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CHROMATIC NUMBER OF 2-STRONG PRODUCT GRAPH

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Abstract Cartesian product, tensor product, and strong product are well-known product graphs, which have been studied in detail. Some of these products are generalized by defining 2-Cartesian product, 2-tensor product, and distance product. Additionally, certain basic graph parameters of these products have been studied. Recently, a new graph product, 2-strong product graph has been defined, and the connectedness of this graph has been discussed. In this paper, we studied chromatic number and clique number of 2-strong product graph.

1 Introduction

The Cartesian product, tensor product, and strong product are well-known product graphs that have been studied in detail [8]. In [2], [3] & [13], some of these products have been generalized by defining the 2-Cartesian product, 2-tensor product, and distance product. Graph parameters such as connectedness, bipartiteness, independence number, etc., have been discussed for these 2-Cartesian product, 2-tensor product, and distance product graphs ([1], [2], [3], [13]). Recently, in [14], the 2-strong product graph was introduced and its connectedness was discussed. It is natural to study some more graph parameters of 2-strong product graph. We will focus on chromatic number and clique number and discuss both these concepts for 2-strong product graph.

The chromatic number of a graph has been generalized in many different ways in the literature ([10],[11]). In fact, for the usual strong product graph, there is no result that gives the chromatic number of strong product graph in terms of the chromatic number of factor graphs. So, it is natural to obtain a lower bound and upper bound for it and that work has been done in [12] and [19]. Further, exact distance-p graph and chromatic number of exact distance-p graph has been well studied in literature ([5], [6], [9], [15], [16], [17]). We used this concept with $p = 2$ to obtain a lower bound and upper bound for the chromatic number of the 2-strong product graph. Also, we give a condition under which the chromatic number of the 2-strong product graph can be expressed in terms of the chromatic number of exact distance-2 graph of its factor graphs. Furthermore, we determine the exact chromatic number of the 2-strong product graph for certain non-bipartite graphs.

The clique number of exact distance-p graphs has also been studied in [7] and [15]. Taking this for $p = 2$, we derive the clique number of 2-strong product graph. Utilizing this concept, we also establish a lower bound for the chromatic number of the 2-strong product graph.

Throughout this work, we symbolize a cycle with n vertices as C_n, a complete bipartite graph as $K_{m,n}$, and a path with n vertices as P_n. A graph is considered connected if every pair of vertices is connected by a path. The distance $d_G(x, x')$ between two vertices x and x' in graph G is defined as the length of the shortest path between them.

We limit our discussion to finite graphs that are simple and connected. For other basic graph definitions, we refer to [4] and [20].

2 Bounds for $\chi(G \boxtimes_2 H)$

In [5], $\chi(G, p)$ is defined as the minimum number of colors in a vertex coloring of G such that the colors of two vertices x and x' are different, in the case where $d_G(x, x') = p$. Equivalently, in [15], the same concept is defined as the chromatic number of the exact distance-p graph of G, i.e., $\chi(G^{[\natural p]}) = \chi(G, p)$.

In the literature, k-distance coloring function f on $V(G)$ is also defined, where $f(x) \neq f(x')$ if $d(x, x') \leq k$ for two vertices x and x' [18]. In [6], exact distance-p n coloring of G is defined as a map $f : V(G) \to X$, with $|X| = n$ such that for two vertices x & x', $f(x) \neq f(x')$, provided that $d_G(x, x') = p$. The exact distance-p chromatic number, symbolized as $\chi^{[\natural p]}(G)$, is defined as the smallest of these n, and this concept is discussed in the context of planar graphs.

We consider this concept for $p = 2$ and denote the exact distance-2 chromatic number as $\chi_2(G)$, i.e., $\chi_2(G) = \chi^{[\natural 2]}(G) = \chi(G, 2)$. Note that when $p = 1$, this corresponds to the usual chromatic number $\chi(G)$.

In general, there is no direct relation between $\chi(G)$ and $\chi_2(G)$.

Example 2.1. (i) Let $G = C_{2m+1}$. Then $\chi(G) = \chi_2(G) = 3$.

(ii) For the graph G in Figure 1, $\chi(G) = 3$ and $\chi_2(G) = 2$. Thus $\chi(G) > \chi_2(G)$.

(iii) For the graph H in Figure 1, $\chi(H) = 3$ and $\chi_2(H) = 7$. Thus, $\chi(H) < \chi_2(H)$.

