Theorem 2.1] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big| \int_\Omega uv\, dx \Big| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. [24, Theorem 1.11] If $p_1,\ p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then we have $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Now, let $p \in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

equipped with the norm

$$|u|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Furthermore, we have the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$(see [28]).

Remark 2.6. Note that for all $u \in W^{1,p(x)}(\Omega)$, we have

$$|u|_{p(x)} \leq |u|_{1,p(x)} \ \text{ and } \ |\nabla u|_{p(x)} \leq |u|_{1,p(x)}.$$

Next, for all $u \in W^{1,p(x)}(\Omega)$, we introduce the following notation

$$\rho_{1,p(x)}(u) = \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u).$$

Then, from [24, Theorem 1.3], we have the following result.

Proposition 2.7. *If $u \in W^{1,p(x)}(\Omega)$, then the following properties hold true*

$$|u|_{1,p(x)} < 1(resp. = 1; > 1) \ \Leftrightarrow \ \rho_{1,p(x)}(u) < 1(resp. = 1; > 1), \tag{2.8}$$

$$|u|_{1,p(x)} > 1 \ \Rightarrow \ |u|^{p^-}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq |u|^{p^+}_{1,p(x)}, \tag{2.9}$$

$$|u|_{1,p(x)} < 1 \ \Rightarrow \ |u|^{p^+}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq |u|^{p^-}_{1,p(x)}. \tag{2.10}$$

Proposition 2.8. *[24, 28] The space $\Big(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\Big)$ is a separable and reflexive Banach space.*

3 A review on the topological degree theory

We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle \cdot\,,\,\cdot\rangle$ and given a nonempty subset $\mathcal{D}$ of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 3.1. Let Y be another real Banach space. An operator $F : \mathcal{D} \subset X \to Y$ is said to be

(i) bounded, if it maps any bounded set to a bounded set.

(ii) demicontinuous, if $(u_n) \subset \mathcal{D}$, and $u_n \to u$ in X as $n \to \infty$, then $F(u_n) \rightharpoonup F(u)$ in Y.

(iii) compact, if it is continuous and the image of any bounded set in X is relatively compact in Y.

Definition 3.2. A mapping $F : \mathcal{D} \subset X \to X^*$ is said to be

(i) of class (S_+), if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$ in X and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$ in X.

(ii) quasimonotone, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$ in X, we have $\limsup\limits_{n\to\infty}\langle Fu_n,\, u_n - u\rangle \geq 0$.

Definition 3.3. Let $T : \mathcal{D}_1 \subset X \to X^*$ be a bounded operator such that $\mathcal{D} \subset \mathcal{D}_1$. For any operator $F : \mathcal{D} \subset X \to X$ we say that

(i) F of class $(S_+)_T$, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$ in X, $y_n := Tu_n \rightharpoonup y$ in X^* and $\limsup\limits_{n\to\infty}\langle Fu_n,\, y_n - y\rangle \leq 0$, we have $u_n \to u$ in X.

(ii) F has the property $(QM)_T$, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$ in X, $y_n := Tu_n \rightharpoonup y$ in X^*, we have $\limsup\limits_{n\to\infty}\langle Fu_n, y - y_n\rangle \geq 0$.

In the sequel, we consider the following classes of operators:

$$\mathcal{F}_1(\mathcal{D}) := \Big\{F : \mathcal{D} \to X^* : F \textit{ is bounded, demicontinuous and of class } (S_+)\Big\},$$

$$\mathcal{F}_T(\mathcal{D}) := \Big\{F : \mathcal{D} \to X : F \textit{ is demicontinuous and of class } (S_+)_T\Big\},$$

$$\mathcal{F}_{T,B}(\mathcal{D}) := \Big\{F \in \mathcal{F}_T(\mathcal{D}) : F \textit{ is bounded }\Big\},$$

for any $\mathcal{D} \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\mathcal{D})$.
Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) : E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ *is called an essential inner map to F.*

Lemma 3.4. *[27, Lemma 2.3] Let E be a bounded open set in a real reflexive Banach space X, and let $T \in \mathcal{F}_1(\overline{E})$ be a continuous operator. Let $S : D(S) \subset X^* \to X$ be a demicontinuous operator, such that $T(\overline{E}) \subset D(S)$. Then, the following statements hold.*

(i) If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.

