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On strong countably McCoy rings

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Abstract The purpose of this paper is to study the strong version of countably $\mathcal{A}$-rings (or the countably McCoy rings) introduced by T. Lucas in [21]. Moreover, we introduce and investigate the module theoretic version of the strongly countably $\mathcal{A}$-ring notion, namely the strongly countably $\mathcal{A}$-modules. We focus especially on the behavior of the stronlgy countably $\mathcal{A}$-property vis-à-vis the polynomial ring, the power series ring, the idealization and the amalgamated duplication of a ring along an ideal.

1 Introduction

Throughout this paper, all rings are supposed to be commutative with unit element and all R-modules are unital. Let R be a commutative ring and M an R-module. We denote by $\mathrm{Z}_R(M) = \{r \in R : rm = 0$ for some nonzero element $m \in M\}$ the set of zero divisors of R on M and by $\mathrm{Z}(R) := \mathrm{Z}_R(R)$ the set of zero divisors of the ring R. In [4], the notions of $\mathcal{A}$-module and $\mathcal{SA}$-module are extensively studied. In fact, an R-module M satisfies Property ($\mathcal{A}$), or M is an $\mathcal{A}$-module over R (or $\mathcal{A}$-module if no confusion is likely), if for every finitely generated ideal I of R with $I \subseteq \mathrm{Z}_R(M)$), there exists a nonzero $m \in M$ with $Im = 0$, or equivalently, $\mathrm{ann}_M(I) \neq 0$. M is said to satisfy strong Property ($\mathcal{A}$), or is an $\mathcal{SA}$-module over R (or an $\mathcal{SA}$-module if no confusion is likely), if for any $r_1, \cdots, r_n \in \mathrm{Z}_R(M)$, there exists a nonzero $m \in M$ such that $r_1 m = \cdots = r_n m = 0$. The ring R is said to satisfy Property ($\mathcal{A}$), or an $\mathcal{A}$-ring, (respectively, $\mathcal{SA}$-ring) if R is an $\mathcal{A}$-module (resp., an $\mathcal{SA}$-module). One may easily check that M is an $\mathcal{SA}$-module if and only if M is an $\mathcal{A}$-module and $\mathrm{Z}_R(M)$ is an ideal of R. It is worthwhile reminding the reader that the Property ($\mathcal{A}$) for commutative rings was introduced by Quentel in [25] who called it Property (C) and Huckaba used the term Property ($\mathcal{A}$) in [18, 19]. In [13], Faith called rings satisfying Property ($\mathcal{A}$) McCoy rings. The Property ($\mathcal{A}$) for modules was introduced by Darani [11] who called such modules F-McCoy modules (for Faith McCoy terminology). He also introduced the strong Property ($\mathcal{A}$) under the name super coprimal and called a module M coprimal if $\mathrm{Z}_R(M)$ is an ideal. In [22], the strong Property ($\mathcal{A}$) for commutative rings was independently introduced by Mahdou and Hassani in [22] and further studied by Dobbs and Shapiro in [12]. Note that a finitely generated module over a Noetherian ring is an $\mathcal{A}$-module (for example, see [20, Theorem 82]) and thus a Noetherian ring is an $\mathcal{A}$-ring. Also, it is well known that a zero-dimensional ring R is an $\mathcal{A}$-ring as well as any ring R whose total quotient ring $Q(R)$ is zero-dimensional. In fact, it is easy to see that R is an $\mathcal{A}$-ring if and only if so is $Q(R)$ [9, Corollary 2.6]. Any polynomial ring $R[X]$ is an $\mathcal{A}$-ring [18] as well as any reduced ring with a finite number of minimal prime ideals [18]. In [7], we generalize a result of T.G. Lucas which states that if R is a reduced commutative ring and M is a flat R-module, then the idealization $R \ltimes M$ is an $\mathcal{A}$-ring if and only if R is an $\mathcal{A}$-ring [21, Proposition 3.5]. In effect, we drop the reduceness hypotheses and prove that, given an arbitrary commutative ring R and any submodule M of a flat R-module F, $R \ltimes M$ is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring) if and only if R is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring). In [8], we present an answer to a problem raised by D.D. Anderson and S. Chun in [4] on characterizing when is the idealization $R \ltimes M$ of a ring R on an R-module M an $\mathcal{A}$-ring (resp., an $\mathcal{SA}$-ring) in terms of module-theoretic properties of R and M. Also, we were concerned with presenting a complete answer to an open question asked by these two authors which reads the following: What modules over a given ring R are homomorphic images of modules satisfying the strong Property $(\mathcal{A})$? [4, Question 4.4 (1)]. The main theorem of [9] extends a result of Hong, Kim, Lee and Ryu in [17] which proves that a direct product $\prod R_i$ of rings is an $\mathcal{A}$-ring if and only if so is any R_i [17, Proposition 1.3]. In this regard, we show that if $\{R_i\}_{i\in I}$ is a family of rings and $\{M_i\}_{i\in I}$ is a family of modules such that each M_i is an R_i-module, then the direct product $\prod\limits_{i\in I} M_i$ of the M_i is an $\mathcal{A}$-module over $\prod\limits_{i\in I} R_i$ if and only if each M_i is an $\mathcal{A}$-module over R_i, $i \in I$. Finally, our main concern in [1] is to introduce and investigate a new class of rings lying properly between the class of $\mathcal{A}$-rings and the class of $\mathcal{SA}$-rings. The new class of rings, termed the class of $\mathcal{PSA}$-rings, turns out to share common characteristics with both $\mathcal{A}$-rings and $\mathcal{SA}$-rings. Numerous properties and characterizations of this class are given as well as the module-theoretic version of $\mathcal{PSA}$-rings is introduced and studied. For further works related to the Property $(\mathcal{A})$ and $(\mathcal{SA})$, we refer the reader to [2, 3, 4, 5, 16, 17, 21, 23, 24].

