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ON A GENERALIZATION OF QUASI-ARMENDARIZ RINGS

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Abstract. Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. Properties of the ring $[[R^{S,\leq}, \omega]]$ of skew generalized power series with coefficients in R and exponents in S are considered. This paper is devoted to the study of linearly (S, ω)-quasi-Armendariz ring, which is unify the notions of linearly (S, ω)-Armendariz ring and (S, ω)-quasi-Armendariz ring. It is shown that, if R is linearly (S, ω)-quasi-Armendariz ring, U is a nonempty subset in R is a two-sided ideal of R, $A = \ell_R(U)$ and $\omega_s|_U$ is surjective for all $s \in S$, then R/A is linearly $(S, \overline{\omega})$-quasi-Armendariz. Also, we prove that, R is semiprime if and only if R is reduced linearly (S, ω)-quasi-Armendariz. Moreover, Under a necessary and sufficient conditions, if R is linearly (S, ω)-quasi-Armendariz, then Q is linearly $(S, \widetilde{\omega})$-quasi-Armendariz, where Q is the classical left ring of quotients of R. Consequently, some results of linearly (S, ω)-quasi-Armendariz are given.

1 Introduction

Throughout this paper all rings considered here are associative with identity. We will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $\ell_R(X)(r_R(X))$. We will denote by $\text{End}(R)$ the monoid of ring endomorphisms of R, and by $\text{Aut}(R)$ the group of ring automorphisms of R. Any concept and notation not defined here can be found in Marks et al. [1] and Mazurek and Ziembowski [2].

Rege and Chhawchharia [3] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. (The converse is always true.) The name "Armendariz ring" was chosen because Armendariz [4, Lemma 1] had noted that a reduced ring satisfies this condition. Reduced rings (i.e., rings with no nonzero nilpotent elements). Some properties of Armendariz rings have been studied in E. P. Armendariz [4], Anderson and Camillo [5], Kim and Lee [6], Huh, Lee and Smoktunowicz [7], and Lee and Wong [8].

By Kim et al. in [9]. A ring R is said to be power-serieswise Armendariz if whenever power series $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[[x]]$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for all i, j. Armendariz rings were generalized to quasi-Armendariz rings by Hirano [10]. A ring R is called quasi-Armendariz provided that $a_iRb_j = 0$ for all i, j whenever $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)R[x]g(x) = 0$. Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to Elliott and Ribenboim [11].

Let $(S, \leq)$ is a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \rightarrow R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let $X_s(f,g) = \{(u,v) \in S \times S | u+v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from Ribenboim [12, 4.1] that $X_s(f,g)$ is finite. This fact allows one to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

Clearly, supp$(fg) \subseteq supp(f) + supp(g)$, thus by Ribenboim [13, 3.4] supp(fg) is artinian and narrow, hence $fg \in [[R^{S,\leq}]]$. With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes an associative ring, with identity element e, namely $e(0) = 1, e(s) = 0$ for every $0 \neq s \in S$. Which is called the ring of generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in Ribenboim ([12]−[14]), Elliott and Ribenboim [11] and Varadarajan ([16], [17]). For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] \cong R[S]$, the monoid ring of S over R. Further examples are given in Ribenboim [13]. To any $r \in R$ and $s \in S$, we associate the maps $c_r, e_s \in [[R^{S,\leq}]]$ defined by

$$c_r(x) = \begin{cases} r, & x = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad e_s(x) = \begin{cases} 1, & x = s, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq}]]$, $s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}]]$, and $c_r e_s = e_s c_r$. Recall that a monoid S is torsion-free if the following property holds: If $s, t \in S$, if k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$.

Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to$End(R) a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω, that is, $\omega_s = \omega(s)$. Let A be the set of all functions $f : S \to R$ such that the support supp$(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f,g) = \{(u,v) \in supp(f) \times supp(g) : s = uv\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v))$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called the ring of skew generalized power series with coefficients in R and exponents in S, see [2] and denoted by $[[R^{S,\leq}, \omega]]$ (or by $R[[S, \omega]]$ when there is no ambiguity concerning the order $\leq$).