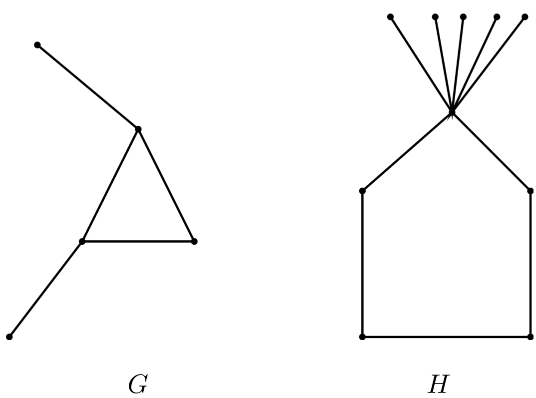

Figure 1.

Definition 2.2. [14] The 2-strong product graph, $G \boxtimes_2 H$ of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is defined as a graph with vertex set $V(G \boxtimes_2 H) = V(G) \times V(H)$ and two vertices (x, y) & (x', y') are adjacent if $d_G(x, x') \in \{0, 2\}$ & $d_H(y, y') \in \{0, 2\}$.

In this section, we will discuss the chromatic number of $G \boxtimes_2 H$. Note that, in the case of the usual strong product $G \boxtimes H$, the value of $\chi(G \boxtimes H)$ cannot be determined in terms of $\chi(G)$ and $\chi(H)$. Therefore, here as well, we establish upper and lower bounds for $\chi(G \boxtimes_2 H)$ using the concepts of $\chi_2(G)$ and $\chi_2(H)$.

Theorem 2.3. *For two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, $\chi(G \boxtimes_2 H) \leq \chi_2(G)\chi_2(H)$.*

Proof. Let $f_G : V(G) \to X$ be an exact distance-2 $\chi_2(G)$-coloring of G and $f_H : V(H) \to Y$ be an exact distance-2 $\chi_2(H)$-coloring of H with $|X| = \chi_2(G)$ and $|Y| = \chi_2(H)$. Define $f : V(G \boxtimes_2 H) \to X \times Y$ by $f(x, y) = (f_G(x), f_H(y))$. We prove that f is a $\chi_2(G)\chi_2(H)$-coloring of $G \boxtimes_2 H$. Let $(x, y)(x', y') \in E(G \boxtimes_2 H)$, i.e., $d_{G\boxtimes_2 H}((x, y)(x', y')) = 1$.

First, suppose $d_G(x, x') = 2 = d_H(y, y')$. Then, as f_G and f_H are exact distance-2 $\chi_2(G)$ and exact distance-2 $\chi_2(H)$ coloring of G and H respectively, $f_G(x) \neq f_G(x')$ and $f_H(y) \neq f_H(y')$. Therefore, $(f_G(x), f_H(y)) \neq (f_G(x'), f_H(y'))$. If $d_G(x, x') = 2$ and $d_H(y, y') = 0$, then we get $f_G(x) \neq f_G(x')$ and so, $f(x, y) \neq f(x', y)$. Similarly, for $d_G(x, x') = 0$ and $d_H(y, y') = 2$, we get the result. Thus, $\chi(G \boxtimes_2 H) \leq \chi_2(G)\chi_2(H)$. □

Theorem 2.4. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs with $N^2(x) \neq \emptyset$ for some $x \in V(G)$ and $N^2(y) \neq \emptyset$ for some $y \in V(H)$, where $N^2(x) = \{x' \in V(G) : d_G(x, x') = 2\}$ and $N^2(y) = \{y' \in V(H) : d_H(y, y') = 2\}$. Then, $\chi(G \boxtimes_2 H) \geq \max\{\chi_2(G), \chi_2(H)\} + 2$.*

Proof. Let us, without loss of generality assume that $\max\{\chi_2(G), \chi_2(H)\} = \chi_2(G) = p$. Suppose that $\chi(G \boxtimes_2 H) \leq p + 1$. Then, there exists a function $f : V(G \boxtimes_2 H) \to X = \{1, 2, 3, \ldots, p + 1\}$ with $f(x, y) \neq f(x', y')$ whenever $(x, y)(x', y') \in E(G \boxtimes_2 H)$. Now, let $y \in V(H)$ be such that $N^2(y) \neq \emptyset$. Then, there exists $y' \in V(H)$ such that, $d_H(y, y') = 2$. Define $f_G : V(G) \to X' = \{1, 2, 3, \ldots, p - 1\}$ by

$$f_G(x) = \begin{cases} \min\{f(x, y), f(x, y')\}, & \text{if} \quad \min\{f(x, y), f(x, y')\} < p \\ p - 1, & \text{if} \quad \min\{f(x, y), f(x, y')\} = p. \end{cases}$$

Note that $f(x, y) = f(x, y') = p + 1$, cannot be possible as (x, y) and (x, y') forms an edge in $G \boxtimes_2 H$ as $d_H(y, y') = 2$. Thus, $\min\{f(x, y), f(x, y')\} \neq p + 1$.