The main goal of this paper is to study the strong version of the class of countably $\mathcal{A}$-rings ($\mathcal{CA}$-rings for short) introduced by T. Lucas in [21]. The new class of $\mathcal{SCA}$-rings turns out to lie properly between the class of $\mathcal{SA}$-rings (or strong McCoy rings) and the class of total-$\mathcal{SA}$-rings. Furthermore, we introduce the module theoretic version of the countably $\mathcal{SCA}$-ring notion. We focus especially on the behavior of the $\mathcal{SCA}$-property vis-à-vis the polynomial ring, the power series ring, the idealization and the amalgamted duplication of a ring along an ideal. It is known that the polynomial ring $R[X]$ is an $\mathcal{SA}$-ring if and only if R is so. First, we extend this result to the polynomial ring $R[\{X_i\}_{i\in\Lambda}]$ for an arbitrary family of indeterminates $\{X_i\}_{i\in\Lambda}$. Also, we prove that, given a family $\{X_i\}_{i\in\Lambda}$ of indeterminates over R, $R[\{X_i\}_{i\in\Lambda}]$ is an $\mathcal{SCA}$-ring if and only if so is R. Regarding the power series ring, recall that a longstanding question, which is still open, asks whether $R[[X]]$ is always an $\mathcal{A}$-ring. In this aspect, recall that McCoy's theorem on polynomial rings don't carry over to power series ring $R[[X]]$ over R (see [13, Example 3^2]). Then several authors showed interest in determining the commutative rings R that satisfy the extension of McCoy's theorem to $R[[X]]$ and that we will call throughout the $R[[X]]$-McCoy's theorem. In this regard, Fields proved that if R is Noetherian, then R satisfies the $R[[X]]$-McCoy's theorem [14, Theorem 5]. Also, Gilmer, Grams and Parker proved that if either R is reduced or the total quotient ring of R is a von Neumann regular ring, then R satisfies the $R[[X]]$-McCoy's theorem (see [15]). We prove, in this context, that $R[[X]]$ is an $\mathcal{SCA}$-ring implies that R is an $\mathcal{SCA}$-ring, and that if, moreover, R satisfies the $R[[X]]$-McCoy's theorem, then $R[[X]]$ is an $\mathcal{SCA}$-ring if an only if R is an $\mathcal{SCA}$-ring. This stands as a partial affirmative answer to the above question on the $\mathcal{A}$-property of $R[[X]]$. Moreover, we give an example of an $\mathcal{A}$-ring R such that $R[[X]]$ is a not an $\mathcal{SCA}$-ring. In Section 4, we aim at seeking when the idealization $R \ltimes M$ of a ring R on an R-module M is an $\mathcal{SCA}$-ring. We characterize the $\mathcal{SCA}$-Property of $R \ltimes M$ in terms of properties of R and M. In particular, we prove that if R is a domain, then $R \ltimes M$ is an $\mathcal{SCA}$-ring if and only if M is an $\mathcal{SCA}$-module. Finally, in Section 5, we study the behavior of the $\mathcal{SCA}$-property with respect to the amalgamated duplication $R \bowtie I$ of a ring R along an ideal I. Our main theorem in this section proves that $R \bowtie I$ is an $\mathcal{SCA}$-ring if and only if R is an $\mathcal{SCA}$-ring and $I \subseteq \mathrm{Z}(R)$.

2 $\mathcal{SCA}$-rings and $\mathcal{SCA}$-modules

Recall that the countably McCoy rings were introduced by T. Lucas in [21] in his investigation on the graph of power series rings. In this section, we aim at studying the strong version of this class of rings. Also, we introduce and investigate the strongly countably McCoy modules. We prove that the class of $\mathcal{SCA}$-rings is a proper intermediate class between the class of $\mathcal{SA}$-rings and the class of total-$\mathcal{SA}$-rings. Also, it is worth reminding that the polynomial ring $R[X]$ is an $\mathcal{SA}$-ring if and only so is R. In this context, we prove that, given a family $\{X_i\}_{i\in\Lambda}$ of indeterminates over R, $R[\{X_i\}_{i\in\Lambda}]$ is an $\mathcal{SCA}$-ring if and only if so is R.

Definition 2.1. Let R be a ring and M an R-module.

(i) R is said to be a strongly countably McCoy ring or a strongly countably $\mathcal{A}$-ring ($\mathcal{SCA}$-ring for short), if for any countably generated ideal $J = (a_1, a_2, \cdots, a_n, \cdots)$ such that $a_n \in \mathrm{Z}(R)$ for each integer $n \geq 1$, we have $\mathrm{ann}_R(J) \neq 0$.

(ii) M is said to be a strongly countably McCoy module or a strongly countably $\mathcal{A}$-module ($\mathcal{SCA}$-module for short), if for any countably generated ideal $J = (a_1, a_2, \cdots, a_n, \cdots)$ such that $a_n \in \mathrm{Z}_R(M)$ for each integer $n \geq 1$, we have $\mathrm{ann}_M(J) \neq 0$.

The next proposition records the simple fact that the class of strongly countably McCoy rings is a proper intermediate class between the class of total strongly McCoy rings and the class of strongly McCoy rings. Recall that a ring R (resp., an R-module M) is said to be a total-$\mathcal{SA}$-ring (resp., total-$\mathcal{SA}$-module) if for any nonempty subset H of $\mathrm{Z}(R)$ (resp., $\mathrm{Z}_R(M)$), we have $\mathrm{ann}_R(H) \neq 0$ (resp., $\mathrm{ann}_M(H) \neq 0$).

Proposition 2.2. *1) $\mathcal{SCA}$-rings $\subseteq$ $\mathcal{CA}$-rings $\subseteq$ $\mathcal{A}$-rings.*
2) total-$\mathcal{SA}$-rings $\subsetneq$ $\mathcal{SCA}$-rings $\subsetneq$ $\mathcal{SA}$-rings.
3) Let R be a ring. Then
a) $\mathcal{SCA}$-modules $\subseteq$ $\mathcal{CA}$-modules $\subseteq$ $\mathcal{A}$-modules.
b) total-$\mathcal{SA}$-modules $\subseteq$ $\mathcal{SCA}$-modules $\subseteq$ $\mathcal{SA}$-modules.

Proof. 1) It is clear.
2) The large inclusions are direct. The strict inclusions are proved by Example 2.3 and Example 4.6.
3) It is direct. □

We next provide an example of an $\mathcal{SA}$-ring which is not an $\mathcal{SCA}$-ring.

Example 2.3. Let $R = \dfrac{\mathbb{Q}[X_n]_{n\in\mathbb{N}}}{(X_n^2)_n}$. Then:

(i) R is a countable local 0-dimensional ring.

(ii) R is an $\mathcal{SA}$-ring.

(iii) R is not an $\mathcal{SCA}$-ring.

