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The Square Power Graph of a Finite Abelian Group

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Abstract Let G be a finite abelian group. The square power graph of G is a special type of an undirected simple finite graph whose vertex set is group G itself and two distinct vertices $a, b \in G$ are adjacent if and only if $a + b = 2c$ for some $c \in G$ and $2c \neq e$ (e is identity of group G). In this paper, we have studied the structure and various structural properties of square power graph such as degree of a vertex, diameter, girth, connectedness, independent number, clique number, etc for finite abelian group. We have introduced and studied extended square power graph. We have also discussed about prime labeling of square power graph and extended square power graph.

1 Introduction

Using various group properties, we can define numerous graphs for finite groups such as commuting graphs, power graphs, additive power graphs, and prime graphs. In recent years, a number of scholars have investigated into the various characteristics of graphs of groups. Those power graphs of finite groups, which are cographs, were explored by P. J. Cameron et al. in 2021 [1]. The various properties of power graphs, enhanced power graphs, deep commuting graphs, commuting graphs, and non-generating graphs were also researched by P. J. Cameron[2]. A. Sehgal and S. N. Singh presented a formula to determine the degree of a vertex in a finite abelian group's power graph[3] and for $k^{th}-$power graph of finite abelian group in [4]. A. Sehgal et al. studied co-prime order graphs of finite abelian groups and established a solution to calculate the degree of each vertex in the finite abelian group's co-prime order graph[5]. For a detailed survey on graph labeling and graceful labeling of power graph we refer to [6, 7]. A. V. Kanetkar shows that prime labeling for specific grids is possible[8]. D. Beutner and H. Harborth studied graceful labeling of nearly complete graphs [9]. D. Sinha and D. Sharma characterized absorption cayley graphs and their properties such as connectedness, degree, diameter, planarity, girth, regularity, etc in [10]. General Randic energy of the various graphs obtained through graph operations is studied in [11]. Various structural properties of line graphs associated to the unit graphs of rings, square element graphs over semi-groups and idempotent graph of rings are studied in [12, 13, 14]

Structure of square power graph of $\mathbb{Z}_n$ and $\mathbb{Z}_2^m \times \mathbb{Z}_{2^n}$ is studied in [15]. R. R. Prathap and T. T. Chelvam studied various properties such as connectedness, diameter, girth, independent number, clique number, chromatic number etc of cubic power graph and complement graph of square power graph of finite abelian group in [16, 17] respectively. Throughout this paper we have studied structure and various properties of square power graph $\Gamma_{sq}(G)$ such as degree, radius, diameter, girth, connectedness, independent number, clique number, etc. We have established prime labeling of square power graph under different conditions. We have also defined and studied various properties of extended square power graph $\Gamma_{sqe}(G)$.

2 Preliminaries

Let us discuss some basic terms of graph theory and related to groups which are used throughout this paper. A simple undirected finite graph with V vertex set and E edge set is denoted as $\Gamma = (V, E)$. Any graph is said to be connected if there is a path between every pair of different vertices. If every pair of distinct vertices in graph Γ are joined by an edge then Γ is said to be complete. Complete graph with n vertices is denoted as K_n. The length of shortest path between two vertices a and b in graph is known as distance between vertices a and b, denoted as $d(a, b)$ and in case no such path exists between a and b then $d(a, b) = \infty$. Eccentricity of vertex a in graph Γ is the maximum distance of a to any vertex in Γ denoted by $e(a)$ i.e $e(a) = max\{d(a, b) : b \in G\}$. Minimum eccentricity among elements of G in Γ is known as radius of Γ, denoted as radius(Γ) and whereas maximum eccentricity is known as diameter of Γ denoted as $diam(\Gamma)$. If eccentricity of every vertex of graph Γ has same value then graph Γ is said to be self-centered i.e if $radius(\Gamma) = diam(\Gamma)$ then Γ is said to be self-centered. The length of the shortest cycle in graph Γ is known as girth of Γ and denoted as $gr(\Gamma)$. If there exist no cycle in Γ then we have $gr(\Gamma) = \infty$. If $K_{1,n-1}$ is a sub-graph of graph Γ with n vertices then Γ is said to be refinement of star graph $K_{1,n-1}$ (here Γ is not complete graph). Number of vertices adjacent with vertex a is known as degree of vertex a, denoted as $deg_{\Gamma(G)}(a)$. A clique is a subset of $V(\Gamma)$ such that sub-graph with vertex set as that subset, is complete. Number of vertices in clique of Γ with maximum vertices is known as clique number, denoted as $\omega(\Gamma)$. A set of vertices in which no two vertices are adjacent is known as independent set and number of vertices in maximal independent set is known as independent number. If all the vertices of connected graph Γ have even degree then Γ is said to be eulerian graph.
All groups considered in this paper are nontrivial finite abelian groups. We have used 0 or e for identity element of G. The additive power graph $\Gamma(G)$ of finite abelian group G is the simple undirected graph whose vertex set is group itself and two distinct vertices a,b are adjacent if and only if $a + b = nc$ for some $n \geq 2$ and $c \in G$ with $nc \neq e$, where e is identity of G. When $n = 2$, additive power graph is studied as square power graph with notation $\Gamma_{sq}(G)$. We have $G_1 = \{2c : c \in G\} \subseteq G$ for a finite abelian group G and $G_2 = G \setminus G_1$ so $G = G_1 \cup G_2$. It should be noted that two distinct elements $a, b \in G$ have edge between them in $\Gamma_{sq}(G)$ if and only if $a + b \in G_1 \setminus \{e\}$. When $gcd(|G|, 2) = 1$ for a finite abelian group G, then we have $G_1 = \{2a : a \in G\} = G$.
Also we define a new graph, extended square power graph Γ_{sqe} which is simple undirected finite graph whose vertex set is group G itself and two distinct vertices a, b are adjacent if and only if $a + b = 2c$ where $c \in G$ i.e two distinct vertices $a, b \in G$ are adjacent in $\Gamma_{sq}(G)$ if and only if $a + b \in G_1$. As graphs discussed in this paper are simple so whenever we are using two vertices a and b, we mean distinct vertices. We have used complement of component of graph to simplify the representation of the structure of $\Gamma_{sq}(G)$. Throughout this paper we have used p, p_i for prime numbers and $G(p_i)$ for p_i-group.

3 Structural Properties

Theorem 3.1. *Let G be a finite abelian group, such that $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ and $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ where $p_i \neq 2$. Then*
(i) $\Gamma_{sq}(G) = \overline{[K_1 \cup \frac{\prod_{i=1}^{n} p_i^{m_i} - 1}{2} K_2]}$
(ii) $\Gamma_{sq}(G)$ *is connected.*
(iii) $deg_{\Gamma_{sq}(G)}(a) = \begin{cases} |G| - 1 \text{ if } a = a^{-1} \\ |G| - 2 \text{ otherwise} \end{cases}$
(iv) $radius\big(\Gamma_{sq}(G)\big) = 1$ *and* $diam\big(\Gamma_{sq}(G)\big) = 2$
(v) $gr\big(\Gamma_{sq}(G)\big) = \begin{cases} \infty \text{ if } G \cong \mathbb{Z}_3 \\ 3 \text{ otherwise} \end{cases}$
(vi) $\Gamma_{sq}(G)$ *is a refinement of the star graph* $K_{1,|G|-1}$.

Proof. Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ and $p_i \neq 2$. So we have $gcd(|G|, 2) = 1$ and so $G_1 = G$.