We prove that f_G is exact distance-2 $|f_G(V(G))|$ coloring of G, i.e., $f_G(x) \neq f_G(x')$ if $d_G(x, x') = 2$. Let $x, x' \in V(G)$ with $d_G(x, x') = 2$. Then, $(x, y), (x, y'), (x', y)$, and (x', y') vertices in $G \boxtimes_2 H$ forms a complete graph K_4. Thus, $f(x, y), f(x, y'), f(x', y)$ & $f(x', y')$ are all distinct. So, $\min\{f(x, y), f(x, y')\}$ and $\min\{f(x', y), f(x', y')\}$ are different. Hence, if $\min\{f(x, y), f(x, y')\} < p$ and $\min\{f(x', y), f(x', y')\} < p$, then $f_G(x) \neq f_G(x')$. Now, suppose $\min\{f(x, y), f(x, y')\} = p$. Then $\{f(x, y), f(x, y')\} = \{p, p + 1\}$ and so the maximum value of $f(x', y)$ and $f(x', y')$ is $p - 1$. Thus, $f_G(x') = \min\{f(x', y), f(x', y')\} \leq p - 2 < p - 1 = f_G(x)$. Therefore, $f_G(x) \neq f_G(x')$. Similarly, if $\min\{f(x', y), f(x', y')\} = p$, then we get $f_G(x) < f_G(x')$ and so $f_G(x) \neq f_G(x')$. Therefore, $\chi_2(G) \leq |f_G(V(G))| \leq p - 1 < p = \chi_2(G)$, which is a contradiction. Hence, $\chi(G \boxtimes_2 H) \geq p + 2 = \max\{\chi_2(G), \chi_2(H)\} + 2$. □

Note that, for usual strong product, similar results are known [19].

Example 2.5. (i) Let G and H both be graph G in Figure 1. Then, $\max\{\chi_2(G), \chi_2(H)\} + 2 = 4$, $\chi(G \boxtimes_2 H) \leq \chi_2(G)\chi_2(H) = 4$. Thus, $\max\{\chi_2(G), \chi_2(H)\} + 2 = \chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$.

(ii) Let $G = H = C_5$. Then, $G \boxtimes_2 H = G \boxtimes H$ and $\chi(G \boxtimes H) = 5$ [19]. So, $\chi(G \boxtimes_2 H) = 5$ and $\max\{\chi_2(G), \chi_2(H)\} + 2 = 5$. Also $\chi_2(G)\chi_2(H) = 9$. Thus, $\max\{\chi_2(G), \chi_2(H)\} + 2 = \chi(G \boxtimes_2 H) < \chi_2(G)\chi_2(H)$.

(iii) Let G and H both be graph H in Figure 1. Then $\chi_2(G) = \chi_2(H) = 7$ and hence $\max\{\chi_2(G), \chi_2(H)\} + 2 = 9$ and $\chi_2(G)\chi_2(H) = 49$. Note that, $K_{49} = K_7 \boxtimes K_7 \subset G \boxtimes_2 H$. Thus $\chi(G \boxtimes_2 H) \geq 49$ and so, $\chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$. Therefore, $\max\{\chi_2(G), \chi_2(H)\} + 2 < \chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$.

3 $\omega(G \boxtimes_2 H)$ & $\chi(G \boxtimes_2 H)$

We recall that for a graph $G = (V(G), E(G))$, $W \subset V(G)$ is defined as a clique if $W = V(H)$, where H is a complete subgraph of G. The clique number, denoted as $\omega(G)$, represents the number of vertices in the largest clique within graph G [4]. Similarly, a 2-clique is defined as a subset W of $V(G)$ such that $d_G(x, x') = 2$ for each distinct x, x' in W. The 2-clique number is defined as the number of vertices in the largest 2-clique. This concept is also defined in [7] as the clique number $\omega(G^{[\natural 2]})$ of the exact distance-2 graph. We denote this quantity as $\omega_2(G)$.

In this section, we determine the value of $\omega(G \boxtimes_2 H)$. Additionally, we present a condition under which equality is achieved in Theorem 2.3.