Proof. Observe that R is a countable local ring of Krull dimension 0. Let $x_n = \overline{X_n}$ for each integer $n \geq 0$. Let $I = (x_n)_{n\in N}$ be the unique maximal ideal of R. Assume that $\mathrm{ann}_R(I) \neq (0)$. Then $I = \mathrm{Z}(R)$. Let $f(X_1, X_2, \cdots, X_p) \in \mathbb{Q}[X_1, X_2, \cdots, X_p]$ such that $0 \neq \overline{f(X_1, \cdots, X_p)} \in \mathrm{ann}_R(I)$ for some positive integer p. Then, we may assume, without loss of generality, that $f(X_1, \cdots, X_p) = \sum\limits_{1\leq i_1,\cdots,i_s\leq p} a_{i_1}\cdots a_{i_s} X_{i_1}\cdots X_{i_s}$, that is, the degree of f on each indeterminate X_i is $\deg_{X_i}(f) \leq 1$. Now, $\overline{f}I = (\overline{0})$, then, in particular, $\overline{f}x_{p+1} = \overline{0}$. Thus $fX_{p+1} \in (\{X_n^2\}_{n\in\mathbb{N}})$. This leads to a contradiction since $\deg_{X_i}(f) \leq 1$ for each $i = 1, \cdots, p$, so that $\deg_{X_i}(fX_{p+1}) \leq 1$ for each $i = 1, \cdots, p, p+1$. It follows that R is not a $\mathcal{CA}$-ring and thus R is not an $\mathcal{SCA}$-ring. We prove that R is an $\mathcal{SA}$-ring. Let $g_1, g_2, \cdots, g_m \in \mathrm{Z}(R)$. Then there exists an integer $n \geq 1$ such that $g_i = \overline{f_i(X_1, \cdots, X_n)}$ with $f_i(X_1, \cdots, X_n) \in (X_1, X_2, \cdots, X_n)\mathbb{Q}[X_1, X_2, \cdots, X_n]$ for $i = 1, \cdots, m$. Let $y = x_1x_2\cdots x_n$. Then $y \neq \overline{0}$ and it is easy to verify that $yg_i = \overline{0}$ for each $i = 1, \cdots, m$. Therefore, R is an $\mathcal{SA}$-ring, as desired.

□

Let R be a commutative ring and M be an R-module. Put $S_M := R\backslash \mathrm{Z}_R(M)$ the set of non-zero divisors of M. We define the total quotient ring of M over R to be the localization ring $Q_R(M) := S_M^{-1}R$ and the total quotient module of M to be the $Q_R(M)$-module $Q(M) := S_M^{-1}M$. It is well known that M is an $\mathcal{A}$-module over R if and only if $Q(M)$ is $\mathcal{A}$-module over $Q_R(M)$ [4, Theorem 2.1 (3)]. We next prove an analog result of this theorem for the $\mathcal{CA}$-property.

Theorem 2.4. *Let R be a ring and M be an R-module. Then the following assertions are equivalent:*

(i) M is an $\mathcal{SCA}$-module over R.

(ii) M is a $\mathcal{CA}$-module over R and $\mathrm{Z}_R(M)$ is an ideal of R.

(iii) M is a $\mathcal{CA}$-module over R and R has only one maximal prime of M.

(iv) M *is a* $\mathcal{CA}$*-module over* R *and* $Q_R(M)$ *is a local ring.*

(v) $Q(M)$ *is an* $\mathcal{SCA}$*-module over* $Q_R(M)$*.*

(vi) $Q(M)$ *is a* $\mathcal{CA}$*-module over* $Q_R(M)$ *and* $Q_R(M)$ *is a local ring.*

Proof. 1) $\Rightarrow$ 2) Assume that M is $\mathcal{SCA}$-module. Then M is a $\mathcal{CA}$-module. Also, by proposition 2.2, M is an $\mathcal{SA}$-module and thus, by [9, Theorem 2.4], $\mathrm{Z}_R(M)$ is an ideal of R.
2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 4) hold by [9, Lemma 2.5].
4) $\Rightarrow$ 5) Assume that M is a $\mathcal{CA}$-module over R and $Q_R(M)$ is a local ring. Let $J := \left(\frac{a_1}{s_1}, \cdots, \frac{a_n}{s_n}, \cdots\right)$ be a countably generated ideal of $Q_R(M)$ such that each $\frac{a_i}{s_i} \in \mathrm{Z}_{Q_R(M)}(Q(M))$ with $a_i \in \mathrm{Z}_R(M)$ and $s_i \in S_M$ for each i (see [9, Lemma 2.2]). Consider the countably generated ideal $I = (a_1, \cdots, a_n, \cdots)$ of R and observe that each $a_i \in \mathrm{Z}_R(M)$. Note that, as $Q_R(M)$ is a local ring, by [9, Lemma 2.5], $\mathrm{Z}_R(M)$ is an ideal of R. Therefore $I \subseteq \mathrm{Z}_R(M)$. Since M is a $\mathcal{CA}$-module, it follows that $\mathrm{ann}_M(I) \neq 0$. Hence, there exists $0 \neq m \in M$ such that $Im = 0$. Then $\frac{m}{1} \neq 0$ in $Q(M)$ and $J\frac{m}{1} = (0)$, as $J = S_M^{-1}I$, so that, $\mathrm{ann}_{Q(M)}(J) \neq (0)$. It follows that Q(M) is an $\mathcal{SCA}$-module over $Q_R(M)$.
5) $\Rightarrow$ 6) It is clear using [9, Lemma 2.5].
6) $\Rightarrow$ 1) Let $I = (a_1, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that each $a_i \in \mathrm{Z}_R(M)$. Then $S_M^{-1}I = \left(\frac{a_1}{1}, \cdots, \frac{a_n}{1}, \cdots\right)$ is countably generated ideal of $Q_R(M)$ with each $0 \neq \frac{a_i}{1} \in \mathrm{Z}_{Q_R(M)}(Q(M))$. As $Q_R(M)$ is local with maximal ideal is $\mathrm{Z}_{Q_R(M)}(Q(M))$, we get $S_M^{-1}I \subseteq \mathrm{Z}_{Q_R(M)}(Q(M))$. By hypothesis, $Q(M)$ is a $\mathcal{CA}$-module over $Q_R(M)$, it follows that $\mathrm{ann}_{Q(M)}(S_M^{-1}I) \neq 0$. Then, there exists $0 \neq \frac{m}{s} \in Q(M)$ with $S_M^{-1}I\frac{m}{s} = (0)$. Hence, it is easy to check that $m \neq 0$ and $Im = 0$. It follows that M is $\mathcal{SCA}$-module completing the proof of the theorem. □

We deduce the following characterizations of $\mathcal{SCA}$-rings.

