Palestine Journal of Mathematics

Vol 13(1)(2024) , 181–200

GENERALIZED q-DIFFERENCE EQUATION FOR THE GENERALIZED q-OPERATOR ${}_t\Phi_s(D_q)$ AND ITS APPLICATIONS IN q-POLYNOMIALS

Faiz A. Reshem and Husam L. Saad

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 05A30; Secondary 33D45.

Keywords and phrases: generalized q-operator, generalized q-difference equation, generalized q-polynomials, generating function, Srivastava-Agarwal type generating function, transformational identity.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

.

Abstract The solution to a generalized q-difference equation is described in q-operator form in this paper, which is a generalization of Fang's work [9]. We solve a general q-difference equation to get a general q-operator identity. The generalized q-polynomial $\phi_n^{(A,B)}(b, c|q)$ is defined. We present two types generating function and a generalization of the transformational identity for the polynomials $\phi_n^{(A,B)}(b, c|q)$ using the q-difference equation. By assigning specific values to the parameters in the findings for the polynomials $\phi_n^{(A,B)}(b, c|q)$, we get two types generating functions and the transformational identity for polynomials $H_n(a_0, \dots, a_s; b_1, \dots, b_s; b, c)$, $P_n(x, y, a)$, $h_n(x, y, a, b|q)$ and $h_n(a_1, \cdots, a_r; b_1, \cdots, b_s; x, y|q)$.

1 Introduction

In this paper, the notations that was used in [10] is followed and we assume that $|q| < 1$. We're going to mention to a few notations that we depend on during this paper.

The q-shifted factorial is defined by [10]:

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Also the multiple q-shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \dots (a_m;q)_n.$$

The basic hypergeometric series ${}_t\phi_s$ is given by [10]:

$${}_t\phi_s\left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, b_2, \dots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_0, a_1 \dots, a_{t-1}; q)_n}{(q, b_1, b_2 \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} x^n,$$

where $q \neq 0$ when $t > s+1$. Note that

$${}_{s+1}\phi_s\left(\begin{matrix} a_1, a_2, \dots, a_{s+1} \\ b_1, b_2, \dots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{s+1}; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} x^n, \quad |x| < 1.$$

The q-binomial coefficients is given by [10]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad 0 \leqslant k \leqslant n.$$

Cauchy identity is provided by:

$$\sum_{m=0}^{\infty} \frac{(a;q)_m}{(q;q)_m}\, x^m = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \tag{1.1}$$

Euler found the following as a special case of Cauchy identity (1.1) [10]:

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \quad |x|<1.$$

We shall employ the following identities [10]:

$$(a;q)_n = (q^{1-n}/a;q)_n(-a)^n q^{\binom{n}{2}}. \tag{1.2}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}. \tag{1.3}$$

$$(a;q)_{-n} = \frac{(-q/a)^n}{(q/a;q)_n} q^{\binom{n}{2}}. \tag{1.4}$$

$$(aq^{-n};q)_k = \frac{(a;q)_k(q/a;q)_n}{(q^{1-k}/a;q)_n} q^{-nk}. \tag{1.5}$$

$$(a;q)_n(aq^n;q)_k = (a;q)_k(aq^k;q)_n. \tag{1.6}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^{-n};q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}+nk}. \tag{1.7}$$

For the fundamentals of q-calculus, see [16, 17, 18, 29].

The q-Chu-Vandermonde sum is [26]

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{1.8}$$

Heine's transformations of ${}_2\phi_1$ series [10, Appendix III, equations (III.1), (III.2)] are:

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(b, az;q)_\infty}{(c, z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/b, z \\ az \end{matrix}; q, b\right). \tag{1.9}$$

$$= \frac{(c/b, bz;q)_\infty}{(c, z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} abz/c, b \\ bz \end{matrix}; q, c/b\right). \tag{1.10}$$

Jackson's transformation of ${}_2\phi_1$ [10, Appendix III, equation (III.4)] is:

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2\left(\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz\right). \tag{1.11}$$

Transformation of ${}_3\phi_2$ series [10, Appendix III, equation (III.12)] is:

$${}_3\phi_2\left(\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, q\right) = \frac{(e/c;q)_n}{(e;q)_n} c^n\, {}_3\phi_2\left(\begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix}; q, \frac{bq}{e}\right). \tag{1.12}$$

The Cauchy polynomials is defined by [11, 12, 20, 23]:

$$P_k(x;y) = (x-y)(x-qy)\ldots(x-yq^{k-1}) = (y/x;q)_k x^k,$$

which has the generating function [1, 2]

$$\sum_{k=0}^{\infty} P_k(x;y)\frac{t^k}{(q;q)_k} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.13}$$