We will use the symbol 1 to denote the identity elements of the monoid S, the ring R, and the ring $[[R^{S,\leq}, \omega]]$ as well as the trivial monoid homomorphism $1 : S \to$End(R) that sends every element of S to the identity endomorphism. A subset $P \subseteq R$ will be called S-invariant if for every $s \in S$ it is ω_s-invariant (that is, $\omega_s(P) \subseteq P$). To each $r \in R$ and $s \in S$, we associate elements $c_r, e_s \in [[R^{S,\leq}, \omega]]$ defined by

$$c_r(x) = \begin{cases} r, & \text{if} \quad x = 1, \\ 0, & \text{if} \quad x \in S\backslash\{1\}, \end{cases} \qquad e_s(x) = \begin{cases} 1, & \text{if} \quad x = s, \\ 0, & \text{if} \quad x \in S\backslash\{s\}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq}, \omega]]$ and $s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}, \omega]]$, and $e_s c_r = c_{\omega_s(r)} e_s$.

If R is a ring and S is a strictly ordered monoid, then the ring R is called a generalized Armendariz ring if for each $f, g \in [[R^{S,\leq}]]$ such that $fg = 0$ implies that $f(u)g(v) = 0$ for each $u \in supp(f)$ and $v \in supp(g)$. In [18] Liu called such ring S-Armendariz ring. If R is a ring, S be a torsion-free and cancellative monoid and $\leq$ a strict order on S, then the ring R is called a generalized quasi-Armendariz ring if for each $f, g \in [[R^{S,\leq}]]$ such that $f[[R^{S,\leq}]]g = 0$, then $f(u)Rg(v) = 0$ for each $u, v \in S$. Ali and Elshokry in [19] called such S-quasi-Armendariz ring and a generalization of it in [28]. Marks et al. [1] a ring R is called (S, ω)-Armendariz, if whenever $f, g \in [[R^{S,\leq}, \omega]]$, $fg = 0$ implies $f(u)\omega_u(g(v)) = 0$ for all $u, v \in S$. Also, they defined that a ring R is linearly (S, ω)-Armendariz, if for all $s \in S\backslash\{1\}$ and $a_0, a_1, b_0, b_1 \in R$, such that $(c_{a_0} + c_{a_1}e_s)(c_{b_0} + c_{b_1}e_s) = 0$, then $a_0b_0 = a_0b_1 = a_1\omega_s(b_0) = a_1\omega_s(b_1) = 0$. A common generalization of S-quasi-Armendariz ring and (S, ω)-Armendariz introduced by Paykan and Moussavi [20], said that, a ring R is called (S, ω)-quasi-Armendariz, if whenever $f, g \in [[R^{S,\leq}, \omega]]$ such that $f[[R^{S,\leq}, \omega]]g = 0$, then $f(u)R\omega_u(g(v)) = 0$ for each $u, v \in S$.

This paper is devoted to the study of linearly (S, ω)-quasi-Armendariz which is unify the notions of linearly (S, ω)-Armendariz and (S, ω)-quasi-Armendariz ring. It is shown that, $(\ddagger)$ If R is linearly (S, ω)-quasi-Armendariz ring, U is a nonempty subset in R is a two-sided ideal of R, $A = \ell_R(U)$ and $\omega_s|_U$ is surjective for all $s \in S$, then R/A is linearly $(S, \overline{\omega})$-quasi-Armendariz. $(\ddagger')$ For a two-sided ideal I of R, if R/I is a linearly $(S, \overline{\omega})$-quasi-Armendariz ring and I is a semiprime ring without identity, then R is linearly (S, ω)-quasi-Armendariz. Moreover, $(\ddagger'')$ Under a necessary and sufficient conditions, if R is a linearly (S, ω)-quasi-Armendariz, then Q is a linearly $(S, \widetilde{\omega})$-quasi-Armendariz, where Q is the classical left ring of quotients of R. Consequently, some results of a linearly (S, ω)-quasi-Armendariz are given.

Clark defined quasi-Baer rings in [21]. A ring R is called quasi-Baer if the left annihilator of every left ideal of R is generated by an idempotent. Birkenmeier, Kim and Park in [23] introduced the concept of principally quasi-Baer rings. A ring R is called left principally quasi-Baer (or simply left $p.q.$-Baer) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right $p.q.$-Baer rings can be defined. A ring is called $p.q.$-Baer if it is both right and left $p.q.$-Baer. Observe that biregular rings and quasi-Baer rings are $p.q.$-Baer. For more details and examples of left $p.q.$-Baer rings, (see [22] and [23]). A ring R is called a right (resp., left) PP-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R).

2 Linearly (S, ω)-Quasi-Armendariz Rings

In this section we introduce the concept of linearly (S, ω)-quasi-Armendariz ring and study its properties. Observe that the notion of linearly (S, ω)-quasi-Armendariz rings not only generalizes that of (S, ω)-quasi-Armendariz rings, but also extends that of linearly (S, ω)-Armendariz rings. We start by the following definition.