Corollary 2.5. *Let* R *be a ring. Then the following assertions are equivalent:*

(i) R *is an* $\mathcal{SCA}$*-ring;*

(ii) R *is a* $\mathcal{CA}$*-ring over* R *and* $\mathrm{Z}(R)$ *is an ideal of* R*;*

(iii) R *is a* $\mathcal{CA}$*-ring and* R *has a unique maximal prime;*

(iv) R *is a* $\mathcal{CA}$*-ring and* $Q(R)$ *is a local ring;*

(v) $Q(R)$ *is an* $\mathcal{SCA}$*-ring;*

(vi) $Q(R)$ *is a* $\mathcal{CA}$*-ring which is local ring.*

Recall that the polynomial ring $R[X]$ over a ring R is an $\mathcal{SA}$-ring if and only if R is so. We extend this result to the polynomial ring $R[\{X_i\}_{i\in\Lambda}]$ for an arbitrary family of indeterminates $\{X_i\}_{i\in\Lambda}$. Also, we prove that, given a family $\{X_i\}_{i\in\Lambda}$ of indeterminates over R, $R[\{X_i\}_{i\in\Lambda}]$ is an $\mathcal{SCA}$-ring if and only if so is R.

Theorem 2.6. *Let* R *be a ring and* $\{X_i\}_{i\in\Lambda}$ *be a family of indeterminates over* R*. Then* $R[\{X_i\}_{i\in\Lambda}]$ *is an* $\mathcal{SCA}$*-ring (resp.,* $\mathcal{SA}$*-ring) if and only if* R *is an* $\mathcal{SCA}$*-ring (resp.,* $\mathcal{SA}$*-ring).*

Proof. Assume that $R[\{X_i\}_{i\in\Lambda}]$ is an $\mathcal{SCA}$-ring. Let $I = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that the $a_i \in \mathrm{Z}(R)$. Since $IR[\{X_i\}_{i\in\Lambda}] = (a_1, a_2, \cdots, a_n, \cdots)R[\{X_i\}_{i\in\Lambda}]$ is a countably generated ideal of $R[\{X_i\}_{i\in\Lambda}]$ with the $a_i \in \mathrm{Z}(R[\{X_i\}_{i\in\Lambda}])$ and $R[\{X_i\}_{i\in\Lambda}]$ is an $\mathcal{SCA}$-ring, then there exists $i_1, i_2, \cdots, i_n \in \Lambda$ and $f \in R[X_{i_1}, X_{i_2}, \cdots, X_{i_n}] \setminus \{0\}$ such that $fIR[\{X_i\}_{i\in\Lambda}] = 0$. Thus $fI = (0)$. Let $aX_{i_1}^{m_{i_1}}...X_{i_n}^{m_{i_n}}$ be a nonzero monomial of f, that is, $a \neq 0$. Hence $aX_{i_1}^{m_1}...X_{i_n}^{m_{i_n}}I = (0)$ so that $aI = (0)$ with $a \neq 0$. It follows that R is an $\mathcal{SCA}$-ring. Conversely, assume that R is an $\mathcal{SCA}$-ring. Let $J = (f_1, \cdots, f_i, \cdots)$ be a countably generated ideal of R such that $f_i \in \mathrm{Z}(R[\{X_i\}_{i\in\Lambda}])$ for each integer $i \geq 1$. Let $I = (\mathrm{c}(f_1), \mathrm{c}(f_2), \cdots, \mathrm{c}(f_i), \cdots)$ be the ideal of R generated by the contents of the f_i. Then I is a countably generated ideal of R with $\mathrm{c}(f_i) \subseteq \mathrm{Z}(R)$ for each integer $i \geq 1$, by McCoy's theorem. Hence, since R is an $\mathcal{SCA}$-ring, we get $\mathrm{ann}_R(I) \neq (0)$. Let $b \in R \setminus \{0\}$ such that $bI = (0)$. Then it is easy to see that $bJ = (0)$. This proves that $R[\{X_i\}_{i\in\Lambda}]$ is an $\mathcal{SCA}$-ring.

□

Corollary 2.7. *Let R be a ring and $X_1, X_2, \cdots, X_n$ be a finite set of indeterminates over R. Then $R[X_1, X_2, \cdots, X_n]$ is an $\mathcal{SCA}$-ring if and only if R is an $\mathcal{SCA}$-ring.*

3 $\mathcal{SCA}$-property and power series ring

This section aims at investigating the behavior of the power series ring $R[[X]]$ with respect to the property $\mathcal{SCA}$. In this regard, recall that, given a ring R, McCoy proved that if $f \in \mathrm{Z}(R[X])$, then there exists $a \in R \setminus \{0\}$ such that $af = 0$. This theorem don't carry over to power series ring $R[[X]]$ over R (see [13, Example 3[2]]). The question that arises is what are the commutative rings R that satisfy the extension of McCoy's theorem to $R[[X]]$ and that we will call throughout the $R[[X]]$-McCoy's theorem. In this regard, Fields proved that if R is Noetherian, then R satisfies the $R[[X]]$-McCoy's theorem. Also, Gilmer, Grams and Parker proved that if either R is reduced, or the total quotient ring of R is a von Neumann regular ring or each zero divisor f of $R[[X]]$ is annihilated by an element of $R[X]$, then R satisfies the $R[[X]]$-McCoy's theorem. On the other hand, it is an open question to know whether the power series ring $R[[X]]$ is an $\mathcal{A}$-ring. The main theorem of this section answers positively this question when R is a $\mathcal{SCA}$-ring such that $\mathrm{Z}(R) = \mathrm{Rad}(R)$. Also, it permits to construct an example of a ring R such that $R[[X]]$ is not a $\mathcal{SCA}$-ring.

We begin by announcing the main theorem of this section. Given a ring R, we denote by $\mathrm{Z}(R)[X]$ (resp., $\mathrm{Z}(R)[[X]]$) the subset of $R[X]$ (resp., of $R[[X]]$) consisting of elements f of $R[X]$ (resp., of $R[[X]]$) such that the coefficients of f are elements of $\mathrm{Z}(R)$.

Theorem 3.1. *Let R be a ring.*

(i) If $R[[X]]$ is an $\mathcal{SCA}$-ring, then R is an $\mathcal{SCA}$-ring.

(ii) Assume that $\mathrm{Z}(R[[X]]) \subseteq \mathrm{Z}(R)[[X]]$. Then $R[[X]]$ is an $\mathcal{SCA}$-ring if and only if R is an $\mathcal{SCA}$-ring.

(iii) Assume that $\mathrm{Z}(R) = \mathrm{Rad}(R)$. Then $R[[X]]$ is an $\mathcal{SCA}$-ring if and only if R is an $\mathcal{SCA}$-ring.

It is worthwhile noting that in the case of a polynomial ring $R[X]$ over a ring R, we always have by McCoy's theorem that $\mathrm{Z}(R[X]) \subseteq \mathrm{Z}(R)[X]$. This is no longer true in the case of a power series ring $R[[X]]$, in the sense that, $\mathrm{Z}(R[[X]]) \nsubseteq \mathrm{Z}(R)[[X]]$, in general (see [14, Example 3.[2]]).