(ii) If S is of class (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.

Definition 3.5. Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. Then the affine homotopy $\mathcal{H} : [0, 1] \times \overline{E} \to X$ defined by

$$\mathcal{H}(t, u) := (1 - t)Fu + tSu, \quad \text{for all} \quad (t, u) \in [0, 1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 3.6. [27, Lemma 2.5] *The above affine homotopy is of class* $(S_+)_T$.

As in [27] we give the topological degree for the class $\mathcal{F}(X)$.

Theorem 3.7. *Let*

$$M = \Big\{(F, E, h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ *that satisfies the following properties:*

(i) (Normalization) For any $h \in F(E)$, *we have*

$$d(I, E, h) = 1.$$

(ii) (Additivity) Let $F \in \mathcal{F}_{T,B}(\overline{E})$. *If* E_1 *and* E_2 *are two disjoint open subsets of* E *such that* $h \notin F(\overline{E}\backslash(E_1 \cup E_2))$, *then we have*

$$d(F, E, h) = d(F, E_1, h) + d(F, E_2, h).$$

(iii) (Homotopy invariance) If $\mathcal{H} : [0, 1] \times \overline{E} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* h*:* $[0, 1] \to X$ *is a continuous path in* X *such that* $h(t) \notin \mathcal{H}(t, \partial E)$ *for all* $t \in [0, 1]$, *then*

$$d(\mathcal{H}(t, \cdot), E, h(t)) = C \text{ for all } t \in [0, 1].$$

(iv) (Existence) If $d(F, E, h) \neq 0$, *then the equation* $Fu = h$ *has a solution in* E.

(v) (Boundary dependence) If $F, S \in \mathcal{F}_{\mathrm{T}}(\overline{\mathrm{E}})$, $F = S$ *on* ∂E, *and* $h \notin F(\partial E)$, *then*

$$d(F, E, h) = d(S, E, h).$$

Definition 3.8. [27, Definition 3.3] The above degree is defined as follows:

$$d(F, E, h) \ := d_B(F|_{\overline{E_0}}, E_0, h),$$

where d_B is the Berkovits degree [8] *and* E_0 *is any open subset of* E *with* $F^{-1}(h) \subset E_0$ *and* F *is bounded on* $\overline{E}_0$.

4 Assumptions and main result

In this section, we will discuss the existence of weak solutions of (1.1).

We assume that $\Omega \subset \mathbb{R}^N (N > 1)$ *is a bounded domain with a Lipschitz boundary* $\partial\Omega$, $p \in C_+(\overline{\Omega})$ *satisfy the* log*-Hölder continuity condition* (**??**), $\delta \in L^\infty(\Omega)$, $\alpha \in C_+(\overline{\Omega})$ *with* $2 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < p^- \leq p(x) \leq p^+ < \infty$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *and* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *are functions such that:*

(A_1) f *is a Carathéodory function.*

(A_2) *There exists* $\varrho > 0$ *and* $\gamma \in L^{p'(x)}(\Omega)$ *such that*

$$|f(x, \zeta, \xi)| \leq \varrho(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}).$$

(A_3) g *is a Carathéodory function.*

(A_4) *There exists* $\sigma > 0$ *and* $\nu \in L^{p'(x)}(\Omega)$ *such that*

$$|g(x, \zeta)| \leq \sigma(\nu(x) + |\zeta|^{s(x)-1}),$$

for a.e. $x \in \Omega$ *and all* $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, *where* $q, s \in C_+(\overline{\Omega})$ *with* $2 \leq q^- \leq q(x) \leq q^+ < p^-$ *and* $2 \leq s^- \leq s(x) \leq s^+ < p^-$.

Remark 4.1.