(i) Identity element e of group G is adjacent with every vertex in $\Gamma_{sq}(G)$ as $e + a \in G_1 \setminus \{e\}$ $\forall\ a \in G \setminus \{e\}$. There exist no non-identity element $a \in G$ such that $a = a^{-1}$ so we have for every non-identity element $a \in G$ adjacent with every $b \in G \setminus \{a^{-1}\}$ as $a + b \in G_1 \setminus \{e\}$. So in complement we have identity element adjacent with no vertex and non-identity vertex only adjacent with their inverse in $\Gamma_{sq}(G)$. Hence $\Gamma_{sq}(G) = \overline{[K_1 \cup \frac{\prod_{i=1}^{n} p_i^{m_i} - 1}{2} K_2]}$.
(ii) As discussed in above part (i), identity element is adjacent with every element of G, so we have path between every pair of vertices in $\Gamma_{sq}(G)$. Hence $\Gamma_{sq}(G)$ is connected.
(iii) As we have only identity element of G which is self inverse and also it is adjacent with all non-identity elements of G in $\Gamma_{sq}(G)$ so we have $deg_{\Gamma_{sq}(G)}(a) = |G| - 1$ when $a = a^{-1}$ (in this case only possible value of a is identity element of G). Also for every non-identity element $a \in G$ we have $a \neq a^{-1}$ and adjacent with every element of $G \setminus \{a, a^{-1}\}$. So we get $deg_{\Gamma_{sq}(G)}(a) = |G| - 2$ when $a \neq a^{-1}$.
(iv) Since every non-identity element a of G is adjacent with identity element e of G, as described in (i), so $d(e, a) = 1\ \forall a \in G \setminus \{e\}$. So we have $e_{\Gamma_{sq}}(e) = 1$. Every $a \in G \setminus \{e\}$ is adjacent with every $b \in G \setminus \{a^{-1}\}$ so we have $d(a, b) = 1$ and $d(a, a^{-1}) = 2$ for every $a \in G \setminus \{e\}$, so shortest path between vertices a, a^{-1} is $a - e - a^{-1}$ which give us $e_{\Gamma_{sq}}(a) = 2$. Hence we have $radius\big(\Gamma_{sq}(G)\big) = 1$ $diam\big(\Gamma_{sq}(G)\big) = 2$.
(v) Case 1. When $G \cong \mathbb{Z}_3$ then we have $\Gamma_{sq}(G) = P_3$ as e identity vertex adjacent with other two elements x, x^2 and x is not adjacent with x^2 as they are inverse of each other, so we have $x - e - x^2$ path as $\Gamma_{sq}(G)$. Hence $gr(\Gamma_{sq}(G)) = \infty$.
Case 2. When $G \ncong \mathbb{Z}_3$ then we have $e, a, b \in G$ such that $a + b \neq e$. Then we have $e - a - b - e$ cycle of length 3 in $\Gamma_{sq}(G)$, so $gr(\Gamma_{sq}(G)) = 3$.
(vi) From above discussion, we have $\Gamma_{sq}(G)$ is not a complete graph and as there exists $a \in G$ such that $a \neq a^{-1}$ and so we have pair of vertices a, a^{-1} not adjacent with each other. Now as identity vertex in $\Gamma_{sq}(G)$, adjacent with $|G| - 1$ elements of $G \setminus \{e\}$ and also $a, b(\neq a^{-1}) \in G$ adjacent with each other. So $K_{1,|G|-1}$ is sub-graph of $\Gamma_{sq}(G)$. Hence $\Gamma_{sq}(G)$ is refinement of the star graph with center identity element e. □

Theorem 3.2. *Let G be a finite abelian group with identity e and* $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$*;* $n \in \mathbb{N}$ *where* $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ *and* $p_i \neq 2$ *then* $\Gamma_{sq}(G) \cong \Gamma_{sq}(\mathbb{Z}_m)$ *where* $m = \prod_{i=1}^{n} p_i^{m_i}$.

Proof. For $m = \prod_{i=1}^{n} p_i^{m_i}$ where $p_i \neq 2$, $\Gamma_{sq}(\mathbb{Z}_m)$ is a connected graph in which identity element e is adjacent with every $a \in \mathbb{Z}_m \setminus \{e\}$ and every $a \in \mathbb{Z}_m \setminus \{e\}$ adjacent with every $b \in \mathbb{Z}_m \setminus \{a, a^{-1}\}$. In $\mathbb{Z}_m$ we have only identity element e which is self inverse. So in complement of $\Gamma_{sq}(\mathbb{Z}_m)$ we have vertex e adjacent with no vertex and every non-identity a vertex adjacent with only with its inverse a^{-1} vertex. Hence $\Gamma_{sq}(\mathbb{Z}_m) = \overline{[K_1 \cup \frac{m-1}{2} K_2]}$. Also from Theorem 3.1$(i)$ we have $\Gamma_{sq}(G) = \overline{[K_1 \cup \frac{\prod_{i=1}^{n} p_i^{m_i} - 1}{2} K_2]}$. So we have $\Gamma_{sq}(G) \cong \Gamma_{sq}(\mathbb{Z}_m)$. □

Theorem 3.3. *Let G be a finite abelian group with identity e and* $G = G(p_1) \times G(p_2) \times \cdots \times G(p_s) \times \cdots \times G(p_r) \times \cdots \times G(p_n)$*;* $n \in \mathbb{N}$ *where* $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$*;* $p_i = 2$ *for* $1 \leq i \leq r$ *and* $p_i \neq 2$ *for* $(r+1) \leq i \leq n$*. Then we have*
(i)$\Gamma_{sq}(G)$ *is disconnected.*
(ii)$deg_{\Gamma_{sq}(G)}(a) = \begin{cases} \frac{|G|}{2^r} - 2 \text{ if } a^{-1} \neq a \\ \frac{|G|}{2^r} - 1 \text{ if } a^{-1} = a \end{cases}$
(iii)$\Gamma_{sq}(G)$ *is*
$$\begin{cases} (a)\ 2^r \overline{[K_1 \cup (\frac{|G| - 2^r}{2^{r+1}}) K_2]} \text{ when } m_i = 1 \text{ for } 1 \leq i \leq r \text{ and } n \neq r \\ (b)\ 2^n K_1 \text{ when } m_i = 1 \text{ and } n = r \\ (c)\ \overline{[2^r K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-1}) K_2]} \cup (2^r - 1) \overline{[\frac{|G|}{2^{r+1}} K_2]} \text{ when } m_i \geq 2 \text{ for } 1 \leq i \leq r \\ (d)\ 2^s \overline{[2^{r-s} K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-s-1}) K_2]} \cup (2^r - 2^s) \overline{[\frac{|G|}{2^{r+1}} K_2]} \\ \quad \text{when } m_i = 1 \text{ for } 1 \leq i \leq s, m_i \geq 2 \text{ for } s+1 \leq i \leq r \text{ and } G \ncong \mathbb{Z}_2^s \times \mathbb{Z}_4 \\ (e)\ 2^s \overline{[K_2]} \cup 2^{s+1} \overline{[K_1]} \text{ when } G \cong \mathbb{Z}_2^s \times \mathbb{Z}_4 \end{cases}$$
(iv)$radius(\Gamma_{sq}(G)) = \infty$ *and* $diam(\Gamma_{sq}(G)) = \infty$.

$(v) gr\big(\Gamma_{sq}(G)\big) = \begin{cases} \infty \text{ if } G \in \{\mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_2^n, \mathbb{Z}_2^n \times \mathbb{Z}_3, \mathbb{Z}_2^n \times \mathbb{Z}_4\} \\ 3 \text{ otherwise} \end{cases}$

Proof. Let G be a finite abelian group with identity e and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_r) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; $p_i = 2$ for $1 \leq i \leq r$ and $p_i \neq 2$ for $(r+1) \leq i \leq n$. Then we have $G_1 \neq G$.