Proof. 1) Assume that $R[[X]]$ is an $\mathcal{SCA}$-ring. Let $I = (\{a_n\}_{n\in\mathbb{N}})$ be a countably generate ideal of R such that $a_n \in \mathrm{Z}(R)$ for each positive integer n. Then $J = IR[[X]] = (\{a_n\}_{n\in\mathbb{N}})R[[X]]$ is a countably generated ideal of $R[[X]]$ generated by the a_n and each $a_n \in \mathrm{Z}(R[[X]])$. Hence there exists $f \in R[[X]]$ such that $f(0) \neq 0$ and $fJ = (0)$. Therefore $f(0)I = 0$ and $f(0) \neq 0$ and thus $\mathrm{ann}_R(I) \neq (0)$. It follows that R is an $\mathcal{SCA}$-ring.
2) Assume that R is an $\mathcal{SCA}$-ring. Let $J = (\{f_n\}_{n\in\mathbb{N}})$ be a countably generated ideal of $R[[X])$ such that the $f_n \in \mathrm{Z}(R[[X]])$. Then, by hypothesis, the coefficients of f_n belong to $\mathrm{Z}(R)$ for each integer $n \geq 1$. Consider the countably generated ideal $K = (\mathrm{c}(f_1), \mathrm{c}(f_2), \cdots, \mathrm{c}(f_n), \cdots)$ of R generated by the coefficients of the f_n. Now, as R is an $\mathcal{SCA}$-ring, K is a countably generated ideal of R and the coefficients of the f_n are elements of $\mathrm{Z}(R)$, we get $\mathrm{ann}_R(K) \neq \{0\}$. It follows that $\mathrm{ann}_{R[[X]]}(J) \neq \{0\}$ yielding that $R[[X]]$ is an $\mathcal{SCA}$-ring.
3) Suppose that $\mathrm{Z}(R) = \mathrm{Rad}(R)$. Note that, By [14, Theorem 3], $\mathrm{Z}(R[[X]]) \subseteq \mathrm{Z}(R)[[X]]$. Then, applying (2), we get the desired equivalence completing the proof of the theorem. □

It is easy to see that if R satisfies the $R[[X]]$-McCoy's theorem, in particular, if R is reduced, then $\mathrm{Z}(R[[X]]) \subseteq \mathrm{Z}(R)[[X]]$. The next result is a direct consequence of Theorem 4.1.

Corollary 3.2. *Let R be a ring.*

(i) Assume that R satisfies the $R[[X]]$-McCoy's theorem. Then $R[[X]]$ is an $\mathcal{SCA}$-ring if and only if R is an $\mathcal{SCA}$-ring.

(ii) Assume that R is a reduced ring. Then $R[[X]]$ is an $\mathcal{SCA}$-ring if and only if R is an $\mathcal{SCA}$-ring.

The following example shows that there exists an $\mathcal{A}$-ring R such that $R[[X]]$ is not a $\mathcal{SCA}$-ring.

Example 3.3. Let $R = \dfrac{\mathbb{Q}[X_n]_{n\in\mathbb{N}}}{(X_n^2)_n}$. Then:

(i) R is a countable local 0-dimensional ring.

(ii) R is an $\mathcal{A}$-ring.

(iii) $R[[X]]$ is not an $\mathcal{SCA}$-ring.

Proof. R is a local ring of Krull dimension 0. Then $\mathrm{Z}(R) = \mathrm{Rad}(R)$.
1) Since R is zero-dimensional, then R is an $\mathcal{A}$-ring.
2) By Example 2.3, R is not a $\mathcal{CA}$-ring and thus R is not an $\mathcal{SCA}$-ring. It follows, by Theorem 3.1, that $R[[X]]$ is not an $\mathcal{SCA}$-ring, as desired. □

4 $\mathcal{SCA}$-Property and idealization

We are interested in this section in determining when the idealization $R \ltimes M$ of a ring R on an R-module M is an $\mathcal{SCA}$-ring. We characterize the $\mathcal{SCA}$-property of $R \ltimes M$ in terms of properties of R and M. In particular, we prove that if R is a domain, then $R \ltimes M$ is a $\mathcal{SCA}$-ring if and only if M is a $\mathcal{SCA}$-module.

Our first results investigate some characterizations of modules satisfying the $\mathcal{SCA}$-property.

Theorem 4.1. *Let R be a ring. Let M be an R-module and N a submodule of M such that $\mathrm{Z}_R(M) = \mathrm{Z}_R(N)$. If N is an $\mathcal{SCA}$-module, then M is so.*

Proof. Let $J = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that $a_n \in \mathrm{Z}_R(M) = \mathrm{Z}_R(N)$ for each integer $n \geq 1$. Suppose that N is a $\mathcal{SCA}$-module. Then $\mathrm{ann}_N(J) \neq 0$. Hence, as $N \subseteq M$, $\mathrm{ann}_M(J) \neq 0$. It follows that M is a $\mathcal{SCA}$-module, as desired. □

Corollary 4.2. *Let M and N be R-modules such that $\mathrm{Z}_R(N) \subseteq \mathrm{Z}_R(M)$. If M is an $\mathcal{SCA}$-module, then $M \oplus N$ is so.*

Proof. Note that $\mathrm{Z}_R(M) = \mathrm{Z}_R(M \oplus N)$ and M is a submodule of $M \oplus N$. Then, apply Theorem 4.1 to get the desired result. □

Corollary 4.3. *Let M be an R-module. Then M is an $\mathcal{SCA}$-module if and only if $\bigoplus_I M$ is so.*

Proof. Assume that M is a $\mathcal{SCA}$-module. We have $\mathrm{Z}_R(\bigoplus_I M) = \mathrm{Z}_R(M)$ and $M \subseteq \bigoplus_I M$. Then, by Theorem 4.1, $\bigoplus_I M$ is a $\mathcal{SCA}$-module. Conversely, assume that $\bigoplus_I M$ is a $\mathcal{SCA}$-module. Let $J = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that $a_n \in \mathrm{Z}_R(M) = \mathrm{Z}_R(\bigoplus_I M)$ for each integer $n \geq 1$. Then $\mathrm{ann}_{\bigoplus_I M}(J) \neq 0$. So there exists $0 \neq m = (m_i)_i \in \bigoplus_I M$ such that $(Jm_i)_i = Jm = 0$. Therefore $Jm_i = 0$ for each $i \in I$. Then, there exists $k \in I$ such that $m_k \neq 0$ and $Jm_k = 0$. It follows that $\mathrm{ann}_M(J) \neq 0$. Hence M is a $\mathcal{SCA}$-module completing the proof. □

Anderson and Chun proved in [4] that if R is an integral domain and M is an R-module, then the idealization $R \ltimes M$ is an $\mathcal{A}$-ring (resp., an $\mathcal{SA}$-ring) if and only if M is an $\mathcal{A}$-module (resp., $\mathcal{SA}$-module) [4, Theorem 2.12]. Also, we proved in [8] that $R \ltimes M$ is an $\mathcal{A}$-ring if and only if $R \oplus M$ is an $\mathcal{A}$-module [8, Theorem 2.1]. The next theorem examines this result for the $\mathcal{SCA}$-property.