Definition 2.1. Let R be a ring, $(S, \leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. We say that R is linearly (S, ω)-quasi-Armendariz, if for all $s \in S\backslash\{1\}$ and $a_0, a_1, b_0, b_1 \in R$, whenever $(c_{a_0} + c_{a_1}e_s)[[R^{S,\leq}, \omega]](c_{b_0} + c_{b_1}e_s) = 0$, then $a_0Rb_0 = a_0Rb_1 = a_1R\omega_s(b_0) = a_1R\omega_s(b_1) = 0$.

It can be easily checked that both (S, ω)-quasi-Armendariz rings and linearly (S, ω)-Armendariz rings are linearly (S, ω)-quasi-Armendariz. But there exist linearly (S, ω)-quasi-Armendariz rings which are not linearly (S, ω)-Armendariz e.g., $Mat_2(R)$ over a linearly (S, ω)-quasi-Armendariz ring R is linearly (S, ω)-quasi-Armendariz by [26, Theorem 2.3], but $Mat_2(R)$ is not linearly (S, ω)-Armendariz by [3] (or [6, Example 1]), even in the case where R is commutative and $[[R^{S,\leq}, \omega]] = R[x]$. Also, the construction in [8, Example 3.2] shows that there exist commutative linearly (S, ω)-quasi-Armendariz rings which are not (S, ω)-quasi-Armendariz, even in the case, S be the additive monoid $\mathbb{N} \cup \{0\}$, with the trivial order, R be a ring, $\omega : S \to End(R)$ be trivial. Then R is an (S, ω)-quasi-Armendariz ring if and only if R is a quasi-Armendariz ring in the usual sense. This is so because in this case the skew generalized power series ring $[[R^{S,\leq}, \omega]]$ is isomorphic to the ordinary polynomial ring $R[x]$.

Let R be a ring, $(S_1, \leq_1), (S_2, \leq_2), \ldots, (S_n, \leq_n)$ be strictly ordered monoid, and $\omega^i : S_i \to End(R)$ be a monoid homomorphism for every i. Define $\omega : S \to End(R)$ as

$$\omega(s_1, s_2, \ldots, s_n) = \omega_{s_1}\omega_{s_2}\cdots\omega_{s_n}.$$

That is,

$$\omega_{(s_1, s_2, \ldots, s_n)} = \omega_{s_1}\omega_{s_2}\cdots\omega_{s_n}.$$

Then ω is well-defined.

Lemma 2.2. *If R is S_i-compatible for each i, then R is S-compatible.*

Proof. Let $a, b \in R$. Then for any $s_i \in S$,
$ab = 0 \Leftrightarrow a\omega_{s_n}(b) = 0$
$\Leftrightarrow a\omega_{s_{n-1}}\omega_{s_n}(b) = 0$
...
$\Leftrightarrow a\omega_{s_1}\omega_{s_2}\cdots\omega_{s_n}(b) = 0$
$\Leftrightarrow a\omega_{(s_1,s_2,\ldots,s_n)}(b) = 0.$
Thus, R is S-compatible. □

The next Lemma appeared in Lemma 1.2 [24].

Lemma 2.3. *For any ring R the following are equivalent:*
(1) *For each element $a \in R$, a^r is an ideal of R, where $a^r = \{b \in R : ab = 0\}$.*
(2) *Any annihilator right ideal of R is an ideal of R.*
(3) *Any annihilator left ideal of R is an ideal of R.*
(4) *For any $a, b \in R$, $ab = 0$ implies $aRb = 0$.*

Every reduced ring (i.e., if there exists no nonzero nilpotent elements) is semicommutative but the converse does not hold in general. There exists a linearly (S,ω)-quasi-Armendariz ring which is not semicommutative Example 14 [7], even in the case where R is commutative and $[[R^{S,\leq},\omega]] = R[x]$, and commutative (hence semicommutative) rings need not to be linearly (S,ω)-quasi-Armendariz. Here we have the following.