(i) As $G_1 \neq G$ and so every $a \in G_1$ is not adjacent with any $b \in G \setminus G_1$ i.e for any $a \in G_1$ and $b \in G \setminus G_1$ we have $a + b \notin G_1 \setminus \{e\}$. So we have pair of vertices a, b in $\Gamma_{sq}(G)$ such that no path exists between them. Hence $\Gamma_{sq}(G)$ is disconnected.

(ii) Let $G_{ie} = \{2a : a \in G(p_i)\}$ and $G_{io} = G(p_i) \setminus G_{ie}$ for $1 \leq i \leq r$. Now we have $G_1 = \prod_{i=1}^{r} G_{ie} \times G(p_{r+1}) \times \cdots \times G(p_n)$. Two disjoint elements $a = (a_1, a_2, a_3, \cdots, a_r, \cdots, a_n), b = (b_1, b_2, b_3 \cdots, b_r, \cdots, b_n)$ of G are adjacent in $\Gamma_{sq}(G)$ i.e $a + b \in G_1 \setminus \{e\}$ only when both $a_i, b_i \in G_{ie}$ or $a_i, b_i \in G_{io}$ for $1 \leq i \leq r$ and $a^{-1} \neq b$. So we have 2^r disjoint components in $\Gamma_{sq}(G)$, each with $\frac{\prod_{i=1}^{n} p_i^{m_i}}{2^r}$ vertices. Now we have any vertex a for which $a \neq a^{-1}$ of any component of $\Gamma_{sq}(G)$ adjacent with every vertex other than a and a^{-1} in that component and any vertex a for which $a = a^{-1}$ of any component of $\Gamma_{sq}(G)$ adjacent with every vertex other that itself in that component. Hence we get $deg_{\Gamma_{sq}(G)}(a) = \frac{|G|}{2^r} - 2$ if $a^{-1} \neq a$ and $\frac{|G|}{2^r} - 1$ if $a^{-1} = a$.

(iii) (a) When $m_i = 1$ for $1 \leq i \leq r$ and $n \neq r$, then we have $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cdots \times \mathbb{Z}_2 \times G(p_{r+1}) \times \cdots \times G(p_n) = \mathbb{Z}_2^r \times G(p_{r+1}) \times \cdots \times G(p_n)$. As discussed in above part, in $\Gamma_{sq}(G)$ we have 2^r disjoint component and each component with $\frac{|G|}{2^r}$ number of vertices. In G we have 2^r number of elements which are self inverse and in each component of $\Gamma_{sq}(G)$ we have one element which is self inverse. So in every component we have one element adjacent with every other vertex in that component and $\frac{|G|}{2^r} - 1$ elements in that component adjacent with all vertices in that component other than itself and its inverse. Hence $\Gamma_{sq}(G) = 2^r\overline{[K_1 \cup (\frac{\frac{|G|}{2^r}-1}{2})K_2]} = 2^r\overline{[K_1 \cup (\frac{|G|-2^r}{2^{r+1}})K_2]}$. Hence the required result.

(b) When $m_i = 1$ and $n = r$ then we have $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = \mathbb{Z}_2^n$. Then we have $G_1 = \{e\}$, where e is identity element of G. So we have no pair of vertices in $\Gamma_{sq}(G)$ for which $a + b \in G_1 \setminus \{e\}$ i.e no two vertices in $\Gamma_{sq}(G)$ which are adjacent. Hence $\Gamma_{sq}(G) = 2^n K_1$.

(c) When $m_i \geq 2$ for $1 \leq i \leq r$, in this case we have all the self-inverse 2^r elements of G, in single component of $\Gamma_{sq}(G)$. So one component of $\Gamma_{sq}(G)$, is $\overline{[2^r K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-1})K_2]}$. In remaining $2^r - 1$ components of $\Gamma_{sq}(G)$ we have no self-inverse element vertex. Hence $2^r - 1$ components of $\Gamma_{sq}(G)$ are of form $\overline{[\frac{|G|}{2^{r+1}} K_2]}$. Hence in this case we have $\Gamma_{sq}(G) = \overline{[2^r K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-1})K_2]} \cup (2^r - 1)\overline{[\frac{|G|}{2^{r+1}} K_2]}$.

(d) When $m_i = 1$ for $1 \leq i \leq s$, $m_i \geq 2$ for $s + 1 \leq i \leq r$ and $G \ncong \mathbb{Z}_2^s \times \mathbb{Z}_4$

In this case we have 2^r components of $\Gamma_{sq}(G)$, out of which 2^s components have all the self-inverse elements of G in such a way that each of them contains 2^{r-s} number of self-inverse element vertices. So we have 2^s components of $\Gamma_{sq}(G)$ of form $\overline{[2^{r-s} K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-s-1})K_2]}$. In remaining $2^r - 2^s$ components of $\Gamma_{sq}(G)$ we have no self-inverse element vertices. So we have $2^r - 2^s$ components of form $\overline{[\frac{|G|}{2^{r+1}} K_2]}$. Hence in this case we have

$\Gamma_{sq}(G) = 2^s\overline{[2^{r-s} K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-s-1})K_2]} \cup (2^r - 2^s)\overline{[\frac{|G|}{2^{r+1}} K_2]}$.

(e) When $G \cong \mathbb{Z}_2^s \times \mathbb{Z}_4$

Here we have $r = s + 1$. In this case instead of having 2^{s+1} components in $\Gamma_{sq}(G)$, we have $2^s + 2^{s+1}$ components out of which 2^s components have two self-inverse element vertices each and remaining 2^{s+1} components have one element (which is not self-inverse) vertex each. So we have 2^s number of K_2 components and 2^{s+1} number of K_1 components. Hence $\Gamma_{sq}(G) = 2^s[K_2] \cup 2^{s+1}[K_1]$.

(iv) As $\Gamma_{sq}(G)$ is disconnected so we have $radius(\Gamma_{sq}(G)) = \infty$ and $diam(\Gamma_{sq}(G)) = \infty$.

(v) Case 1. When $G \in \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_2^n, \mathbb{Z}_2^n \times \mathbb{Z}_3, \mathbb{Z}_2^n \times \mathbb{Z}_4\}$ then by using Theorem 3.3(iii) structure of $\Gamma_{sq}(G)$ is shown in Theorem 4.1 and Theorem 4.3. We have no cycle in $\Gamma_{sq}(G)$ when $G \in \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_2^n, \mathbb{Z}_2^n \times \mathbb{Z}_3, \mathbb{Z}_2^n \times \mathbb{Z}_4\}$. So $gr\big(\Gamma_{sq}(G)\big) = \infty$ when $G \in \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_2^n, \mathbb{Z}_2^n \times \mathbb{Z}_3, \mathbb{Z}_2^n \times \mathbb{Z}_4\}$.