Theorem 4.4. *Let R be a commutative ring and M an R-module. Then $R \ltimes M$ is an $\mathcal{SCA}$-ring if and only if $R \oplus M$ is an $\mathcal{SCA}$-module over R.*

Proof. Suppose that $T := R \ltimes M$ is an $\mathcal{SCA}$-ring. Let $I = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that $a_i \in \mathrm{Z}_R(R \oplus M) = \mathrm{Z}(R) \cup \mathrm{Z}_R(M)$ for each integer $i \geq 1$. Then $(a_i, 0) \in \mathrm{Z}(T)$ for each integer $i \geq 1$. Consider the ideal $J := ((a_1, 0), (a_2, 0), \cdots, (a_n, 0), \cdots)T$ with $(a_i, 0) \in \mathrm{Z}(T)$ for each integer $i \geq 1$. As T is an $\mathcal{SCA}$-ring, there exists $(a, m) \in T$ with $(a, m) \neq (0, 0)$ such that $J(a, m) = (0, 0)$. Then $a_i a = 0$ and $a_i m = 0$ for each integer $i \geq 1$. Hence $a_i(a, m) = 0$ for each integer $i \geq 1$. It follows that $I(a, m) = (0, 0)$ and $(a, m) \neq (0, 0)$, that is, $\mathrm{ann}_{R\oplus M}(I) \neq (0, 0)$. Therefore $R \oplus M$ is an $\mathcal{SCA}$-module over R. Conversely, assume that $R \oplus M$ is an $\mathcal{SCA}$-module over R. Let, $J = ((a_1, m_1), (a_2, m_2), \cdots, (a_n, m_n), \cdots)T$ be a countably generated ideal of T such that each $(a_i, m_i) \in \mathrm{Z}(T)$. Then $a_i \in \mathrm{Z}(R) \cup \mathrm{Z}_R(M) = \mathrm{Z}_R(R \oplus M)$. Let $I = (a_1, a_2, \cdots, a_n, \cdots)$ be the finitely generated ideal of R generated by the a_i. As $R \oplus M$ is an $\mathcal{SCA}$-module over R, there exists $(a, m) \in R \oplus M$ with $(a, m) \neq (0, 0)$ and $I(a, m) = (0, 0)$. Then $a_i a = 0$ and $a_i m = 0$ for each integer $i \geq 1$. Two cases arise.
Case 1: $m \neq 0$. Then $(a_i, m_i)(0, m) = (0, 0)$ for each $i \geq 1$ and thus $J(0, m) = (0, 0)$.
Case 2: $m = 0$. Then $a \neq 0$ and $a_i a = 0$ for each $i \geq 1$. Thus $(a_i, m_i)(a, 0) = (a_i a, a m_i) = (0, a m_i)$ for each integer $i \geq 1$. If $a m_i = 0$ for each integer $i \geq 1$, then $(a_i, m_i)(a, 0) = (0, 0)$ for each integer $i \geq 1$, so that $J(a, 0) = (0, 0)$ and $(a, 0) \neq (0, 0)$. Now, suppose that there exists $j \in \mathbb{N} \setminus \{0\}$ such that $a m_j \neq 0$. Then it is easy to verify that $(a_i, m_i)(0, a m_j) = (0, 0)$ for each integer $i \geq 1$ as $a a_i = 0$ for each i. Therefore $J(0, a m_j) = (0, 0)$ and $(0, a m_j) \neq (0, 0)$. It follows that T is an $\mathcal{SCA}$-ring, as desired. □

Corollary 4.5. *Let R be an integral domain and M an R-module. Then $R \ltimes M$ is an $\mathcal{SCA}$-ring if and only if M is an $\mathcal{SCA}$-module.*

Proof. Assume that $R \ltimes M$ is an $\mathcal{SCA}$-ring. Then, by Theorem 4.4, $R \oplus M$ is an $\mathcal{SCA}$-module. Let $I = (a_1, \cdots, a_n, \cdots)$ be a nonzero countably generated ideal of R such that $a_n \in \mathrm{Z}_R(M) = \mathrm{Z}_R(R \oplus M)$ for each integer $n \geq 1$ since R is an integral domain. Then $\mathrm{ann}_{R\oplus M}(I) \neq 0$ and so there is a nonzero element $(r, m) \in R \oplus M$ such that $Ir = 0$ and $Im = 0$. Now, since R is an integral domain and $I \neq (0)$, then $r = 0$ and thus $m \neq 0$. It follows that $\mathrm{ann}_M(I) \neq 0$. Hence M is an $\mathcal{SCA}$-module, as desired. Conversely, assume that M is a $\mathcal{SCA}$-module. Then, since $\mathrm{Z}_R(R \oplus M) = \mathrm{Z}_R(M)$, by Theorem 4.1, $R \oplus M$ is an $\mathcal{SCA}$-module. Finally, apply Theorem 4.4 to complete the proof.

□

Now, we are able to present an $\mathcal{SCA}$-ring which is not a total-$\mathcal{SA}$-ring. First, we provide a local domain R admitting an $\mathcal{SCA}$-module which is not a total-$\mathcal{SA}$-module.