The q-differential operator is defined by [30]

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}. \tag{1.14}$$

The Leibniz rule for D_q is [19]

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}. \tag{1.15}$$

Let k be a nonnegative integer. The following identities are easy to verify [7, 27, 30]:

$$\begin{aligned} D_q^k\left\{\frac{1}{(xt;q)_\infty}\right\} &= \frac{t^k}{(xt;q)_\infty}. \\ D_q^k\{P_n(x,y)\} &= \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y). \\ D_q^k\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\} &= t^k (v/t;q)_k \frac{(xvq^k;q)_\infty}{(xt;q)_\infty}. \end{aligned} \tag{1.16}$$

In 2003. Chen and Gu [6] introduced the Cauchy operator

$$\mathbb{T}(a,b;D_q) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (bD_q)^k.$$

In 2009, Lu [14] obtained the q-difference equation:

Theorem 1.1. [14]. *Let $f(a,b,c)$ be a three variables analytic function in a neighborhood of $(a,b,c) = (0,0,0) \in \mathbb{C}^3$, satisfying the q-difference equation*

$$(c-b)f(a,b,c) = abf(a,bq,cq) - bf(a,b,cq) + (c-ab)f(a,bq,c).$$

Then

$$f(a,b,c) = \mathbb{T}(a,b;D_q)\{f(a,0,c)\}.$$

In 2014, Li and Tan [13] described the generalized q-exponential operator:

$$\mathbb{T}\left[\begin{matrix} w, r \\ v \end{matrix}; q; cD_b\right] = \sum_{n=0}^{\infty} \frac{(w,r;q)_n}{(q,v;q)_n} (cD_b)^n.$$

In 2014, Cao [4] stated the q-difference equation:

Theorem 1.2. [4]. *Let $f(w,r,v,b,c)$ be a five-variable analytic function in neighborhood of $(w,r,v,b,c) = (0,0,0,0,0) \in \mathbb{C}^5$. If $f(w,r,v,b,c)$ satisfies the q-difference equation*

$$\begin{aligned} &b[f(w,r,v,b,c) - (1+q^{-1}v)f(w,r,v,b,cq) + q^{-1}vf(w,r,v,b,cq^2)] \\ &\quad = c\big\{[f(w,r,v,b,c) - f(w,r,v,qb,c)] - (w+r)[f(w,r,v,b,qc) - f(w,r,v,qb,qc)] \\ &\qquad + wr[f(w,r,v,b,q^2c) - f(w,r,v,qb,q^2c)]\big\}. \end{aligned}$$

Then

$$f(w,r,v,b,c) = \mathbb{T}\left[\begin{matrix} w, r \\ v \end{matrix}; q; cD_b\right]\{f(w,r,v,b,0)\}.$$

In 2014, Fang [9] defined the generalized q-operator $F(a_0,\ldots,a_s,b_1,\ldots,b_s,cD_{q,b})$:

$$F(a_0,\ldots,a_s,b_1,\ldots,b_s,cD_{q,b}) = \sum_{n=0}^{\infty} \frac{(a_0,a_1\ldots,a_s;q)_n}{(q,b_1,\ldots,b_s;q)_n} (cD_{q,b})^n. \tag{1.17}$$

Fang [9] presented the following generalized q-difference equation:

Theorem 1.3. [9]. *Let $f(a_0,\dots,a_s,b_1,\dots,b_s,b,c)$ be a $2s+3$-variable analytic function in a neighborhood of $(a_0,\dots,a_s,b_1,\dots,b_s,b,c)=(0,0,\dots,0)\in\mathbb{C}^{2s+3}$, $s\in\mathbb{N}$, satisfying q-difference equation*

$$b\sum_{j=0}^{s+1}\frac{(-1)^jB_j}{q^j}f(a_0,\dots,a_s,b_1,\dots,b_s,b,cq^j)-c\sum_{j=0}^{s+1}(-1)^jA_j[f(a_0,\dots,a_s,b_1,\dots,b_s,b,cq^j) \\ -f(a_0,\dots,a_s,b_1,\dots,b_s,bq,cq^j)]=0, \tag{1.18}$$

where

$$b=q,\quad B_0=A_0=1,\quad B_1=\sum_{j=0}^{s}b_j,\quad B_2=\sum_{0\le i<j\le s}b_ib_j$$
$$B_3=\sum_{0\le i<j<k\le s}b_ib_jb_k,\dots,\quad B_{s+1}=b_0b_1\dots b_s,\quad A_1=\sum_{i=0}^{s}a_i$$
$$A_2=\sum_{0\le i<j\le s}a_ia_j,\quad A_3=\sum_{0\le i<j<k\le s}a_ia_ja_k,\dots,\quad A_{s+1}=a_0a_1\dots a_s.$$