Example 3.1. (i) For $G = P_n$, $(n \geq 3)$ or C_n, $(n \geq 4, n \neq 6)$, $\omega_2(G) = 2 = \omega(G)$.

(ii) For $G = K_{m,n}$, $(m, n > 2)$, $\omega_2(G) = \max\{m, n\} > 2 = \omega(G)$.

(iii) For G shown in Figure 2, $\omega_2(G) = 2 < 4 = \omega(G)$.

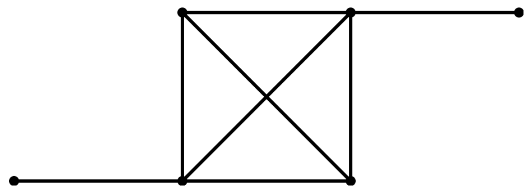

Figure 2.

Remark 3.2. For two graphs G and H, if $N^2(x) \neq \emptyset$ and $N^2(y) \neq \emptyset$ for some $x \in V(G)$ and $y \in V(H)$, then $\omega(G \boxtimes_2 H) \geq 4$.

Now, we obtain clique number of $G \boxtimes_2 H$ in terms of $\omega_2(G)$ and $\omega_2(H)$.

Proposition 3.3. *For two graphs G and H, $\omega(G \boxtimes_2 H) = \omega_2(G)\omega_2(H)$.*

Proof. Let W_1 and W_2 be two 2-cliques in G and H respectively. Then $W_1 \times W_2$ is a clique in $G \boxtimes_2 H$. Thus, $\omega(G \boxtimes_2 H) \geq \omega_2(G)\omega_2(H)$. Now let Q be a clique in $G \boxtimes_2 H$. If $P_G(Q) = \{x : (x, y) \in Q\}$ and $P_H(Q) = \{y : (x, y) \in Q\}$, then $P_G(Q)$ is a 2-clique in G, because if x, x', $(x \neq x')$ are in $P_G(Q)$, then $(x, y), (x', y') \in Q$, which is a clique in $G \boxtimes_2 H$ and so $d_G(x, x') = 2$. Similarly, $P_H(Q)$ is a 2-clique in H. Consequently, $\omega(G \boxtimes_2 H) \leq \omega_2(G)\omega_2(H)$. □

Corollary 3.4. *Maximal cliques in $G \boxtimes_2 H$ are of the form $W = W_G \times W_H$, where W_G is maximal 2-clique in G and W_H is maximal 2-clique in H.*

Proof. Using Proposition 3.3, we get the result. □

Corollary 3.5. *Let G and H be two graphs. If $\chi_2(G) = \omega_2(G)$ & $\chi_2(H) = \omega_2(H)$, then $\omega(G \boxtimes_2 H) = \chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$.*

Proof. As $\omega(G \boxtimes_2 H) \leq \chi(G \boxtimes_2 H)$, from Theorem 3.3 & Theorem 2.3, it follows that, $\omega(G \boxtimes_2 H) = \chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$. □

Using 2-clique number of graphs, we obtain an additional lower bound for $\chi(G \boxtimes_2 H)$.

Theorem 3.6. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs with $N^2(x) \neq \emptyset$ for some $x \in V(G)$ and $N^2(y) \neq \emptyset$ for some $y \in V(H)$. Then,*

$$\chi(G \boxtimes_2 H) \geq \max\{\chi_2(G) + 2\omega_2(H) - 2, \chi_2(H) + 2\omega_2(G) - 2\}.$$

Proof. Let $W = \{y_1, y_2, \ldots, y_n\}$ be a 2-clique in H with $n = \omega_2(H)$. Note that, $\omega_2(G) \geq 2$ and hence, by Proposition 3.3, $\chi(G \boxtimes_2 H) \geq 2n$. Let $\chi(G \boxtimes_2 H) = 2n + \epsilon$ and let $f : V(G \boxtimes_2 H) \to \{1, 2, \ldots, 2n + \epsilon\}$ be a proper coloring of $G \boxtimes_2 H$.

For $x \in V(G)$, let $m_x = \min\{f(x, y_i) : 1 \leq i \leq n\}$. Then, $m_x \leq n + \epsilon + 1$. For, if $m_x \geq n + \epsilon + 2$, then $f(x, y_i) \geq n + \epsilon + 2$, for every i. But, as $(x, y_1), (x, y_2), \ldots, (x, y_n)$ forms a clique in $G \boxtimes_2 H$, $f(x, y_i)$ are all distinct. Thus, in order to color $(x, y_1), \ldots, (x, y_n)$, we need n colors larger than $n + \epsilon + 1$. But, there are only $n - 1$ colors, $n + \epsilon + 2, n + \epsilon + 3, \ldots, n + \epsilon + n$ to color n vertices, which is not possible. Thus $m_x \leq n + \epsilon + 1$.