Example 4.6. Let k be a field, Λ an uncountable set and $\{X_i\}_{i\in\Lambda}$ be a set of indeterminates over k. Let $R = k[[\{X_i\}_{i\in\Lambda}]]$ and note that R is a local domain of maximal ideal $m = (X_i)_{i\in\Lambda}$. Let Ω be the set of all countable subsets of Λ and, for each $A \in \Omega$, let $P_A = (X_j)_{j\in A}$ be the countably generated prime ideal of R generated by the X_j with $j \in A$. Consider the R-module $M = \bigoplus_{A\in\Omega} \dfrac{R}{P_A}$. Observe that $\mathrm{Z}_R(M) = \bigcup_{A\in\Omega} P_A$ and that the maximal ideal $m = (X_i)_{i\in\Lambda} \subseteq \bigcup_{A\in\Omega} P_A = \mathrm{Z}_R(M)$. Let $H = \{f_1, f_2, \cdots, f_n, \cdots\} \subseteq \mathrm{Z}_R(M)$ be a countable subset of $\mathrm{Z}_R(M)$. Note that there exists $A \in \Omega$ such that $H \subseteq P_A$. As $P_A(\overline{1}_{R/P_A}) = \overline{0}$, we get that $H(\overline{1}_{R/P_A}) = \overline{0}$. It follows that M is an $\mathcal{SCA}$-module over R. We prove that M is not a total-$\mathcal{A}$-module. In effect, assume that there exists $0 \neq a = (\overline{a_1}, \cdots, \overline{a_n}) \in M$ such that $ma = (0)$. Let $\overline{a}_1 \neq \overline{0}$ and $\overline{a_1} \in \dfrac{R}{P_A}$ for some $A \in \Omega$. Hence $ma_1 \subseteq P_A$ and thus $m \subseteq P_A$ as $a_1 \notin P_A$. Therefore $m = P_A$. This leads to a contradiction since m is not a countably generated ideal. It follows that $\mathrm{ann}_M(m) = (0)$. Consequently, M is not a total-$\mathcal{A}$-module and thus M is not a total-$\mathcal{SA}$-module. Therefore, by applying [8, Corollary 2.4] and Corollary 4.5, we get $T = R \ltimes M$ is an $\mathcal{SCA}$-ring which is not a total-$\mathcal{SA}$-ring, as desired.

Proposition 4.7. *Let R be a ring and M a free R-module. Then $R \ltimes M$ is an $\mathcal{SCA}$-ring if and only if R is so.*

Proof. Assume that $R \ltimes M$ is an $\mathcal{SCA}$-ring. By Theorem 4.4, $R \oplus M =: N$ is an $\mathcal{SCA}$-module. Since M is a free R-module, then $N = R^{(\Lambda)}$ is a free R-module for some set Λ. Hence, by Corollary 4.3, R is an $\mathcal{SCA}$-ring. Conversely, assume that R is a $\mathcal{SCA}$-ring. Then, by Corollary 4.3, M is an $\mathcal{SCA}$-module as M is a free R-module. Also, note that $\mathrm{Z}(R) = \mathrm{Z}_R(M) = \mathrm{Z}_R(R \oplus M)$. Hence, by Theorem 4.4, $R \oplus M$ is an $\mathcal{SCA}$-module. It follows, by Theorem 4.4, that $R \ltimes M$ is an $\mathcal{SCA}$-ring, as desired. □

Proposition 4.8. *Let R be a ring and $N \subseteq M$ be R-modules such that M is an essential extension of N. Then M is an $\mathcal{SCA}$-module if and only if N is so.*

Proof. Note that, by [4, Theorem 2.2 (4)], $\mathrm{Z}_R(N) = \mathrm{Z}_R(M)$. Then, by Theorem 4.1, N is an $\mathcal{SCA}$-module implies that M is so. Conversely, assume that M is an $\mathcal{SCA}$-module. Let $I \subseteq \mathrm{Z}_R(N) = \mathrm{Z}_R(M)$ be a countably generated ideal of R. Then, since M is an $\mathcal{SCA}$-module, we get $\mathrm{ann}_M(I) \neq 0$. As M is an essential extension of N, then $\mathrm{ann}_M(I) \cap N \neq 0$. Thus there exists $0 \neq x \in N$ such that $Ix = 0$ so that $\mathrm{ann}_N(I) \neq 0$. It follows that N is an $\mathcal{SCA}$-module establishing the desired equivalence. □

5 $\mathcal{SCA}$-property and amalgamated duplication

The main purpose of this section is to characterize when is the amalgamated duplication $R \bowtie I$ of a ring R along an ideal I an $\mathcal{SCA}$-ring.

If R is a ring and I an ideal of R, for easiness, we adopt the following notation: for any subset E of R, we consider the next two subsets of $R \bowtie I$:

$$\begin{cases} E \bowtie I &= \quad \{(e, e+i) \in R \bowtie I : e \in E \text{ and } i \in I\} \\ I \bowtie E &:= \quad \{(e+i, e) : e \in E \text{ and } i \in I\} \end{cases}$$

and the subset of R:

$$E + I := \{e + i : e \in E \text{ and } i \in I\}.$$

We begin by stating the main theorem of this section.

Theorem 5.1. *Let R be a ring and I an ideal of R. Then the following statements are equivalent:*
1) $R \bowtie I$ is an $\mathcal{SCA}$-ring;
2) R is an $\mathcal{SCA}$-ring and $I \subseteq \mathrm{Z}(R)$.

To prove this theorem, we need the following preparatory results.

Lemma 5.2. *Let R be a ring and I an ideal of R. Then*
1) $\mathrm{Z}(R) \bowtie I \subseteq \mathrm{Z}(R \bowtie I)$.
2) $I \bowtie \mathrm{Z}_R(I) \subseteq \mathrm{Z}(R \bowtie I)$.

Proof. 1) Let $a \in \mathrm{Z}(R)$ and $e \in I$. Then two cases arise.
Case 1: $\mathrm{ann}_R(a) \cap \mathrm{ann}_R(e) \neq (0)$. Then let $b \in \mathrm{ann}_R(a) \cap \mathrm{ann}_R(e)$ such that $b \neq 0$. Hence $(b, b)(a, a+e) = (0, 0)$, so that $(a, a+e) \in \mathrm{Z}(R \bowtie I)$.
Case 2: $\mathrm{ann}_R(a) \cap \mathrm{ann}_R(e) = (0)$. Let $b \in \mathrm{ann}_R(a)$ such that $b \neq 0$. Then $b \notin \mathrm{ann}_R(e)$ and thus $be \in I \setminus \{0\}$. Hence $(be, 0)(a, a+e) = (0, 0)$ and $(be, 0) \neq (0, 0)$. Therefore $(a, a+e) \in \mathrm{Z}(R \bowtie I)$.
It follows from the above cases that $\mathrm{Z}(R) \bowtie I \subseteq \mathrm{Z}(R \bowtie I)$.
2) Let $a \in \mathrm{Z}_R(I)$ and $i \in I$. Then there exists $e \in I \setminus \{0\}$ such that $ae = 0$. Hence $(a+i, a)(0, e) = (0, 0)$ and thus $(a+i, a) \in \mathrm{Z}(R \bowtie I)$. It follows that $I \bowtie \mathrm{Z}_R(I) \subseteq \mathrm{Z}(R \bowtie I)$, as desired.