Proposition 2.4. *Let $[[R^{S,\leq},\omega]]$ over a ring R be semicommutative, $(S,\leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. If R is (linearly) (S,ω)-quasi-Armendariz, then R is (linearly) (S,ω)-Armendariz.*

Proof. Since the two cases have the same argument, we only give the proof of (S,ω)-Armendariz case. Assume that the skew generalized power series ring $[[R^{S,\leq},\omega]]$ over R is (S,ω)-quasi-Armendariz and semicommutative. Let $f, g \in [[R^{S,\leq},\omega]]$ such that $fg = 0$. Then we get $f[[R^{S,\leq},\omega]]g = 0$ and so $f(u)R\omega_u(g(v)) = f(u)Rg(v) = 0$, by compatibility, for all $u, v \in S$. Thus, $f(u)g(v) = 0$ for all $u, v \in S$, and therefore R is (S,ω)-Armendariz. □

Proposition 2.5. *Let $(S,\leq)$ be a strictly ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and e be a central idempotent of a ring R with $\omega_s(e) = e$. Then R is linearly (S,ω)-quasi-Armendariz if and only if eR and $(1-e)R$ are linearly (S,ω)-quasi-Armendariz.*

Proof. Suppose that R is linearly (S,ω)-quasi-Armendariz. Let $c_{a_0} + c_{a_1}e_s$ and $c_{b_0} + c_{b_1}e_s \in [[(eR)^{S,\leq},\omega]]$ such that $(c_{a_0} + c_{a_1}e_s)[[(eR)^{S,\leq},\omega]](c_{b_0} + c_{b_1}e_s) = 0$. Note that $(c_{a_0} + c_{a_1}e_s)c_e = c_{a_0} + c_{a_1}e_s$ and $c_e(c_{b_0} + c_{b_1}e_s) = c_{b_0} + c_{b_1}e_s$. For any $r \in R$, $(c_{a_0} + c_{a_1}e_s)c_r(c_{b_0} + c_{b_1}e_s) = (c_{a_0} + c_{a_1}e_s)(c_{er})(c_{b_0} + c_{b_1}e_s) = 0$, and so $(c_{a_0} + c_{a_1}e_s)[[R^{S,\leq},\omega]](c_{b_0} + c_{b_1}e_s) = 0$. Since R is linearly (S,ω)-quasi-Armendariz, $a_0Rb_0 = a_0Rb_1 = a_1R\omega_s(b_0) = a_1R\omega_s(b_1) = 0$. Since e is central $a_0(eR)b_0 = 0, a_0(eR)b_1 = 0, a_1(eR)\omega_s(b_0) = 0$ and $a_1(eR)\omega_s(b_1) = 0$. Therefore, eR is linearly (S,ω)-quasi-Armendariz. Similarly, we can show that $(1-e)R$ is linearly (S,ω)-quasi-Armendariz.

Conversely, assume that both eR and $(1-e)R$ are linearly (S,ω)-quasi-Armendariz. Let $c_{a_0} + c_{a_1}e_s$ and $c_{b_0} + c_{b_1}e_s \in [[R^{S,\leq},\omega]]$ be such that $(c_{a_0} + c_{a_1}e_s)[[R^{S,\leq},\omega]](c_{b_0} + c_{b_1}e_s) = 0$. We will show that $a_0Rb_0 = 0, a_0Rb_1 = 0, a_1R\omega_s(b_0) = 0$ and $a_1R\omega_s(b_1) = 0$. For any $r \in R$, $c_e(c_{a_0} + c_{a_1}e_s)(c_{er})c_e(c_{b_0} + c_{b_1}e_s) = c_e((c_{a_0} + c_{a_1}e_s)c_r(c_{b_0} + c_{b_1}e_s)) = 0$ and $c_{1-e}(c_{a_0} + c_{a_1}e_s)c_{1-e}c_r(c_{1-e}(c_{b_0} + c_{b_1}e_s)) = 0$, so

$$c_e(c_{a_0} + c_{a_1}e_s)[[(eR)^{S,\leq},\omega]]c_e(c_{b_0} + c_{b_1}e_s) = 0$$

and

$$c_{1-e}(c_{a_0} + c_{a_1}e_s)[[((1-e)R)^{S,\leq},\omega]]c_{1-e}(c_{b_0} + c_{b_1}e_s) = 0.$$