Case 2. When $G \notin \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_2^n, \mathbb{Z}_2^n \times \mathbb{Z}_3, \mathbb{Z}_2^n \times \mathbb{Z}_4\}$ then we have $e, a, b \in G$ such that $a + b \neq e$. So we have $e - a - b - e$ cycle of length 3 in $\Gamma_{sq}(G)$. Hence $gr(\Gamma_{sq}(G)) = 3$. □

Corollary 3.4. *Let G be a finite abelian group then $\Gamma_{sq}(G)$ is connected if and only if $|G|$ is odd.*

Proof. By using Theorem 3.1(ii) and Theorem 3.3(i) we get the required result. □

Theorem 3.5. *Let G be a finite abelian group then $\Gamma_{sq}(G)$ is not self-centered.*

Proof. Let G be a finite abelian group then
Case 1. When $|G|$ is not divisible by 2 then by using Theorem 3.1 we have graph $\Gamma_{sq}(G)$ connected, in which identity element e is adjacent with every element of G and non-identity element a is adjacent with every $b \in G \setminus \{a, a^{-1}\}$. So we get $e_{\Gamma_{sq}}(e) = 1$ and $e_{\Gamma_{sq}}(a) = 2$ where $a \in G \setminus \{e\}$. Hence $\Gamma_{sq}(G)$ is not self-centered.
Case 2. When $|G|$ is divisible by 2 then we have $\Gamma_{sq}(G)$ is disconnected as shown in Theorem 3.3. So $\Gamma_{sq}(G)$ is not self-centered. □

Theorem 3.6. *(i)Let $G \cong \mathbb{Z}_n$, $n \geq 2$ then we have independent number*

$$\beta(\Gamma_{sq}(G)) = \begin{cases} 2 \text{ when } gcd(2,n) = 1 \\ 2 \text{ when } n = 2 \\ 3 \text{ when } n = 4 \\ 4 \text{ when n is even natural number} \geq 6 \end{cases}$$

(ii) Let G be a finite abelian group with identity e and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_s) \times \cdots \times G(p_r) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; $p_i = 2$ for $1 \leq i \leq r$ and $p_i \neq 2$ for $(r+1) \leq i \leq n$. Then we have independent number

$$\beta(\Gamma_{sq}(G)) = \begin{cases} 2^n \text{ when } m_i = 1 \text{ and } n = r \\ 2^s + 2^{s+1} \text{ when } G \cong \mathbb{Z}_2^s \times \mathbb{Z}_4 \\ 2^{r+1} \text{ otherwise} \end{cases}$$

Proof. **(i) Case 1.** Let $G \cong \mathbb{Z}_n$ where $n \geq 2$ and $gcd(2,n) = 1$ then by using Theorem 3.1 and Theorem 3.2 we get that $\Gamma_{sq}(G)$ is connected and there exists $a \in G$ which is adjacent with every $b \in G \setminus \{a, a^{-1}\}$, here $a^{-1} \neq a$. So we have maximal independent set $\{a, a^{-1}\}$ where $a^{-1} \neq a$. Hence $\beta(\Gamma_{sq}(G)) = 2$ if $gcd(2,n) = 1$.
Case 2. When $n = 2$ we have $\Gamma_{sq}(G) = 2K_1$ and so we have $\beta(\Gamma_{sq}(G)) = 2$ when $n = 2$.
Case 3. When $n = 4$ we have $\Gamma_{sq}(G) = 2K_1 \cup K_2$ and so we have $\beta(\Gamma_{sq}(G)) = 3$ when $n = 4$.
Case 4. When n is even natural number≥ 6 then we have two disjoint components in $\Gamma_{sq}(G)$. One component has vertex set $G_1 = \{2a : a \in G\}$ and another component have $G \setminus G_1$ as its vertex set. Now for every $a, b \in G_1$, we have $a + b \in G_1 \setminus \{e\}$ whenever $a^{-1} \neq b$. And also for every $a, b \in G \setminus G_1$ we have $a + b \in G_1 \setminus \{e\}$ whenever $a^{-1} \neq b$. Hence we have maximal independent set $\{a, a^{-1}, b, b^{-1}\}$ where $a, a^{-1} \in G_1$ and $b, b^{-1} \in G \setminus G_1$. So we get $\beta(\Gamma_{sq}(G)) = 4$ when n is even natural number ≥ 6.
(ii) Case 1. When $m_i = 1$ and $n = r$ then by using Theorem 3.3(iii)(b) we get $\Gamma_{sq}(G) = 2^n K_1$. So we have $\mathbb{Z}_2^n$ as maximal independent set and independent number $\beta(\Gamma_{sq}(G)) = 2^n$.
Case 2. When $G \cong \mathbb{Z}_2^s \times \mathbb{Z}_4$ then by using Theorem 3.3(iii)(e) we have $\Gamma_{sq}(G) = 2^s[K_2] \cup 2^{s+1}[K_1]$. So we have maximal independent set containing 2^{s+1} vertices forming 2^{s+1} K_1 components and one vertex from each 2^s K_2 components. Hence we have $2^s + 2^{s+1}$ number of elements in maximal independent set and so $\beta(\Gamma_{sq}(G)) = 2^s + 2^{s+1}$.

Case 3. Using Theorem 3.3(iii)(a) we have $\Gamma_{sq} = 2^r\overline{[K_1 \cup (\frac{|G|-2^r}{2^{r+1}})K_2]}$ when $m_i = 1$ for $1 \leq i \leq r$ and $n \neq r$. Then maximal independent set contains a pair of a, a^{-1} from each of 2^r component. Hence we have $2 \times 2^r = 2^{r+1}$ elements in maximal independent set and so $\beta(\Gamma_{sq}(G)) = 2^{r+1}$.

Using Theorem 3.3(iii)(c) we have $\Gamma_{sq}(G) = \overline{[2^r K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-1})K_2]} \cup (2^r - 1)\overline{[\frac{|G|}{2^{r+1}} K_2]}$ when $m_i \geq 2$ for $1 \leq i \leq r$. In this case maximal independent set contains two elements from each $2^r - 1$ components and two elements from the remaining component, which gives $\beta(\Gamma_{sq}(G)) = 2 \times (2^r - 1) + 2 = 2^{r+1}$.

Also from Theorem 3.3(iii)(d) we have$\Gamma_{sq}(G) = 2^s\overline{[2^{r-s}K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-s-1})K_2]} \cup (2^r - 2^s)\overline{[\frac{|G|}{2^{r+1}} K_2]}$ when $m_i = 1$ for $1 \leq i \leq s$, $m_i \geq 2$ for $s+1 \leq i \leq r$ and $G \ncong \mathbb{Z}_2^s \times \mathbb{Z}_4$. In this case we have two elements from each of 2^s components and two elements from each of $2^r - 2^s$ components forming maximal independent set. So we have $2 \times (2^r - 2^s) + 2 \times 2^s =$

$2^{r+1} - 2^{s+1} + 2^{s+1} = 2^{r+1}$ elements in maximal independent set. Hence $\beta(\Gamma_{sq}(G)) = 2^{r+1}$. Hence the required result. □

Corollary 3.7. *Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; and $p_i \neq 2$. Then independent number $\beta(\Gamma_{sq}(G)) = 2$.*

Proof. Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; and $p_i \neq 2$. Then by using Theorem 3.2 we get $\Gamma_{sq}(G) \cong \Gamma_{sq}(\mathbb{Z}_m)$ where $m = \prod_{i=1}^{n} p_i^{m_i}$ and so $gcd(m, 2) = 1$. Now by using Theorem 3.6(i) we get $\beta(\Gamma_{sq}(G)) = 2$. □

Theorem 3.8. *(i) Let $G \cong \mathbb{Z}_n$, $n \geq 2$ then we have clique number*

$$\omega(\Gamma_{sq}(G)) = \begin{cases} \frac{n+1}{2} \text{ when } gcd(n,2) = 1 \\ \frac{n+2}{4} \text{ when } n = 2n_1 \\ \frac{n+4}{4} \text{ when } n = 4n_2 \end{cases}$$

where n_1 is odd natural number and $n_2 \in \mathbb{N}$.
(ii) Let G be a finite abelian group with identity e and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_s) \times \cdots \times G(p_r) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; $p_i = 2$ for $1 \leq i \leq r$ and $p_i \neq 2$ for $(r+1) \leq i \leq n$. Then

$$\omega(\Gamma_{sq}(G)) = \begin{cases} \frac{|G|}{2^{r+2}} + \frac{3}{4} \text{ when } m_i = 1 \text{ for } 1 \leq i \leq r \text{ and } n \neq r \\ 1 \text{ when } m_i = 1 \text{ and } n = r \\ \frac{|G|}{2^{r+2}} + \frac{3}{4} \times 2^r \text{ when } m_i \geq 2 \text{ for } 1 \leq i \leq r \\ \frac{|G|}{2^{r+2}} + \frac{3}{4} \times 2^{r-s} \text{ when} \\ m_i = 1 \text{ for } 1 \leq i \leq s, m_i \geq 2 \text{ for } s+1 \leq i \leq r \text{ and } G \ncong \mathbb{Z}_2^s \times \mathbb{Z}_4 \\ 2 \text{ when } G \cong \mathbb{Z}_2^s \times \mathbb{Z}_4 \end{cases}$$