Define $g : V(G) \to \{1, 2, \ldots, \epsilon + 2\}$ by $g(x) = \begin{cases} m_x, & \text{if } m_x \leq \epsilon + 1 \\ \epsilon + 2, & \text{if } \epsilon + 2 \leq m_x \leq n + \epsilon + 1. \end{cases}$

Claim: g is exact 2-distance coloring, i.e., if $x, x' \in V(G)$ with $d_G(x, x') = 2$, then $g(x) \neq g(x')$.

Let $x, x' \in V(G)$ with $d_G(x, x') = 2$. Note that, $\{(x, y_i), (x', y_i) : 1 \leq i \leq n\}$ forms a clique in $G \boxtimes_2 H$ and so all $f(x, y_i)$ and $f(x', y_i)$ are distinct. Thus, $g(x) \neq g(x')$, if $m_x \leq \epsilon + 1$ or $m_{x'} \leq \epsilon + 1$. Suppose, m_x, $m_{x'} \geq \epsilon + 2$, then $f(x, y_i) \geq \epsilon + 2$ and $f(x', y_i) \geq \epsilon + 2$, for every i.

Therefore, colors of $2n$ vertices $(x, y_1), \ldots, (x, y_n)$, $(x', y_1), \ldots (x', y_n)$ are larger than $\epsilon + 1$. But, this is not possible as these $2n$ vertices forms clique and there are only $2n - 1$ colors $\epsilon + 2, \epsilon + 3, \ldots, \epsilon + 2n$ larger than $\epsilon + 1$. Thus, $g(x) \neq g(x')$.

Therefore, $\chi_2(G) \leq \epsilon + 2 = \chi(G \boxtimes_2 H) - 2n + 2 = \chi(G \boxtimes_2 H) - 2\omega_2(H) + 2$. Therefore, $\chi(G \boxtimes_2 H) \geq \chi_2(G) + 2\omega_2(H) - 2$. □

Remark 3.7. Since $\omega_2(G) \geq 2$ and $\omega_2(H) \geq 2$, we have $\chi_2(G) + 2\omega_2(H) - 2 \geq \chi_2(G) + 2$. Consequently, Theorem 3.6 provides a sharper lower bound for $\chi(G \boxtimes_2 H)$ compared to the bound in Theorem 2.4.

4 $\chi(G \boxtimes_2 H)$ for some non-bipartite graphs

In this section, we determine the chromatic number of the 2-strong product graph for certain non-bipartite graphs, such as the wheel graph W_n, the Helm graph H_n, and the closed Helm graph CH_n.

Definition 4.1. [14]

(i) A wheel graph W_n is a graph obtained from star graph $K_{1,n}$ by joining all pendent edges by a cycle.

(ii) The Helm graph H_n is a graph with $2n+1$ vertices, which is obtained from W_n by attaching one pendant edge each to every vertex on the cycle of W_n.

(iii) Closed Helm graph CH_n is obtained from H_n by joining all pendent edges in it by a cycle.

Note that for graph $G = W_m$, H_m or CH_m, we get that,

$$\chi_2(G) = \begin{cases} n+1, & \text{if} \quad m = 2n+1 \ (n \geq 4) \\ n, & \text{if} \quad m = 2n \ (n \geq 5), \end{cases} \quad \& \quad \omega_2(G) = \begin{cases} n, & \text{if} \quad m = 2n+1 \ (n \geq 3) \\ n, & \text{if} \quad m = 2n \ (n \geq 4). \end{cases}$$

If G and H are W_m, H_m or CH_m with m-even, then using Corollary 3.5, we obtain $\chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$. We will now establish that this equality holds for odd values of m, i.e., $\chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$ for odd m as well.

Theorem 4.2. $\chi(W_{2m+1} \boxtimes_2 W_{2n}) = (m+1)n = \chi_2(W_{2m+1})\chi_2(W_{2n}),\ m, n \in \mathbb{N}$, $m \geq 3$ & $n \geq 4$.