Since eR and $(1-e)R$ are linearly (S,ω)-quasi-Armendariz, we have $e(a_0Rb_0)=0, e(a_0Rb_1)=0, e(a_1R\omega_s(b_0))=0, e(a_1R\omega_s(b_1))=0$ and $(1-e)(a_0Rb_0)=0, (1-e)(a_0Rb_1)=0, (1-e)(a_1R\omega_s(b_0))=0, (1-e)(a_1R\omega_s(b_1))=0$ and hence $a_0Rb_0=e(a_0Rb_0)+(1-e)(a_0Rb_0)=0, a_0Rb_1=e(a_0Rb_1)+(1-e)(a_0Rb_1)=0, a_1R\omega_s(b_0)=e(a_1R\omega_s(b_0))+(1-e)(a_1R\omega_s(b_0))=0, a_1R\omega_s(b_1)=e(a_1R\omega_s(b_1))+(1-e)(a_1R\omega_s(b_1))=0$. Therefore, R is linearly (S,ω)-quasi-Armendariz. □

Definition 2.6. Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. We say that R is (S,ω)-semiprime, if whenever $f \in [[R^{S,\leq},\omega]]$ satisfy $f[[R^{S,\leq},\omega]]f=0$, then $f=0$.

The following result appeared in [1].

Lemma 2.7. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Then $[[R^{S,\leq},\omega]]$ is reduced if and only if R is reduced.*

Lemma 2.8. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. Then R is a semiprime ring if and only if $[[R^{S,\leq},\omega]]$ is a semiprime ring.*

Proof. $(\Rightarrow)$ Assume the contrary. Then there exists a nonzero $f \in [[R^{S,\leq},\omega]]$ such that $([[R^{S,\leq},\omega]]f[[R^{S,\leq},\omega]])^2=0$. Thus, $f[[R^{S,\leq},\omega]]f=0$. Let $\pi(f)=s_0$. Then, for any $s_0 \in S$, $f(s_0)R\omega_{s_0}(f(s_0))=f(s_0)Rf(s_0)=0$, by compatibility. Set $I=Rf(s_0)R$. Then I is a nonzero ideal of R and $I^2=0$, which is contradict to the fact that R is a semiprime ring.

$(\Leftarrow)$ Let I be an ideal of ring R with $I^2=0$. Then $[[I^{S,\leq},\omega]]$ is an ideal of the ring $[[R^{S,\leq},\omega]]$. For any $f,g \in [[I^{S,\leq},\omega]]$ and any $s \in S$. Since R is compatible,

$$(fg)(s)=\sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v))=0.$$

Thus $fg=0$, which implies that $[[I^{S,\leq},\omega]]^2=0$. Hence $[[I^{S,\leq},\omega]]=0$ since $[[R^{S,\leq},\omega]]$ is a semiprime ring. Consequently, $I=0$, and so R is a semiprime ring. □

Theorem 2.9. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. Then R is semiprime if and only if R is reduced linearly (S,ω)-quasi-Armendariz.*

Proof. $(\Rightarrow)$. Is trivial.
$(\Leftarrow)$ Let R be a reduced linearly (S,ω)-quasi-Armendariz. In particular for any $C_a \in [[R^{S,\leq},\omega]]$ be such that $C_a[[R^{S,\leq},\omega]]C_a=0$, then $aR\omega_s(a)=0$. Thus, by compatibility and reduced $(aR)^2=0$. Therefore $a=0$. □

Corollary 2.10. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. If R is reduced ring, then R is linearly (S,ω)-quasi-Armendariz.*

Since any reduced ring is a semiprime. Here we have.

Corollary 2.11. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. If R is semiprime, then R is linearly (S,ω)-quasi-Armendariz.*

Lemma 2.12. *[1, Proposition 4.8] Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. The following conditions are equivalent:*
(1) R is linearly (S,ω)-Armendariz and reduced, and ω_s is injective for every $s \in S$;
(2) R is S-rigid and $s^2 \notin \{1,s\}$ for every $s \in S\backslash\{1\}$.

Proposition 2.13. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. The following conditions are equivalent:*
(1) R is linearly (S,ω)-quasi-Armendariz and reduced, and ω_s is injective for every $s \in S$;
(2) R is (S,ω)-semiprime and $s^2 \notin \{1,s\}$ for every $s \in S\backslash\{1\}$.

Proof. It follows from Lemma 2.8, Theorem 2.9 and Lemma 2.12. □

The following definition appeared in Definition 2.19 [27].

Definition 2.14. Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to$End(R) a monoid homomorphism. We say that a ring R is completely S-compatible if, for any ideal I of R, R/I is S-compatible, to indicate the homomorphism ω, we will sometimes say that R is completely (S, ω)-compatible.

Every completely S-compatible ring is S-compatible.