Proof. **(i) Case 1.** Let $G \cong \mathbb{Z}_n$, $n \geq 2$ and $gcd(n, 2) = 1$. By Theorem 3.1 we have any $a \in G$ is adjacent with all other vertices in $\Gamma_{sq}(G)$ whenever $a = a^{-1}$ and whenever $a \neq a^{-1}$ we have a adjacent with all other vertices except a^{-1}. But we have only identity element e in G which is self inverse, so sub-graph formed by $C = \{e\} \cup \{a \text{ or } a^{-1} : a \neq a^{-1}, a \in G\}$ is a maximal complete sub-graph of $\Gamma_{sq}(G)$ and $|C| = 1 + \frac{n-1}{2} = \frac{n+1}{2}$. Hence $\omega(\Gamma_{sq}(G)) = \frac{n+1}{2}$ when $gcd(n, 2) = 1$.
Case 2. Let $G \cong \mathbb{Z}_n$, $n \geq 2$ and $n = 2n_1$ where n_1 is odd natural number. Then we have two disjoint components in $\Gamma_{sq}(G)$, one of them with vertex set $G_1 = \{2a : a \in G\}$ and another component with vertex set $G \setminus G_1$. Also we have only identity element e in G_1, which is self inverse and only one element n_1 in $G \setminus G_1$ which is self inverse. So sub-graph formed by $C = \{e\} \cup \{a \text{ or } a^{-1} : a \neq a^{-1}, a \in G_1\}$ is a maximal complete sub-graph of $\Gamma_{sq}(G)$ and $|C| = 1 + \frac{\frac{n}{2}-1}{2} = \frac{n+2}{4}$. Hence $\omega(\Gamma_{sq}(G)) = \frac{n+2}{4}$ when $n = 2n_1$ where n_1 is odd natural number.
Case 3. Let $G \cong \mathbb{Z}_n$, $n \geq 2$ and $n = 4n_2$ where $n_2 \in \mathbb{N}$. Then we have two disjoint components of $\Gamma_{sq}(G)$, one of them with vertex set $G_1 = \{2a : a \in G\}$ and another component with vertex set $G \setminus G_1$. We have only two elements $e, 2n_2 \in G$ which are self inverse, also $e, 2n_2 \in G_2$. So we have sub-graph formed by $C = \{e, 2n_2\} \cup \{a \text{ or } a^{-1} : a \neq a^{-1}, a \in G_1\}$ is a maximal complete sub-graph of $\Gamma_{sq}(G)$ and $|C| = 2 + \frac{\frac{n}{2}-2}{2} = \frac{n+4}{4}$. Hence $\omega(\Gamma_{sq}(G)) = \frac{n+4}{4}$ when $n = 4n_2$ where $n_2 \in \mathbb{N}$.
(ii) Case 1. When $m_i = 1$ for $1 \leq i \leq r$ and $n \neq r$ then from Theorem 3.3(iii)(a) we have $\Gamma_{sq}(G) = 2^r[K_1 \cup (\frac{|G|-2^r}{2^{r+1}})K_2]$. So all the components are of same structure. If we remove one of every pair of vertices a, a^{-1} in any component the we will get the required clique with maximum number of vertices in $\Gamma_{sq}(G)$. So we have $1 + \frac{|G|-2^r}{2^{r+2}} = \frac{|G|}{2^{r+2}} + \frac{3}{4}$ number of vertices in clique with maximum vertices in $\Gamma_{sq}(G)$. Hence $\omega(\Gamma_{sq}(G)) = \frac{|G|}{2^{r+2}} + \frac{3}{4}$.
Case 2. When $m_i = 1$ and $n = r$ then from Theorem 3.3(iii)(b) we have $\Gamma_{sq}(G) = 2^n K_1$. Hence $\omega(\Gamma_{sq}(G)) = 1$.
Case 3. When $m_i \geq 2$ for $1 \leq i \leq r$ then from Theorem 3.3(iii)(c) we have
$\Gamma_{sq} = [2^r K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-1})K_2] \cup (2^r - 1)[\frac{|G|}{2^{r+1}} K_2]$, In this case we have two types of components in $\Gamma_{sq}(G)$. From one type of component we get maximal complete sub-graph with

$2^r + \frac{|G|}{2^{r+2}} - 2^{r-2} = \frac{|G|}{2^{r+2}} + \frac{3}{4} \times 2^r$ number of vertices and from another type we get maximal complete sub-graph with $\frac{|G|}{2^{r+2}}$. Hence $\omega(\Gamma_{sq}(G)) = \frac{|G|}{2^{r+2}} + \frac{3}{4} \times 2^r$.

Case 4. When $m_i = 1$ for $1 \leq i \leq s$, $m_i \geq 2$ for $s+1 \leq i \leq r$ and $G \ncong \mathbb{Z}_2^s \times \mathbb{Z}_4$ then from Theorem 3.3(iii)(d) we have $\Gamma_{sq}(G) = 2^s\overline{[2^{r-s}K_1 \cup (\frac{|G|}{2^{r+1}} - 2^{r-s-1})K_2]} \cup (2^r - 2^s)\overline{[\frac{|G|}{2^{r+1}}K_2]}$. In this case we have two types of components in $\Gamma_{sq}(G)$. From one type of components we get maximal complete sub-graph with $2^{r-s} + \frac{|G|}{2^{r+2}} - 2^{r-s-2} = \frac{|G|}{2^{r+2}} + \frac{3}{4} \times 2^{r-s}$ number of vertices and where as from another type of components we get maximal complete sub-graph with $\frac{|G|}{2^{r+2}}$ number of vertices. Hence $\omega(\Gamma_{sq}(G)) = \frac{|G|}{2^{r+2}} + \frac{3}{4} \times 2^{r-s}$.

Case 5. When $G \cong \mathbb{Z}_2^s \times \mathbb{Z}_4$ then from Theorem 3.3(iii)(e) we have $\Gamma_{sq}(G) = 2^s[K_2] \cup 2^{s+1}[K_1]$. In this case we have only K_1 and K_2 type of components in $\Gamma_{sq}(G)$. Hence $\omega(\Gamma_{sq}(G)) = 2$.

□

Corollary 3.9. *Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in N$ where $G(p_i) = Z_{p_i^{m_i}}$, $m_i \in N$; $p_i \neq 2$. Then the clique number $\omega(\Gamma_{sq}(G)) = \frac{\prod_{i=1}^n p_i^{m_i}+1}{2}$.*

Proof. Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; $p_i \neq 2$. By using Theorem 3.2 we have $\Gamma_{sq}(G) \cong \Gamma_{sq}(\mathbb{Z}_m)$ where $m = \prod_{i=1}^n p_i^{m_i}$ and so $gcd(m, 2) = 1$. Now by using Theorem 3.8(i) we have $\omega(\Gamma_{sq}(G)) = \frac{m+1}{2} = \frac{\prod_{i=1}^n p_i^{m_i}+1}{2}$. □

Theorem 3.10. *Let G be any finite abelian group the $\Gamma_{sq}(G)$ is not Eulerian.*

Proof. **Case 1.** If $|G|$ is not divisible by 2 then by using Theorem 3.1 we have $deg_{\Gamma_{sq}(G)}(a) = |G| - 1$ if $a = a^{-1}$ and $|G| - 2$ otherwise. So we have vertices in $\Gamma_{sq}(G)$ with odd degree. Hence $\Gamma_{sq}(G)$ is not Eulerian.