□

Lemma 5.3. *Let R be a ring and I an ideal of R. Let a be a regular element of R and $e \in I$. Then*

$$(a, a+e) \in \mathrm{Z}(R \bowtie I) \Leftrightarrow a + e \in \mathrm{Z}_R(I).$$

Proof. $\Leftarrow$) Assume that $a+e \in \mathrm{Z}_R(I)$. Then there exists $i \in I \setminus \{0\}$ such that $(a+e)i = 0$. Hence $(a, a+e)(0, i) = (0, 0)$ and $(0, i) \neq (0, 0)$. It follows that $(a, a+e) \in \mathrm{Z}(R \bowtie I)$.
$\Rightarrow$) Suppose that $(a, a+e) \in \mathrm{Z}(R \bowtie I)$. Then there exists $(b, b+j) \in R \bowtie I \setminus \{(0,0)\}$ such that $(b, b+j)(a, a+e) = (0, 0)$. Hence $ba = 0$ and $(b+j)(a+e) = 0$. Since $a \notin \mathrm{Z}(R)$, we get $b = 0$. Therefore $j(a+e) = 0$. It follows that $a+e \in \mathrm{Z}_R(I)$, as desired. □

We recall that $S = R \setminus \mathrm{Z}(R)$ stands for the set of all regular elements of R.

Corollary 5.4. *Let R be a ring and I an ideal of R. Then*

$$\mathrm{Z}(R \bowtie I) = (\mathrm{Z}(R) \bowtie I) \bigsqcup \left((S \bowtie I) \bigcap (R \times \mathrm{Z}_R(I)) \right).$$

Proof. It follows from Lemma 5.2 and Lemma 5.3. □

Lemma 5.5. *Let R be a ring and I an ideal of R. If R is an $\mathcal{SA}$-ring and $I \subseteq \mathrm{Z}(R)$, then*

$$\mathrm{Z}(R) \bigcap (S + I) = \emptyset.$$

Proof. Assume that R is an $\mathcal{SA}$-ring and that $I \subseteq \mathrm{Z}(R)$. Then, in particular, $\mathrm{Z}(R)$ is an ideal of R. Suppose that $(S+I) \bigcap \mathrm{Z}(R) \neq \emptyset$. Then there exists $b \in S$ and $i \in I$ such that $b+i \in \mathrm{Z}(R)$. Since $i \in I \subseteq \mathrm{Z}(R)$, $b+i \in \mathrm{Z}(R)$ and $\mathrm{Z}(R)$ is an ideal of R, we get $b \in \mathrm{Z}(R)$ which is absurd. Hence $(S+I) \bigcap \mathrm{Z}(R) = \emptyset$, as desired. □

Proof of Theorem 5.1. 1) $\Rightarrow$ 2) Let $J = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that $a_n \in \mathrm{Z}(R)$ for each integer $n \geq 1$. By Lemma 5.2, we have $(a_k, a_k) \in \mathrm{Z}(R \bowtie I)$ for each integer $k \geq 1$. Now, since $\mathrm{Z}(R \bowtie I)$ is an ideal of $R \bowtie I$, we get $K = ((a_1, a_1), ..., (a_n, a_n), \cdots)R \bowtie I \subseteq \mathrm{Z}(R \bowtie I)$. Therefore, as $R \bowtie I$ is an $\mathcal{SCA}$-ring, there exists $(a, a+e) \in (R \bowtie I) \setminus \{(0,0)\}$ such that $(a, a+e)K = (0, 0)$. If $a \neq 0$, then $aa_k = 0$ for each integer $k \geq 1$, so that, $aJ = 0$ and thus $\mathrm{ann}(J) \neq (0)$. If $a = 0$, then $e \neq 0$ and $eJ = 0$, that is, $\mathrm{ann}(J) \neq (0)$. It follows that R is an $\mathcal{SCA}$-ring. On the other hand, we claim that

$$(S \bowtie I) \bigcap (R \times \mathrm{Z}_R(I)) = \emptyset.$$

In fact, assume that $(S \bowtie I) \cap (R \times \mathrm{Z}_R(I)) \neq \emptyset$. Then there exists $b \in S$ and $e \in I$ such that $b + e \in \mathrm{Z}_R(I)$. Note that, by Corollary 5.4, $(b, b+e) \in \mathrm{Z}(R \bowtie I)$ and $(0, e) \in \mathrm{Z}(R \bowtie I)$. Since $\mathrm{Z}(R \bowtie I)$ is an ideal, we get $(b, b) \in \mathrm{Z}(R \bowtie I)$ which is absurd as $b \notin \mathrm{Z}(R)$. Hence $(S \bowtie I) \cap (R \times \mathrm{Z}_R(I)) = \emptyset$ proving our claim. Now, assume that $I \nsubseteq \mathrm{Z}(R)$. Then there exists $j \in I$ such that $j \notin \mathrm{Z}(R)$. Hence $(j, 0) \in (S \bowtie I) \cap (R \times \mathrm{Z}_R(I))$ which leads to a contradiction in view of the above claim. It follows that $I \subseteq \mathrm{Z}(R)$.
2) $\Rightarrow$ 1) Suppose that R is an $\mathcal{SCA}$-ring and $I \subseteq \mathrm{Z}(R)$. Then, by Lemma 5.5, we have $(S + I) \cap \mathrm{Z}(R) = \emptyset$. Therefore $(S \bowtie I) \cap (R \times \mathrm{Z}_R(I)) = \emptyset$. It follows, by corollary 5.4, that $\mathrm{Z}(R \bowtie I) = \mathrm{Z}(R) \bowtie I$. Now, let $\{(a_1, a_1 + e_1), \cdots, (a_n, a_n + e_n), \cdots\}$ be a countable subset of $\mathrm{Z}(R \bowtie I)$. Then $a_n \in \mathrm{Z}(R)$ for each integer $n \geq 1$. Therefore, as R is an $\mathcal{SCA}$-ring, there exists $a \in R \setminus \{0\}$ such that $aa_k = 0$ for each integer $k \geq 1$. If $ae_k = 0$ for each $k \geq 1$, then, for each integer $k \geq 1$,

$$(a, a)(a_k, a_k + e_k) = (0, 0) \text{ and } (a, a) \in R \bowtie I \setminus \{(0, 0)\}.$$

Assume that there exists an integer $t \geq 1$ such that $ae_t \neq 0$. Hence $(ae_t, 0)(a_k, a_k + e_k) = (0, 0)$ for each integer $k \geq 1$ and $(ae_t, 0) \in R \bowtie I \setminus \{(0, 0)\}$. It follows from both cases that $\mathrm{ann}_{R \bowtie I}((a_1, a_1 + e_1), ..., (a_n, a_n + e_n), \cdots) \neq \{(0, 0)\}$. Consequently, $R \bowtie I$ is an $\mathcal{SCA}$-ring completing the proof of the theorem . □

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