Theorem 2.15. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism and R is completely S-compatible.*

(1) *If R is linearly (S, ω)-quasi-Armendariz ring, U is a nonempty subset in R is a two-sided ideal of R, $A = \ell_R(U)$ and $\omega_s|_U$ is surjective for all $s \in S$. Then R/A is linearly $(S, \overline{\omega})$-quasi-Armendariz.*

(2) *For a two-sided ideal I of R, if R/I is a linearly $(S, \overline{\omega})$-quasi-Armendariz ring and I is a semiprime ring without identity, then R is linearly (S, ω)-quasi-Armendariz.*

Proof. (1) Assume that $A = r_R(U)$ is a two sided of linearly (S, ω)-quasi-Armendariz ring R for $\emptyset \neq U \subseteq R$. Let $\bar{a} = a + A$ for $a \in R$. Suppose $c_{\bar{a}_0} + c_{\bar{a}_1} e_s$ and $c_{\bar{b}_0} + c_{\bar{b}_1} e_s \in [[\overline{R}^{S,\leq}, \overline{\omega}]]$ with $(c_{\bar{a}_0} + c_{\bar{a}_1} e_s)[[\overline{R}^{S,\leq}, \overline{\omega}]](c_{\bar{b}_0} + c_{\bar{b}_1} e_s) = \bar{0}$. We claim that $\bar{a}_0(R/A)\bar{b}_0 = 0$, $\bar{a}_0(R/A)\bar{b}_1 = 0$, $\bar{a}_1(R/A)\overline{\omega}_s(\bar{b}_0) = 0$ and $\bar{a}_1(R/A)\overline{\omega}_s(\bar{b}_1) = 0$, where $\overline{\omega}_s(\bar{b}) = \omega_s(b + A)$, for any $s \in S$. Note that the S-compatibility implies $\overline{\omega}_s$ is well-defined. From $(c_{\bar{a}_0} + c_{\bar{a}_1} e_s)[[(R/A)^{S,\leq}, \omega]](c_{\bar{b}_0} + c_{\bar{b}_1} e_s) = \bar{0}$, we get $(c_{\bar{a}_0} + c_{\bar{a}_1} e_s)c_{\bar{r}}(c_{\bar{b}_0} + c_{\bar{b}_1} e_s) = \bar{0}$ for any $\bar{r} \in R/A$. Hence $a_0rb_0, a_0rb_1 + a_1r\omega_s(b_0), a_1r\omega_s(b_1) \in A$, and so $ta_0rb_0 = 0, t(a_0rb_1 + a_1r\omega_s(b_0)) = 0, ta_1r\omega_s(b_1) = 0$ for any $r \in R$ and $t \in U$. Thus, $c_t(c_{a_0} + c_{a_1}e_s)[[R^{S,\leq}, \omega]](c_{b_0} + c_{b_1}e_s) = 0$. Since R is linearly (S, ω)-quasi-Armendariz, we have $t(a_0Rb_0) = 0$, $t(a_0Rb_1) = 0$, $t(a_1R\omega_s(b_0)) = 0$ and $t(a_1R\omega_s(b_1)) = 0$ for any $t \in U$, and hence $a_0Rb_0 \subseteq A$, $a_0Rb_1 \subseteq A$, $a_1R\omega_s(b_0) \subseteq A$ and $a_1R\omega_s(b_1) \subseteq A$. Thus, $\bar{a}_0(R/A)\bar{b}_0 = 0$, $\bar{a}_0(R/A)\bar{b}_1 = 0$, $\bar{a}_1(R/A)\overline{\omega}_s(\bar{b}_0) = 0$ and $\bar{a}_1(R/A)\overline{\omega}_s(\bar{b}_1) = 0$ and therefore R/A is linearly $(S, \overline{\omega})$-quasi-Armendariz.

(2) Let $c_{a_0} + c_{a_1}e_s$ and $c_{b_0} + c_{b_1}e_s \in [[R^{S,\leq}, \omega]]$ such that $(c_{a_0} + c_{a_1}e_s)[[R^{S,\leq}, \omega]](c_{b_0} + c_{b_1}e_s) = 0$. Then we have $a_0rb_0 = 0, a_0rb_1 + a_1r\omega_s(b_0) = 0$ and $a_1r\omega_s(b_1) = 0$ for any $r \in R$, thus $a_0Rb_0 = 0$ and $a_1R\omega_s(b_1) = 0$. We claim that $a_0Rb_1 = 0$. Assume $a_0Rb_1 \neq 0$. Note that $(b_0Ia_0R)^2 = 0$ implies $b_0Ia_0R = 0$ and so $b_0Ia_0 = 0$ since $b_0Ia_0R \subseteq I$ and I is semiprime. Since R/I is linearly $(S, \overline{\omega})$-quasi-Armendariz, we get $a_0Rb_0 \subseteq I$, $a_0Rb_1 \subseteq I$, $a_1R\omega_s(b_0) \subseteq I$ and $a_1R\omega_s(b_1) \subseteq I$. Then