- Note that for all $\omega \in W^{1,p(x)}(\Omega)$ $\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\omega\, dx$ is well defined (see [32]).
- $\delta(x)|u|^{\alpha(x)-2}u$, $\mu\, g(x,u)$ and $\lambda\, f(x,u,\nabla u)$ are belongs to $L^{p'(x)}(\Omega)$ under $u\in W^{1,p(x)}(\Omega)$, the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p,α,q and s because: $r(x)=(q(x)-1)p'(x)\in C_+(\overline{\Omega})$ with $r(x)<p(x)$, $\beta(x)=(\alpha(x)-1)p'(x)\in C_+(\overline{\Omega})$ with $\beta(x)<p(x)$ and $\kappa(x)=(s(x)-1)p'(x)\in C_+(\overline{\Omega})$ with $\kappa(x)<p(x)$.
  Then, by Remark 2.5 we can conclude that $L^{p(x)}\hookrightarrow L^{r(x)}$, $L^{p(x)}\hookrightarrow L^{\beta(x)}$ and $L^{p(x)}\hookrightarrow L^{\kappa(x)}$.
  Hence, since $\omega\in L^{p(x)}(\Omega)$, we have

$$\Big(-\delta(x)|u|^{\alpha(x)-2}u+\mu\, g(x,u)+\lambda\, f(x,u,\nabla u)\Big)\omega\in L^1(\Omega).$$

  This implies that, the integral

$$\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u+\mu\, g(x,u)+\lambda\, f(x,u,\nabla u)\Big)\omega dx$$

  exist.

Then, we shall use the definition of weak solutions for problem (1.1) in the following sense:

Definition 4.2. We say that a function $u\in W^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if for any $\omega\in W^{1,p(x)}(\Omega)$, it satisfies the following:

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\omega dx=\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u+\mu\, g(x,u)+\lambda\, f(x,u,\nabla u)\Big)\omega dx.$$

Before giving the existence result for (1.1), we first give two lemmas that will be used in the proof of this result.
Let us consider the following functional:

$$\mathcal{J}(u):=\int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}\Big)dx.$$

From [32], it is obvious that the derivative operator of the functional $\mathcal{J}$ in the weak sense at the point $u\in W^{1,p(x)}(\Omega)$ is the functional $\mathcal{T}u\in W^{-1,p'(x)}(\Omega)$, given by

$$\langle \mathcal{T}u,\omega\rangle=\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\omega dx,$$

for all $u,\omega\in W^{1,p(x)}(\Omega)$ where $\langle\cdot,\cdot\rangle$ is the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. In addition, the following lemma summarizes the properties of the operator $\mathcal{T}$ (see [32, Proposition 3.1.]).

Lemma 4.3. *The mapping*

$$\begin{aligned}&\mathcal{T}: W^{1,p(x)}(\Omega)\longrightarrow W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{T}u,\omega\rangle=\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\omega dx,\end{aligned}$$

is a continuous, bounded, strictly monotone operator, and is of class (S_+).

Lemma 4.4. *Assume that the assumptions $(A_1)-(A_4)$ hold, then the operator*

$$\begin{aligned}&\mathcal{P}: W^{1,p(x)}(\Omega)\to W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{P}u,\omega\rangle=-\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u+\mu\, g(x,u)+\lambda\, f(x,u,\nabla u)\Big)\omega dx,\end{aligned}$$

for all $u,\omega\in W^{1,p(x)}(\Omega)$, is compact.

Proof. In order to prove this lemma, we proceed in four steps.
Step 1 : Let $\Upsilon : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Upsilon u(x) := -\mu\, g(x, u).$$

In this step, we prove that the operator Υ is bounded and continuous.
First, let $u \in W^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.5) and (2.6), we infer

$$\begin{aligned}
|\Upsilon u|_{p'(x)} &\leq \rho_{p'(x)}(\Upsilon u) + 1 \\
&= \int_\Omega |\mu\, g(x, u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\mu|^{p'(x)} |g(x, u(x)|^{p'(x)} dx + 1 \\
&\leq \Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big) \int_\Omega |\sigma\Big(\nu(x) + |u|^{s(x)-1}\Big)|^{p'(x)} dx + 1 \\
&\leq C\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big) \int_\Omega \Big(|\nu(x)|^{p'(x)} + |u|^{\kappa(x)}\Big) dx + 1 \\
&\leq C\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\Big(\rho_{p'(x)}(\nu) + \rho_{\kappa(x)}(u)\Big) + 1 \\
&\leq C\Big(|\nu|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-}\Big) + 1.
\end{aligned}$$