Case 2. If $|G|$ is divisible by 2 then by Theorem 3.3 we have $\Gamma_{sq}(G)$ is disconnected. Hence $\Gamma_{sq}(G)$ is not Eulerian.

□

Theorem 3.11. *Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ and $p_i \neq 2$ then*
(i)$\Gamma_{sqe}(G)$ is a complete graph and $\Gamma_{sqe}(G) = K_{|G|}$
(ii)$deg_{\Gamma_{sqe}(G)}(a) = |G| - 1$ for every $a \in G$.
(iii)$radius(\Gamma_{sqe}(G)) = 1$ and $diam(\Gamma_{sq}(G)) = 1$
(iv)$\Gamma_{sqe}(G)$ is self-centered.
(v)$gr(\Gamma_{sqe}(G)) = 3$.
(vi)Independent number, $\beta(\Gamma_{sqe}(G)) = 1$.
(vii)Clique number, $\omega(\Gamma_{sqe}(G)) = |G|$.

Proof. **(i)** Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ and $p_i \neq 2$ then we have $G_1 = G$ and so we have every $a \in G$ adjacent with every element of $G \setminus \{a\}$ and so $\Gamma_{sqe}(G)$ is complete and equal to $K_{|G|}$.

(ii) As discussed in above part we have every element $a \in G$ adjacent with every element $b \in G \setminus \{a\}$. So every vertex in $\Gamma_{sqe}(G)$ adjacent with $|G| - 1$ vertices. Hence $deg_{\Gamma_{sqe}(G)}(a) = |G| - 1$ for every $a \in G$.

(iii) From (i) and (ii) we have $d(a, b) = 1$ for every $a, b \in G$ and so $e(a) = 1$ for every $a \in G$. Hence $radius(\Gamma_{sqe}(G)) = 1$ and $diam(\Gamma_{sqe}(G)) = 1$.

(iv) As $\Gamma_{sqe}(G) = K_{|G|}$ so $e_{\Gamma_{sqe}}(a) = 1$ for every $a \in G$. Hence $\Gamma_{sqe}(G)$ is self-centered.

(v) We have $e, a, b \in G$ such that we have cycle $e - a - b - e$ of length 3. So $gr(\Gamma_{sqe}(G)) = 3$.

(vi) As $\Gamma_{sqe}(G) = K_{|G|}$ so we have maximal independent set $\{a\}$ for $a \in G$. Hence independent number, $\beta(\Gamma_{sqe}(G)) = 1$.

(vii) As $\Gamma_{sqe}(G) = K_{|G|}$ so we have Clique number, $\omega(\Gamma_{sqe}(G)) = |G|$. □

Corollary 3.12. *Let $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ and $p_i \neq 2$ then $\Gamma_{sqe}(G) \cong \Gamma_{sqe}(\mathbb{Z}_m)$ where $m = \prod_{i=1}^n p_i^{m_i}$.*

Proof. As $p_i \neq 2$ and $m = \prod_{i=1}^{n} p_i^{m_i}$ so $gcd(2,m) = 1$. Now when $gcd(2,m) = 1$, we have every pair of distinct vertices adjacent in $\Gamma_{sqe}(\mathbb{Z}_m)$. So $\Gamma_{sqe}(\mathbb{Z}_m) = K_m$. Now by using Theorem 3.11(i), we have the required result. □

Theorem 3.13. *Let G be a finite abelian group with identity e and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_r) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; $p_i = 2$ for $1 \leq i \leq r$ and $p_i \neq 2$ for $(r+1) \leq i \leq n$. Then we have*
(i)$\Gamma_{sqe}(G)$ is disconnected.
(ii)Number of components in $\Gamma_{sqe}(G) = 2^r$ and every component of $\Gamma_{sqe}(G)$ is complete.
(iii)$\Gamma_{sqe}(G) = 2^r K_{\frac{|G|}{2^r}}$
(iv)$deg_{\Gamma_{sqe}(G)}(a) = \frac{|G|}{2^r} - 1$.
(v) $gr\big(\Gamma_{sq}(G)\big) = \begin{cases} \infty \text{ if } G \in \{\mathbb{Z}_4, \mathbb{Z}_2^n, \mathbb{Z}_2^n \times \mathbb{Z}_4\} \\ 3 \text{ otherwise} \end{cases}$
(vi)Independent number, $\beta(\Gamma_{sqe}(G)) = 2^r$.
(vii)Clique number, $\omega(\Gamma_{sqe}(G)) = \frac{|G|}{2^r}$.

Theorem 3.14. *Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ then $\Gamma_{sqe}(G)$ is eulerian graph if and only if $p_i \neq 2$.*

Proof. Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ and $\Gamma_{sqe}(G)$ is eulerian then we have $\Gamma_{sqe}(G)$ graph connected and all vertices in it of even degree. As from Theorem 3.11 and Theorem 3.13 we have $\Gamma_{sqe}(G)$ is connected only when $p_i \neq 2$. Hence the result.
Conversely, Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$; $n \in \mathbb{N}$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$ and $p_i \neq 2$. Then by using Theorem 3.11 we have that $\Gamma_{sqe}(G)$ is connected and $deg_{\Gamma_{sqe}(G)}(a) = |G| - 1 \ \forall\ a \in G$. $|G|$ is odd so we have even degree of all vertices in $\Gamma_{sqe}(G)$. So $\Gamma_{sqe}(G)$ is eulerian.
Hence $\Gamma_{sqe}(G)$ is eulerian iff $p_i \neq 2$. □

4 Prime Labeling

In vertex labeling of graph Γ,under specific criteria, we assign integers to all of its vertices. If a graph Γ with n vertices can be labeled with numbers from 1 to n in such a way that every pair of adjacent vertices are labeled with co-prime numbers from 1 to n. Then that labeling is prime labeling and graph Γ is said to be prime graph.

Theorem 4.1. *Let $G \cong \mathbb{Z}_n$ with identity 0 where $n \geq 2$ then*
(i)$\Gamma_{sq}(G)$ is prime graph for $n \in \{2,3,4,5,6,8\}$
(ii) $\Gamma_{sq}(G)$ is not a prime graph otherwise

Proof. Let $G \cong \mathbb{Z}_n$ then for odd values of n we have $\Gamma_{sq}(G)$ connected, in which identity element 0 is adjacent with every $a \in G \setminus \{0\}$ and any element $a \in G \setminus \{0\}$ adjacent with every element of $b \in G \setminus \{a^{-1}\}$. So we have two vertices ($\neq 0$) in $\Gamma_{sq}(G)$ which are not adjacent with each other and adjacent with every other vertex in $\Gamma_{sq}(G)$ for odd values of n.
For even values of $n(\neq 4)$ we have two components of $\Gamma_{sq}(G)$. In each component we have two vertices which are not adjacent with each other and adjacent with every other vertex in that component.
(i) For $n \in \{2,3,4,5,6,8\}$, we can label vertices of $\Gamma_{sq}(G)$ from 1 to n in such a way that numbers labeled to adjacent vertices are co-prime(as shown in Fig. 1). So $\Gamma_{sq}(G)$ is prime graph for $n \in \{2,3,4,5,6,8\}$.