$$(a_1R\omega_s(b_0))(Ra_0Rb_1)^2 = (a_1R)(\omega_s(b_0)Ra_0Rb_1Ra_0)Rb_1 \subseteq a_1R(\omega_s(b_0)Ia_0)Rb_1 = 0$$

by compatibility Lemma 2.5 [1]. From $a_0rb_1 + a_1r\omega_s(b_0) = 0$ for any $r \in R$ and any $s \in S$, we have $0 = (a_0rb_1 + a_1r\omega_s(b_0))(ua_0tb_1)^2 = a_0rb_1(ua_0tb_1)^2$ for any $r, u, t \in R$ and thus $(Ra_0Rb_1)^3 = 0$. Since $Ra_0Rb_1 \subseteq I$ and I is semiprime, $Ra_0Rb_1 = 0$ and so $a_0Rb_1 = 0$, a contradiction. Hence, $a_0Rb_0 = 0$, $a_0Rb_1 = 0$, $a_1R\omega_s(b_0) = 0$ and $a_1R\omega_s(b_1) = 0$ and therefore R is linearly (S, ω)-quasi-Armendariz. □

Remark 2.16. Let $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$. It can be easily checked that R is a linearly (S, ω)-quasi-Armendariz and semicommutative ring, and hence R/A is linearly $(S, \overline{\omega})$-quasi-Armendariz ring for the one-sided annihilator A of a nonempty subset in R by Theorem 2.15(1). Moreover, $R/I \cong \mathbb{Z}_2$ is a linearly $(S, \overline{\omega})$-quasi-Armendariz ring for a semiprime ideal $I = \{0\} \bigoplus \mathbb{Z}_2$ of R, even in the case where R is commutative and $[[R^{S,\leq}, \omega]] = R[x]$.

Corollary 2.17. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism.*

(1) *If a ring R is semicommutative and linearly (S, ω)-quasi-Armendariz, then R/A is linearly $(S, \overline{\omega})$-quasi-Armendariz, where $A = \ell_R(U)$ and $\omega_s|_U$ is surjective for all $s \in S$ and U is a nonempty subset in R.*

(2) *If a ring R is linearly (S,ω)-quasi-Armendariz and satisfies any one of the following conditions, then R/A is linearly $(S,\overline{\omega})$-quasi-Armendariz:*
- *R is an abelian Baer ring and A is the one-sided annihilator of a nonempty subset in R.*
- *R is a quasi-Baer ring and A is the right annihilator of a right ideal in R.*
- *R is an abelian right (resp., left) $p.p.$-ring and A is the right (resp., left) annihilator of an element in R.*
- *R is a right (resp., left) $p.q.$-Baer ring and A is the right (resp., left) annihilator of a principal right (resp., left) ideal in R.*

Proof. (1) By Lemma 2.3, a ring R is semicommutative ring if and only if any one-sided annihilator over R is a two-sided ideal of R, and thus R/A is linearly $(S,\overline{\omega})$-quasi-Armendariz by Theorem 2.15.
(2) If R is abelian or A is the right (resp., left) annihilator of a right (resp., left) ideal in R, then A is a two-sided ideal of R. Thus, R/A is linearly $(S,\overline{\omega})$-quasi-Armendariz by Theorem 2.15. □

Lemma 2.18. *[25, Proposition 2.4] Let S be a right order in a right Artinian ring Q and let $\rho : S \to S$ be a monomorphism.*
(1) *An element $c \in S$ is regular in S if and only if $\rho(c)$ is regular in S.*
(2) *ρ can be uniquely extended to a monomorphism $\tilde{\rho} : Q \to Q$.*

One can find the next definition in [1].

Definition 2.19. Let $(S,\leq)$ be an ordered monoid. We say that $(S,\leq)$ is an artinian narrow unique product monoid (or an $a.n.u.p.$ monoid, or simply $a.n.u.p.$) if for every two artinian and narrow subsets X and Y of S there exists a $u.p.$ element in the product XY. We say that $(S,\leq)$ is quasitotally ordered (and that $\leq$ is a quasitotal order on S) if $\leq$ can be refined to an order $\preceq$ with respect to which S is a strictly totally ordered monoid.