Then, we deduce from Remark 2.6 and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Upsilon u|_{p'(x)} \leq const\Big(|\nu|_{1,p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-}\Big) + 1,$$

that means Υ is bounded on $W^{1,p(x)}(\Omega)$.
Second, we show that the operator Υ is continuous. To this purpose let $u_n \to u$ in $W^{1,p(x)}(\Omega)$. We need to show that $\Upsilon u_n \to \Upsilon u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \text{ and } |u_k(x)| \leq \phi(x), \tag{4.1}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, from (A_2) and (4.1), we have

$$|g(x, u_k(x))| \leq \sigma(\nu(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
On the other hand, thanks to (A_3) and (4.1), we get, as $k \longrightarrow \infty$

$$g(x, u_k(x)) \to g(x, u(x)) \text{ a.e. } x \in \Omega.$$

Seeing that

$$\nu + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega),$$

and

$$\rho_{p'(x)}(\Upsilon u_k - \Upsilon u) = \int_\Omega |g(x, u_k(x)) - g(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.4), we have

$$\Upsilon u_k \to \Upsilon u \text{ in } L^{p'(x)}(\Omega),$$

and consequently

$$\Upsilon u_n \to \Upsilon u \text{ in } L^{p'(x)}(\Omega),$$

that is, Υ is continuous.
Step 2 : We define the operator $\Psi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi u(x) := \delta(x)|u(x)|^{\alpha(x)-2}u(x).$$

We will prove that Ψ is bounded and continuous.
It is clear that Ψ is continuous. Next we show that Ψ is bounded.
Let $u \in W^{1,p(x)}(\Omega)$ and using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi u) + 1 \\
&= \int_\Omega |\delta(x)|u|^{\alpha(x)-2}u|^{p'(x)}dx + 1 \\
&= \int_\Omega |\delta(x)|^{p'(x)}|u|^{(\alpha(x)-1)p'(x)}dx + 1 \\
&\leq ||\delta||^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^{\beta(x)}dx + 1 \\
&= ||\delta||^{p'}_{L^\infty(\Omega)}\rho_{\beta(x)}(u) + 1 \\
&\leq ||\delta||^{p'}_{L^\infty(\Omega)}\Big(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)}\Big) + 1.
\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and Remark 2.6 that

$$|\Psi u|_{p'(x)} \leq const\Big(|u|^{\beta^-}_{1,p(x)} + |u|^{\beta^+}_{1,p(x)}\Big) + 1,$$

and consequently, Ψ is bounded on $W^{1,p(x)}(\Omega)$.
Step 3 : Let us define the operator $\Phi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\lambda\, f(x, u(x), \nabla u(x)).$$

We will show that Φ is bounded and continuous.
Let $u \in W^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Phi u|_{p'(x)} &\leq \rho_{p'(x)}(\Phi u) + 1 \\
&= \int_\Omega |\lambda\, f(x, u(x), \nabla u(x))|^{p'(x)}dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)}|f(x, u(x), \nabla u(x))|^{p'(x)}dx + 1 \\
&\leq \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big)\int_\Omega |\varrho\Big(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\Big)|^{p'(x)}dx + 1 \\
&\leq C\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big)\int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\Big)dx + 1 \\
&\leq C\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big)\Big(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\Big) + 1 \\
&\leq C\Big(|\gamma|^{p'^+}_{p(x)} + |u|^{r^+}_{r(x)} + |u|^{r^-}_{r(x)} + |\nabla u|^{r^+}_{r(x)} + |\nabla u|^{r^-}_{r(x)}\Big) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and Remark 2.6, we have then

$$|\Phi u|_{p'(x)} \leq const\Big(|\gamma|^{p'^+}_{1,p(x)} + |u|^{r^+}_{1,p(x)} + |u|^{r^-}_{1,p(x)}\Big) + 1,$$

and consequently Φ is bounded on $W^{1,p(x)}(\Omega)$.
It remains to show that Φ is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$, we need to show that $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \ \text{ and } \ \nabla u_k(x) \to \nabla u(x), \tag{4.2}$$

$$|u_k(x)| \leq \phi(x) \ \text{ and } \ |\nabla u_k(x)| \leq |\psi(x)|, \tag{4.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1) and (4.2), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \ \text{ a.e. } \ x \in \Omega.$$

On the other hand, from (A_2) and (4.3), we can deduce the estimate

$$|f(x, u_k(x), \nabla u_k(x))| \leq \varrho(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.4) that

$$\Phi u_k \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and consequently

$$\Phi u_n \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and then Φ is continuous.
Step 4: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$.
We then define

$$I^* \circ \Upsilon : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* \circ \Psi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Phi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* \circ \Upsilon$, $I^* \circ \Psi$ and $I^* \circ \Phi$ are compact, that means $\mathcal{P} = I^* \circ \Upsilon + I^* \circ \Psi + I^* \circ \Phi$ is compact. With this last step the proof of Lemma 4.4 is completed. □

We are now in the position to get the existence result of weak solutions for (1.1).