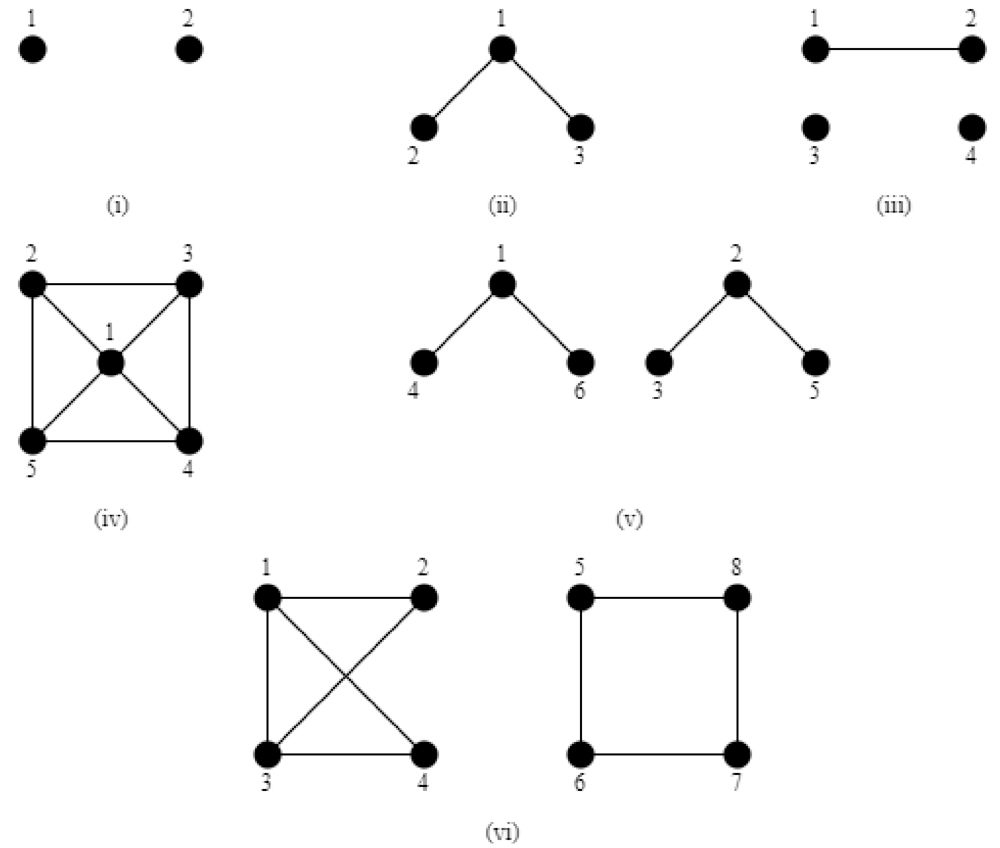

Fig. 1. Prime labeling of (i)$\Gamma_{sq}(\mathbb{Z}_2)$ (ii)$\Gamma_{sq}(\mathbb{Z}_3)$ (iii)$\Gamma_{sq}(\mathbb{Z}_4)$ (iv)$\Gamma_{sq}(\mathbb{Z}_5)$ (v)$\Gamma_{sq}(\mathbb{Z}_6)$ (vi)$\Gamma_{sq}(\mathbb{Z}_8)$

(ii) Case 1. For odd $n \geq 7, \Gamma_{sq}(G)$ is connected. To become $\Gamma_{sq}(G)$ prime graph, its vertices have to be labeled from 1 to n in such a manner that two adjacent vertices are labeled with co-prime numbers. So we have to label three vertices of $\Gamma_{sq}(G)$, which are not adjacent with each other with $2, 4$ and 6 in $\Gamma_{sq}(G)$. But as discussed above we have only two such vertices. So $\Gamma_{sq}(G)$ is not a prime graph for odd $n \geq 7$.
Case 2. For even $n \geq 10, \Gamma_{sq}(G)$ is not connected and have two components and in each component we have two vertices which are not adjacent with each other but adjacent with every other vertex in that component. So we can label maximum four number between 1 to n which are not co-prime with each other in $\Gamma_{sq}(G)$. But between 1 to n for even $n \geq 10$ we have at least five $2, 4, 6, 8, 10$ numbers which are not co-prime. So we can not label $\Gamma_{sq}(G)$ n-vertices in such a way that every pair of adjacent vertices get labeled with co-prime numbers from 1 to n and so $\Gamma_{sq}(G)$ is not a prime graph for even $n \geq 10$.
Hence the required result. □

Corollary 4.2. *Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; and $p_i \neq 2$. Then $\Gamma_{sq}(G)$ is prime graph only when $|G| = 3$ or 5.*

Proof. Let $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; and $p_i \neq 2$. Then by using Theorem 3.2, we have $\Gamma_{sq}(G) \cong \Gamma_{sq}(\mathbb{Z}_m)$ where $m = \prod_{i=1}^{n} p_i^{m_i}$. As $p_i \neq 2$ so we have $gcd(m, 2) \neq 2$. Now by using using Theorem 4.1 we have prime labeling of $\mathbb{Z}_m$ only possible for $m = 3$ or 5 and so $\Gamma_{sq}(G)$ is prime graph only when $|G| = 3$ or 5. □

Theorem 4.3. *(i) If $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_3^m$ where $n \in \mathbb{N}$ and $m \in \{0, 1\}$ then $\Gamma_{sq}(G)$ is prime graph.*
(ii) If $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_4$ where $n \in \mathbb{N}$ then $\Gamma_{sq}(G)$ is prime graph.

Proof. **(i) Case 1.** When $m = 0$, then we have $G \cong \mathbb{Z}_2^n$ and from Theorem 3.3(iii)(b), in $\Gamma_{sq}(G)$ we have 2^n disjoint components, each component with single vertex i.e $\Gamma_{sq}(G) = 2^n K_1$, in

which no pair of vertices is adjacent. Hence $\Gamma_{sq}(G)$ is prime graph.
Case 2. When $m = 1$, then we have $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_3$.
Let us first discuss about structure of $\Gamma_{sq}(G)$, using Theorem 3.3(iii)(a) we have $\Gamma_{sq}(G) = 2^n[\overline{K_1 \cup (\frac{3\times 2^n - 2^n}{2^{n+1}})K_2}] = 2^n[\overline{K_1 \cup K_2}] = 2^n P_3$, in which we have 2^n disjoint P_n as components of $\Gamma_{sq}(G)$. Now for $\Gamma_{sq}(G)$ to be prime graph we have to label its vertices 1 to $2^n\times$ in such a way that every pair of adjacent vertices get labeled with co-prime numbers. As we know two consecutive numbers are co-prime, so we have co-prime numbers $(m, m+1)$ and $(m+1, m+2)$ where $m \in N$. As discussed above we have every component of $\Gamma_{sq}(G)$ of the form P_3 which can be labeled with $m, m+1, m+2$ such that adjacent vertices have co-prime labeling. So we need consecutive three natural numbers for prime labeling of one component of $\Gamma_{sq}(G)$. We have 2^n components in $\Gamma_{sq}(G)$, Hence we can do prime labeling of all components of $\Gamma_{sq}(G)$ using natural numbers 1 to 3×2^n.

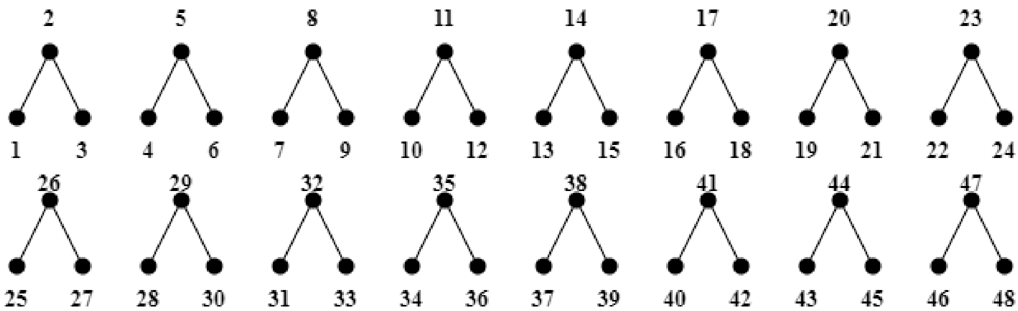

Fig. 2. Prime labeling of $\Gamma_{sq}(\mathbb{Z}_2^4 \times \mathbb{Z}_3)$

(ii) If $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_4$ then by using Theorem 3.3(iii)(e) we have $\Gamma_{sq}(G) = 2^n[K_2 \cup 2^{n+1}[K_1]$, which can be labeled from 1 to 2^{n+2} in such a way that every pair of adjacent vertices get labeled with co-prime numbers. Hence $\Gamma_{sq}(G)$ is prime graph.