For any ordered monoid $(S,\leq)$, the following chain of implications holds:

S is commutative, torsion-free, and cancellative

$$\Downarrow$$

$(S,\leq)$ is quasitotally ordered

$$\Downarrow$$

$(S,\leq)$ is $a.n.u.p. \Rightarrow u.p.$

The converse of the bottom implication holds if $\leq$ is the trivial order. For more details, examples, and interrelationships between these and other conditions on ordered monoids, we refer the reader to [29].

Let R be a semiprime left Goldie ring, and let C denote the set of regular elements of R (that is, elements that are neither left nor right zero-divisors). If $\sigma \in End(R)$ is injective, then $\sigma(C) \subseteq C$ by Lemma 2.18. Therefore, if $Q = Q_{cl}^{\ell}$ is the classical left ring of quotients of R, then one can verify that σ extends (uniquely) to an endomorphism $\widetilde{\sigma}$ of Q defined by $\widetilde{\sigma}(b^{-1}.a) = \sigma(b)^{-1}\sigma(a)$ for all $a \in R$ and $b \in C$.

In this setting, if S is a monoid and $\omega : S \to \mathrm{End}(R)$ is a monoid homomorphism such that ω_s is injective for every $s \in S$, then there is an induced monoid homomorphism $\widetilde{\omega} : S \to \mathrm{End}(Q)$ defined by $\widetilde{\omega}_s = \widetilde{\omega}(s)$ for each $s \in S$.

Notice that $\widetilde{\omega}_s$ is injective for every $s \in S$.

The following result generalizes Theorem 4.17 [1].

Theorem 2.20. *Let R be a semiprime left Goldie ring, $(S,\leq)$ a nontrivial strictly ordered $a.n.u.p.$ monoid, and $\omega : S \to End(R)$ a monoid homomorphism such that ω_s is injective for every $s \in S$. Let $Q = Q_{cl}^{\ell}$ denote the classical left ring of quotients of R, and $\widetilde{\omega} : S \to End(Q)$ the induced S-action. Then the following conditions are equivalent:*
(1) *R is (S,ω)-quasi-Armendariz;*
(2) *R is linearly (S,ω)-quasi-Armendariz;*
(3) *Q is $(S,\widetilde{\omega})$-quasi-Armendariz;*
(4) *Q is linearly $(S,\widetilde{\omega})$-quasi-Armendariz.*

Proof. (1) $\Rightarrow$ (2) Trivial.
(2) $\Rightarrow$ (4) We have to show that for any $p_0, p_1, q_0, q_1 \in Q$ and $s \in S\backslash\{1\}$,
if $(c_{p_0} + c_{p_1} e_s)[[Q^{S,\leq}, \omega]](c_{q_0} + c_{q_1} e_s) = 0$, then $p_0 r q_1 = p_1 r \widetilde{\omega}_s(q_0) = 0.(\ddagger)$

Now, there exist $a_0, a_1, b_0, b_1, u \in R$ such that u is regular and $p_i = u^{-1}a_i, q_i = u^{-1}b_i$ for $i = 1, 2$. Furthermore, for some $d_0, d_1, v \in R$ with v regular, we can write $a_0 u^{-1} = v^{-1} d_0$ and $a_1 \omega_s(u)^{-1} = v^{-1} d_1$. Now it is easy to see that in $[[R^{S,\leq}, \omega]]$ we have $(c_{d_0} + c_{d_1} e_s)[[R^{S,\leq}, \omega]](c_{b_0} + c_{b_1} e_s) = 0$, Since R is linearly (S, ω)-quasi-Armendariz, we obtain $d_0 r b_1 = d_1 r \omega_s(b_0) = 0$. Now $p_0 r q_1 = p_1 r \widetilde{\omega}_s(q_0) = 0$ follows easily, proving ($\ddagger$).
(3) $\Leftrightarrow$ (4) Trivial. □

The following is obtained by applying the method in the proof of Theorem 2.20.

Corollary 2.21. *Let R be a semiprime left Goldie ring, $(S, \leq)$ a nontrivial strictly ordered $a.n.u.p.$ monoid, and $\omega : S \rightarrow End(R)$ a monoid homomorphism such that ω_s is injective for every $s \in S$. Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is linearly (S, ω)-quasi-Armendariz if and only if $\Delta^{-1}R$ is linearly $(S, \widetilde{\omega})$-quasi-Armendariz.*

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