Proof. By Theorem 2.3, $\chi(W_{2m+1} \boxtimes_2 W_{2n}) \leq (m+1)n$. Now, we fix the notations as follows: Let $x_1, x_2, \ldots, x_{2m}$ be the vertices on the cycle of $G = W_{2m}$ and let $\{x_0\}$ be the center vertex. Similarly, let $y_1, y_2, \ldots, y_{2n}$ be vertices on the cycle of $H = W_{2n}$ and $\{y_0\}$ be the center vertex. Let,

$$X_1 = \{(x_i, y_j) : i \in \{1, 3, 5, \ldots, 2m-1\}, j \in \{1, 3, 5, \ldots, 2n-1\}\},$$
$$X_2 = \{(x_i, y_j) : i \in \{1, 3, 5, \ldots, 2m-1\}, j \in \{2, 4, 6, \ldots, 2n\}\},$$
$$X_3 = \{(x_i, y_j) : i \in \{2, 4, 6, \ldots, 2m\}, j \in \{1, 3, 5, \ldots, 2n-1\}\},$$
$$X_4 = \{(x_i, y_j) : i \in \{2, 4, 6, \ldots, 2m\}, j \in \{2, 4, 6, \ldots, 2n\}\}.$$

Let (x_i, y_j) and (x_k, y_l) be two distinct vertices of X_1. Then $d_G(x_i, x_k) = 0$ or 2 and $d_H(y_j, y_l) = 0$ or 2. Thus $(x_i, y_j)(x_k, y_l) \in E(G \boxtimes_2 H)$. Thus X_1 is a clique of size mn in $G \boxtimes_2 H$. Similarly, X_2, X_3, and X_4 each forms a clique of size mn. Let,

$$G_1 = \{(x_{2m+1}, y_j) : j \in \{1, 3, 5, \ldots, 2n-1\}\},$$
$$G_2 = \{(x_{2m+1}, y_j) : j \in \{2, 4, 6, \ldots, 2n\}\}.$$

We show that, a vertex in G_1 or G_2 cannot be given any of mn colors of X_1.
Observe that, $(x_{2m+1}, y_1) \in G_1$ is adjacent to all vertices of clique X_1 except vertices in $A = \{(x_1, y_j) : j \in \{1, 3, 5, \ldots, 2n-1\}\}$ and it is adjacent to all vertices of clique X_3 except vertices in $B = \{(x_{2m}, y_j) : j \in \{1, 3, 5, \ldots, 2n-1\}\}$. Also, $|A| = |B|$ and $A \cup B$ forms a clique in $G \boxtimes_2 H$. Therefore, (x_{2m+1}, y_1) need a new color apart from mn colors of X_1.

But as G_1 forms a clique, all vertices of G_1 needs distinct n colors. Thus, atleast $mn + n$ different colors are required for proper coloring of $G \boxtimes_2 H$. Thus $\chi(G \boxtimes_2 H) \geq n(m+1)$. □

Theorem 4.3. $\chi(W_{2m+1} \boxtimes_2 W_{2n+1}) = (m+1)(n+1) = \chi_2(W_{2m+1})\chi_2(W_{2n+1})$, $m, n \in \mathbb{N}$ & $m, n \geq 3$.

Proof. Let $G = W_{2m+1}$ and $H = W_{2n+1}$. We continue the notations of Theorem 4.2. Let

$$H_1 = \{(x_i, y_{2n+1}) : i \in \{1, 3, 5, \ldots, 2m-1\}\},$$
$$H_2 = \{(x_i, y_{2n+1}) : i \in \{2, 4, 6, \ldots, 2m\}\}$$

and let $p = (x_{2m+1}, y_{2n+1})$. Then, H_1 and H_2 forms clique. We note that, as in Theorem 4.2, vertices in G_1 and G_2 needs color other than that of X_1. By using similar arguments, H_1 and H_2 needs color other than X_1.

First we prove that, the set $G_1 \cup G_2 \cup H_1 \cup H_2$ needs at least $m + n$ new colors.

As G_1 and G_2 forms a clique of size n, at least n colors are required for coloring $G_1 \cup G_2$. Suppose $a_1, a_2, \ldots, a_n$ are different colors of $(x_{2m+1}, y_1), (x_{2m+1}, y_3), \ldots, (x_{2m+1}, y_{2n-1})$ respectively. Then as (x_{2m+1}, y_1) is adjacent to all vertices of G_2, except (x_{2m+1}, y_2), a_1 color can be given to vertex (x_{2m+1}, y_2) only. Similarly a_n color can be given to vertex (x_{2m+1}, y_{2n}) only.