Theorem 4.5. *Assume that the assumptions $(A_1) - (A_4)$ hold, then the problem* (1.1) *possesses at least one weak solutions u in $W^{1,p(x)}(\Omega)$.*

Proof. The basic idea of our proof is to reduce the problem (1.1) to a new one governed by a Hammerstein equation, and apply the theory of topological degree introduced in Section 3 to show the existence of a weak solutions to the state problem.

First, for all $u, \omega \in W^{1,p(x)}(\Omega)$, we define the operators $\mathcal{G}$ and $\mathcal{P}$, as defined in Lemmas 4.3 and 4.4 respectively,

$$\mathcal{G} : W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{G}u, \omega \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u + \frac{|\nabla u|^{2p(x)-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \Big) \nabla \omega dx,$$

$$\begin{aligned}
&\mathcal{P}: W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)\\
&\langle \mathcal{P}u, \omega\rangle = -\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)\omega dx.
\end{aligned}$$

Consequently, the problem (1.1) is equivalent to the equation

$$\mathcal{G}u = -\mathcal{P}u, \quad u \in W^{1,p(x)}(\Omega). \tag{4.4}$$

Taking into account that, by Lemma 4.3, the operator $\mathcal{G}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [35, Theorem 26 A], the inverse operator

$$\mathcal{Q} := \mathcal{G}^{-1} : W^{-1,p'(x)}(\Omega) \to W^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).
On another side, according to Lemma 4.4, we have that the operator $\mathcal{P}$ is bounded, continuous and quasimonotone.
Consequently, following Zeidler's terminology [35], the equation (4.4) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{Q}\omega \;\text{ and }\; \omega + \mathcal{P}\circ\mathcal{Q}\omega = 0, \;\; u \in W^{1,p(x)}(\Omega) \text{ and } \omega \in W^{-1,p'(x)}(\Omega). \tag{4.5}$$

Seeing that (4.4) is equivalent to (4.5), then to solve (4.4) it is thus enough to solve (4.5). In order to solve (4.5), we will apply the Berkovits topological degree introducing in Section 3. First, let us set

$$\mathcal{E} := \Big\{\omega \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \;\text{ such that }\; \omega + t\mathcal{P}\circ\mathcal{Q}\omega = 0\Big\}.$$

Next, we show that $\mathcal{E}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.
Let us put $u := \mathcal{Q}\omega$ for all $\omega \in \mathcal{E}$. Taking into account that $|\mathcal{Q}\omega|_{1,p(x)} = |\nabla u|_{1,p(x)}$, then we have the following two cases:
Case 2: If $|\nabla u|_{1,p(x)} \leq 1$. Then $|\mathcal{Q}\omega|_{1,p(x)} \leq 1$, that means $\Big\{\mathcal{Q}\omega : \omega \in \mathcal{E}\Big\}$ is bounded.
Case 1: If $|\nabla u|_{1,p(x)} > 1$. Then, from (A_2) and (A_4), the inequalities (2.9), (2.7) and (2.6) and the Young's inequality, we get

$$\begin{aligned}
|\mathcal{Q}\omega|^{p^-}_{1,p(x)} &\leq \rho_{p(x)}(\nabla u)\\
&= \langle \mathcal{G}u,\; u\rangle\\
&= \langle \omega,\; \mathcal{Q}\omega\rangle\\
&= -t\langle \mathcal{P}\circ\mathcal{Q}\omega,\; \mathcal{Q}\omega\rangle\\
&= t\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big) u dx\\
&\leq t \max(||\delta||_{L^\infty(\Omega)}, \sigma|\mu|, \varrho|\lambda|)\Big(\rho_{\alpha(x)}(u) + \int_\Omega |\nu(x)u(x)|dx + \int_\Omega |\gamma(x)u(x)|dx\\
&\qquad + \rho_{s(x)}(u) + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big)\\
&\leq C\Big(|u|^{\alpha^-}_{\alpha(x)} + |u|^{\alpha^+}_{\alpha(x)} + |\nu|_{p'(x)}|u|_{p(x)} + |\gamma|_{p'(x)}|u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} + |u|^{q^+}_{q(x)}\\
&\qquad + |u|^{q^-}_{q(x)} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big)\\
&\leq C\Big(|u|^{\alpha^-}_{\alpha(x)} + |u|^{\alpha^+}_{\alpha(x)} + |u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + |\nabla u|^{q^+}_{q(x)}\Big),
\end{aligned}$$