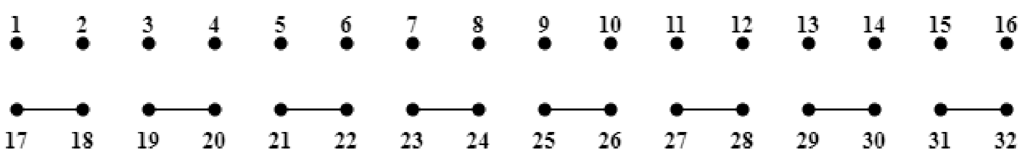

Fig. 3. Prime labeling of $\Gamma_{sq}(\mathbb{Z}_2^3 \times \mathbb{Z}_4)$

□

Theorem 4.4. *Let $G \cong \mathbb{Z}_n$ with identity 0 where $n \geq 2$ then*
(i)$\Gamma_{sqe}(G)$ is prime graph when $n \in \{2, 3, 4\}$.
(ii)$\Gamma_{sqe}(G)$ is not prime graph when $n \geq 5$.

Proof. Let $G \cong \mathbb{Z}_n$ with identity 0 where $n \geq 2$.
(i) When $n \in \{2, 3, 4\}$, $\Gamma_{sqe}(G)$ is prime graph as we have prime labeling shown in Fig. 4.
(ii) For odd $n \geq 5$ we have $\Gamma_{sqe}(G)$ is complete by using Theorem 3.11 and Corollary 3.12, so we have no two distinct vertices in $\Gamma_{sqe}(G)$ which are not adjacent. But we have to label two vertices which are not adjacent with 2, 4 for prime labeling, which is not possible. Hence $\Gamma_{sqe}(G)$ is not prime graph for odd $n \geq 5$. Now for even $n \geq 6$, we have two disjoint components in $\Gamma_{sqe}(G)$ and each component is complete sub-graph of $\Gamma_{sqe}(G)$ with $\frac{|G|}{2}$ number of vertices in each component. Now for prime labeling of $\Gamma_{sqe}(G)$, we at least need three vertices (as even $n \geq 6$) which are not adjacent with each other of $\Gamma_{sqe}(G)$, which can be labeled with 2, 4, 6. We have no such three vertices in $\Gamma_{sqe}(G)$. So $\Gamma_{sqe}(G)$ is not a prime graph for even $n \geq 6$. Hence the required result.

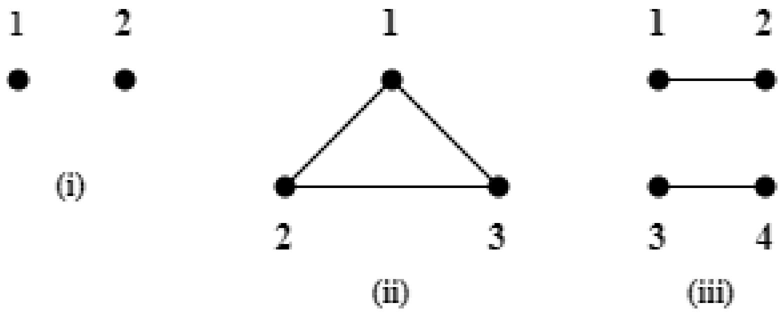

Fig. 4. Prime labeling of (i)$\Gamma_{sqe}(\mathbb{Z}_2)$ (ii)$\Gamma_{sqe}(\mathbb{Z}_3)$ (iii)$\Gamma_{sqe}(\mathbb{Z}_4)$

□

Corollary 4.5. *Let G be a finite abelian group and $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$ where $G(p_i) \cong \mathbb{Z}_{p_i^{m_i}}$, $m_i \in \mathbb{N}$; and $p_i \neq 2$. Then $\Gamma_{sq}(G)$ is prime graph only when $|G| = 3$.*

Proof. By using Theorem 4.4 and Corollary 3.12, we have the required result. □

References

[1] P. J. Cameron, P. Manna, and R. Mehatari, *On finite groups whose power graph is a cograph*, Journal of Algebra, **591**, 59-74 (2022).

[2] P. J. Cameron, *Graphs defined on groups*, International Journal of Group Theory, **11(2)**, 53-107 (2022).

[3] A. Sehgal and S. N. Singh, *The degree of a vertex in the power graph of a finite abelian group*, Southeast Asian Bulletin of Mathematics, **47**, 289–296 (2023).

[4] P. Rana, A. Siwach, A. Sehgal and P. Bhatia, *The degree of a vertex in the kth-power graph of a finite abelian group*, AIP Conference Proceedings, **2782(1)**, 020078 (2023).

[5] A. Sehgal, Manjeet and D. Singh, *Co-prime order graphs of finite Abelian groups and dihedral groups*, Journal of Mathematics and Computer Science, **23(3)**, 196-202 (2021).

[6] A. Sehgal, N. Takshak, P. Maan, and A. Malik, *Graceful labeling of power graph of group $Z_2^{k-1} \times Z_4$*, Asian-European Journal of Mathematics, **15(7)**, 2250121 (2021).

[7] J. A. Gallian, *A dynamic survey of graph labeling*, J. Combin., **5**, 43 (1998).

[8] A. V. Kanetkar, *Prime Labeling of Grids*, AKCE International Journal of Graphs and Combinatorics, **6(1)**, 135-142 (2009).

[9] D. Beutner and H. Harborth, *Graceful labelings of nearly complete graphs*, Results in Mathematics, **41(1)**, 34-39 (2002).

[10] D. Sinha and D. Sharma, *Structural properties of absorption Cayley graphs*, Appl. Math, **10(6)**, 2237-2245 (2016).

[11] S. K. Vaidya and G. K. Rathod, *On General Randic Energy of Graphs*, Palestine Journal of Mathematics, **12-SI(2)** (2023), 49-57.

[12] L. Boro, M. M. Singh, and J. Goswami, *On the line graphs associated to the unit graphs of rings*, Palestine Journal of Mathematics, **11(4)**, 139-145 (2022).

[13] B. Biswas, R. S. Gupta, M. K. Sen, and S. Kar, *Some properties of square element graphs over semigroups*, AKCE International Journal of Graphs and Combinatorics, **17(1)**, 118-130 (2020).

[14] P. Sharma and S. Dutta, *On Idempotent Graph of Rings*, Palestine Journal of Mathematics, **12(1)**, 883-891 (2023).

[15] A. Siwach, P. Rana, A. Sehgal and V. Bhatia, *The square power graph of Zn and Zm2×Z2n group*, AIP Conference Proceedings, **2782(1)**, 020099 (2023).

[16] R. R. Prathap and T. T. Chelvam, *Complement graph of the square graph of finite abelian groups*, Houston Journal of Mathematics, **46(4)**, 845-857 (2020).

[17] R. R. Prathap and T. T. Chelvam, *The cubic power graph of finite abelian groups*, AKCE International Journal of Graphs and Combinatorics, **18(1)**, 16-24 (2021).

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