Again, suppose $b_1, b_2, \ldots, b_m$ are the different colors of $(x_1, y_{2n+1}), (x_3, y_{2n+1}), \ldots, (x_{2m-1}, y_{2n+1})$ respectively. Then, by arguments similar to above, (x_2, y_{2n+1}) can be given color b_1 only and (x_{2m}, y_{2n+1}) can be given color b_m only. But then note that $a_2, a_3, \ldots, a_n, b_2, b_3, \ldots, b_m$ must be distinct as $(G_1 \cup H_1) \setminus \{(x_1, y_{2n+1})\}$ form a clique. Similarly, the colors $a_1, a_2, \ldots, a_{n-1}$, $b_1, b_2, \ldots, b_{m-1}$ must be distinct. Finally, we must have $a_1 \neq b_m$ as a_1 color is given to (x_{2m+1}, y_1) &(x_{2m+1}, y_2) and b_m color is given to (x_{2m-1}, y_{2n+1}) and (x_{2m}, y_{2n+1}), but (x_{2m+1}, y_1) & (x_{2m}, y_{2n+1}) are adjacent. Similarly, $b_1 \neq a_n$.

Thus all colors $a_1, a_2, \ldots, a_n, b_1\, b_2 \ldots, b_m$ are distinct. Therefore, $\chi(G \boxtimes_2 H) \geq mn + (m+n)$.

Next, we show that, the vertex $p = (x_{2m+1}, y_{2n+1})$ requires color other than mn colors of X_1.

Note that, $X_1 \cup X_2 \cup X_3 \cup X_4$ can be colored using mn colors. Suppose, $\{p\} \cup X_1 \cup X_2 \cup X_3 \cup X_4$ can be colored using mn colors. Now, the size of the set $\{p\} \cup X_1 \cup X_2 \cup X_3 \cup X_4$ is $4mn + 1$. So, there must be an independent set say $U \subset \{p\} \cup X_1 \cup X_2 \cup X_3 \cup X_4$ of size atleast 5, i.e., $|U| \geq 5$.

Since each of X_i $(1 \leq i \leq 4)$ forms a clique, $p \in U$ and $|U| = 5$. Now as p is adjacent to all vertices in $X_1 \cup X_2 \cup X_3 \cup X_4$ other than $\{(x_i, y_j) : i \in \{1, 2m\}, 1 \leq j \leq 2n\} \cup \{(x_i, y_j) : 1 \leq i \leq m, j \in \{1, 2n\}\}$, we have $U \subset \{(x_i, y_j) : i \in \{1, 2m\}, 1 \leq j \leq 2n\} \cup \{(x_i, y_j) : 1 \leq i \leq 2m, j \in \{1, 2n\}\} \cup \{p\}$. Note that $|U \setminus \{p\}| = 4$. We fix the following notations:

$$J_1 = \{(x_1, y_j) : j \in \{1, 3, 5, \ldots, 2n-1\}\} \cup \{(x_i, y_1) : i \in \{1, 3, 5, \ldots, 2m-1\}\},$$
$$J_2 = \{(x_1, y_j) : j \in \{2, 4, 6, \ldots, 2n\}\} \cup \{(x_i, y_{2n}) : i \in \{1, 3, 5, \ldots, 2m-1\}\},$$
$$J_3 = \{(x_{2n}, y_j) : j \in \{1, 3, 5, \ldots, 2n-1\}\} \cup \{(x_i, y_1) : i \in \{2, 4, 6, \ldots, 2m\}\},$$
$$J_4 = \{(x_{2m}, y_j) : j \in \{2, 4, 6, \ldots, 2n\}\} \cup \{(x_i, y_{2n}) : i \in \{2, 4, 6, \ldots, 2m\}\}.$$

Then, each of J_i forms a clique and $\{(x_i, y_j) : i \in \{1, 2m\}, 1 \leq j \leq 2n\} \cup \{(x_i, y_j) : 1 \leq i \leq 2m, j \in \{1, 2n\}\} = J_1 \cup J_2 \cup J_3 \cup J_4$. Also, $U \setminus \{p\}$ is an independent subset of $J_1 \cup J_2 \cup J_3 \cup J_4$ of size 4.