then, according to $L^{p(x)} \hookrightarrow L^{\alpha(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$, $L^{p(x)} \hookrightarrow L^{q(x)}$ and Remark 2.6, we get

$$|\mathcal{Q}\omega|^{p^-}_{1,p(x)} \leq C\Big(|\mathcal{Q}\omega|^{\alpha^+}_{1,p(x)} + |\mathcal{Q}\omega|_{1,p(x)} + |\mathcal{Q}\omega|^{s^+}_{1,p(x)} + |\mathcal{Q}\omega|^{q^+}_{1,p(x)}\Big),$$

what implies that $\Big\{\mathcal{Q}\omega : \omega \in \mathcal{E}\Big\}$ is bounded.
On the other hand, we have that the operator is $\mathcal{P}$ is bounded, then $\mathcal{P} \circ \mathcal{Q}\omega$ is bounded. Thus, thanks to (4.5), we have that $\mathcal{E}$ is bounded in $W^{-1,p'(x)}(\Omega)$.
However, $\exists\ R > 0$ such that

$$|\omega|_{-1,p'(x)} < R \ \text{ for all } \ \omega \in \mathcal{E},$$

which leads to

$$\omega + t\mathcal{P} \circ \mathcal{Q}\omega \neq 0, \ \ \omega \in \partial\mathcal{E}_R(0) \ \text{ and } \ t \in [0, 1],$$

where $\mathcal{E}_R(0)$ is the ball of center 0 and radius R in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 3.4, we conclude that

$$I + \mathcal{P} \circ \mathcal{Q} \in \mathcal{F}_{\mathcal{Q}}(\overline{\mathcal{E}_R(0)}) \ \text{ and } \ I = \mathcal{G} \circ \mathcal{Q} \in \mathcal{F}_{\mathcal{Q}}(\overline{\mathcal{E}_R(0)}).$$

On another side, taking into account that I, $\mathcal{P}$ and $\mathcal{Q}$ are bounded, then $I + \mathcal{P} \circ \mathcal{Q}$ is bounded. Hence, we infer that

$$I + \mathcal{P} \circ \mathcal{Q} \in \mathcal{F}_{\mathcal{Q},B}(\overline{\mathcal{E}_R(0)}) \ \text{ and } \ I = \mathcal{G} \circ \mathcal{Q} \in \mathcal{F}_{\mathcal{Q},B}(\overline{\mathcal{E}_R(0)}).$$

Next, we define the homotopy

$$\begin{aligned} \mathcal{H} : [0, 1] \times \overline{\mathcal{E}_R(0)} &\to W^{-1,p'(x)}(\Omega) \\ (t, \omega) &\mapsto \mathcal{H}(t, \omega) := \omega + t\mathcal{P} \circ \mathcal{Q}\omega. \end{aligned}$$

Hence, thanks to the properties of the degree d seen in Theorem 3.7, we obtain

$$d(I + \mathcal{P} \circ \mathcal{Q}, \mathcal{E}_R(0), 0) = d(I, \mathcal{E}_R(0), \ 0) = 1 \neq 0,$$

what implies that there exists $\omega \in \mathcal{E}_R(0)$ which verifies

$$\omega + \mathcal{P} \circ \mathcal{Q}\omega = 0.$$

Finally, we infer that $u = \mathcal{Q}\omega$ is a weak solutions of (1.1). The proof is completed. □

5 Conclusion

This paper aims is to consider a Neumann boundary value problem involving the $p(x)$-Laplacian-like operator. Then, we used the topological degree for a class of demicontinuous operators of generalized (S_+) type and the theory of variable-exponent Sobolev spaces to etablish the existence of weak solutions for the considered problem. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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