Now, we show that $J_1 \cup J_2 \cup J_3 \cup J_4$ cannot contain an independent set of size 4. Suppose, vertex of the form (x_1, y_{2j}) from J_2 is in $U \setminus \{p\}$. Then, (x_2, y_{2n}) is the only possible vertex from J_4 which can be in $U \setminus \{p\}$. Also from J_3, (x_{2m}, y_{2n-1}) is the only vertex which is not adjacent to (x_2, y_{2n}) and finally (x_{2m+1}, y_1) is the only vertex from J_1 which is not adjacent to (x_{2m}, y_{2n-1}). Thus, if (x_1, y_{2j}) is in $U \setminus \{p\}$, then $U \setminus \{p\}$ is $\{(x_1, y_{2j}), (x_2, y_{2n}), (x_{2m}, y_{2n-1}), (x_{2m+1}, y_1)\}$, which is not possible as (x_2, y_{2n}) is adjacent to (x_{2m+1}, y_1).

Similarly, vertex of the form (x_{2i-1}, y_{2n}) is in $U \setminus \{p\}$ is also not possible. Thus, $J_1 \cup J_2 \cup J_3 \cup J_4$ cannot contain independent set of size 4. Thus, there cannot be an independent set of size at least 5 in $\{p\} \cup X_1 \cup X_2 \cup X_3 \cup X_4$. Therefore, p cannot be given colors of X_i $(i \leq i \leq 4)$.

Finally, note that p is adjacent to all vertices in $G_1 \cup G_2$ except, $\{(x_{2m+1}, y_1), (x_{2m+1}, y_{2n})\}$. Thus, p cannot be colored using any a_i $(\leq i \leq n)$. Similarly, p cannot be colored using b_i $(1 \leq i \leq m)$, i.e., p requires color other than $mn + (m+n)$. Thus p requires a new color.

Thus, $\chi(G \boxtimes_2 H) \geq mn + m + n + 1 = (m+1)(n+1)$. Thus we have, $\chi(G \boxtimes_2 H) = (m+1)(n+1)$. □

Next, we obtain $\chi(G \boxtimes_2 H)$ for H_n and CH_n.

Corollary 4.4. *(i)* $\chi(H_{2m+1} \boxtimes_2 H_{2n}) = (m+1)n = \chi_2(H_{2m+1})\chi_2(H_{2n}),\ m, n \in \mathbb{N},$ $m \geq 3$ & $n \geq 4$.

(ii) $\chi(H_{2m+1} \boxtimes_2 H_{2n+1}) = (m+1)(n+1) = \chi_2(H_{2m+1})\chi_2(H_{2n+1}),\ m, n \in \mathbb{N}$ & $m, n \geq 3$.

Proof. (i) By Theorem 2.3, $\chi(H_{2m+1} \boxtimes_2 H_{2n}) \leq (m+1)n$. But, as $W_{2m+1} \boxtimes_2 W_{2n}$ is an induced subgraph of $H_{2m+1} \boxtimes_2 H_{2n}$, $\chi(H_{2m+1} \boxtimes_2 H_{2n}) \geq \chi(W_{2m+1} \boxtimes_2 W_{2n}) \geq (m+1)n$. Therefore, $\chi(H_{2m+1} \boxtimes_2 H_{2n}) = (m+1)n = \chi_2(H_{2m+1})\chi_2(H_{2n})$. By similar arguments, (ii) follows. □

Corollary 4.5. $\chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$, *if* $G = CH_m$ *and* $H = CH_n$, *for every* $m, n \in \mathbb{N},$ & $m, n \geq 9$.

Proof. By Theorem 2.3, $\chi(G \boxtimes_2 H) \leq \chi_2(G)\chi_2(H)$. Now, since W_m is induced subgraph of CH_m, $\chi(G \boxtimes_2 H) \geq \chi(W_m \boxtimes_2 W_n) = \chi_2(W_m)\chi_2(W_n) = \chi_2(G)\chi_2(H)$ for $m, n \geq 7$ and therefore $\chi(G \boxtimes_2 H) = \chi_2(G)\chi_2(H)$. □

5 Conclusion remarks

We studied the chromatic number of the 2-strong product graph. We obtained both the upper bound and the lower bound for the chromatic number of 2-strong product graph in terms of the exact distance-2 chromatic number of the graphs G and H. We attempted to characterize the graphs for which we have the exact results. In this direction, we obtained a sufficient condition for equality in the upper bound. Furthermore, we proved that for non-bipartite graphs such as the wheel graph, helm graph, and closed helm graph, equality is achieved in Theorem 2.3. It would also be interesting to discover a sharper lower bound for the chromatic number of the 2-strong product graph.

Also, we obtained the clique number of the 2-strong product graph in terms of the 2-clique number of the factor graphs. Moreover, we derived a lower bound for the chromatic number of 2-strong product graph in terms of 2-clique number and exact distance-2 chromatic number of graphs G and H.

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