Then

$$f(a_0,\dots,a_s,b_1,\dots,b_s,b,c)=F(a_0,\dots,a_s,b_1,\dots,b_s,cD_{q,b})\{f(a_0,\dots,a_s,b_1,\dots,b_s,b,0)\}.$$

Using the q-difference equation (1.18), Fang [9] proved the following operator identities:

Theorem 1.4. [9]. *For $a_0=q^{-G}$, $G\in\mathbb{N}, b,w,u,v,a_i,b_i\in\mathbb{C}, i=1,2,\dots,s$, then*

$$F(a_0,\dots,a_s,b_1,\dots,b_s,cD_{q,b})\left\{\frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\right\} \\ =\frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,\dots,a_s;q)_n}{(q,b_1,\dots,b_s;q)_n}\left(\frac{c}{b}\right)^n {}_3\phi_2\left(\begin{matrix}q^{-n},bu,bv\\ 0,bw\end{matrix};q,q\right). \tag{1.19}$$

Corollary 1.5. [9]. *If* $\max\{|bu|,|cu|\}<1$, $u,a_i,b_i\in\mathbb{C}$, $i=1,2,\dots,s$, *then*

$$F(a_0,\dots,a_s,b_1,\dots,b_s,cD_{q,b})\left\{\frac{1}{(bu;q)_\infty}\right\}=\frac{1}{(bu;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,\dots,a_s;q)_n}{(q,b_1,\dots,b_s;q)_n}(cu)^n. \tag{1.20}$$

Also, Fang [9] defined the q-polynomials H_n:

$$H_n=H_n(a_0,\dots,a_s,b_1,\dots,b_s;b,c)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a_0,a_1,\dots,a_s;q)_k}{(b_1,\dots,b_s;q)_k}c^kb^{n-k}, \tag{1.21}$$

which have the generating function

$$\sum_{n=0}^{\infty}H_n\frac{u^n}{(q;q)_n}=\frac{1}{(bu;q)_\infty}\sum_{n=0}^{\infty}W_n(cu)^n,\quad \max\{|bu|,|cu|\}<1.. \tag{1.22}$$

In 2019, Srivastava and Arjika [28] defined the homogeneous q-difference operator $\widetilde{E}(a,b;D_q)$ as follows:

$$\widetilde{E}(a,b;D_q)=\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(bD_q)^k. \tag{1.23}$$

They [28] introduced the generalized Cauchy polynomials

$$P_n(x,y,a)=\sum_{k=0}^{\infty}(-1)^kq^{\binom{k}{2}}(a;q)^kx^{n-k}y^k, \tag{1.24}$$

which has the generating function

$$\sum_{n=0}^{\infty} P_n(x,y,a)\frac{t^n}{(q;q)_n} = \frac{1}{(xt;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix}; q, yt\right), \quad |xt|<1 \tag{1.25}$$

and the generalized Hahn polynomials

$$h_n(x,y,a,b|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} b^k (a;q)_k P_{n-k}(y,x), \tag{1.26}$$

which has the generating function:

$$\sum_{n=0}^{\infty} h_n(x,y,a,b|q)\frac{t^n}{(q;q)_n} = \frac{(xt;q)_\infty}{(yt;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix}; q, bt\right), \quad |yt|<1. \tag{1.27}$$

In 2020, Arjika [3] introduced the following result:

Theorem 1.6. [3]. *Let $f(a,x,y)$ be a three veriables analytic function in a neighborhood of $(a,x,y)=(0,0,0)\in\mathbb{C}^3$ satisfying q-difference equation*

$$\begin{aligned} &x[f(a,x,y)-f(a,x,qy)] \\ &\quad = y[f(a,qx,qy)-f(a,x,qy)] - ay[f(a,qx,q^2y)-f(a,x,q^2y)], \end{aligned}$$

Then

$$f(a,x,y) = \widetilde{E}(a,b;D_q)\{f(a,x,0)\}.$$

In 2020, Cao and et al. [5] introduced the q-operator $\mathbb{T}(a,b,c,d,e,yD_x)$ as follows:

$$\mathbb{T}(a,b,c,d,e,yD_x) = \sum_{n=0}^{\infty} \frac{(a,b,c;q)_n}{(q,d,e;q)_n}(yD_x)^n.$$

Also, they gave the following q-difference equation:

Theorem 1.7. [5]. *Let $f(a,b,c,d,e,x,y)$ be a seven-variable analytic function in a neighborhood of $(a,b,c,d,e,x,y)=(0,0,0,0,0,0,0)\in\mathbb{C}^7$. If $f(a,b,c,d,e,x,y)$ satisfies the difference equation*

$$\begin{aligned} &x\big\{f(a,b,c,d,e,x,y)-f(a,b,c,d,e,x,yq) \\ &\quad -(d+e)q^{-1}[f(a,b,c,d,e,x,yq)-f(a,b,c,d,e,x,yq^2)] \\ &\quad +deq^{-2}[f(a,b,c,d,e,x,yq^2)-f(a,b,c,d,e,x,yq^3)]\big\} \\ &= y\big\{f(a,b,c,d,e,x,y)-f(a,b,c,d,e,xq,y) \\ &\quad -(a+b+c)[f(a,b,c,d,e,x,yq)-f(a,b,c,d,e,xq,yq)] \\ &\quad +(ab+ac+bc)[f(a,b,c,d,e,x,yq^2)-f(a,b,c,d,e,xq,yq^2)] \\ &\quad -abc[f(a,b,c,d,e,x,yq^3)-f(a,b,c,d,e,xq,yq^3)]\big\}. \end{aligned}$$

Then

$$f(a,b,c,d,e,x,y) = \mathbb{T}(a,b,c,d,e,yD_x)\{f(a,b,c,d,e,x,0)\}.$$

Saad and Hassan [22] introduced the generalized q-operator as follows:

$$\begin{aligned} &F(a_0,a_1\ldots,a_{t-1},b_1\ldots,b_s,cD_{q,b}) \\ &\quad = \sum_{n=0}^{\infty} \frac{(a_0,a_1\ldots,a_{t-1};q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}(cD_{q,b})^n. \end{aligned} \tag{1.28}$$

Saad and Hassan [22] used the operator F to generalize some well-known q-identities, such as Cauchy identity, Heine's transformation formula and the q-Pfaff-Saalschütz summation formula.

- When $t = s + 1$ in equation (1.28), we get the generalized q-operator $F(a_1, \ldots, a_s, b_1, \ldots, b_s, cD_{q,b})$ created by Fang [9] (equation (1.17)).
- Setting $t = 1$, $s = 1$, $a_0 = a$, $b_1 = 0$ and $c \to -b$ in equation (1.28), we obtain the homogeneous q-difference operator $\widetilde{E}(a, b; D_q)$ defined by Srivastava and Arjika [28] (equation (1.23)).

In 2021, Saad and Hassan [21] described the q-polynomials $h_n(a_1, \ldots, a_r, b_1, \ldots, b_s; x, y|q)$ as follows:

$$h_n(a_1, \ldots, a_r, b_1, \ldots, b_s; x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} W_k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} x^{n-k} y^k, \tag{1.29}$$

where $W_n = \dfrac{(a_1, \cdots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n}$. With the use of the operator F, they gave an operator proof of the generating function

$$\sum_{n=0}^{\infty} h_n(a_1, \ldots, a_r; b_1, \ldots, b_s; x, y|q) \frac{t^n}{(q;q)_n} = \frac{1}{(xt;q)_\infty}\, {}_r\phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, yt \right), \tag{1.30}$$

where $|xt| < 1$.

In 2022, Saad and Reshem [25] used the q-Gospers algorithm [8] to verify that the function $f(a, b, c)$ satisfies the q-difference equation.

Our paper is organized as follows: We solve a generalized q-difference equation and express the solutions in q-operator form in section 2. In section 3, using the q-difference equation, we provide some identities for the q-operator $F(a_0, \ldots, a_{t-1}, b_1 \ldots, b_s, cD_b)$. In section 4, we construct a generalized q-polynomials $\phi_n^{(A,B)}(b, c|q)$ and use the q-difference equation technique to determine the generating function for the polynomials $\phi_n^{(A,B)}(b, c|q)$. In section 5, we introduced two types generating function for $\phi_n^{(A,B)}(b, c|q)$. In section 6, we use the q-difference equation approach to derive a generalization of the transformational identity.

2 A generalized q-Difference Equation

In this section, we solve a generalized q-difference equation and express the solution in the q-operator form which is a generalization of Fang's results [9].

Lemma 2.1. [15]. *If $f(x_1, \ldots, x_k)$ is analytic at the orgin $(0, 0, \ldots, 0) \in \mathbb{C}^k$, then f can be expanded in an absolutely convergent power series*

$$f(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k}^{\infty} \alpha_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

Theorem 2.2. *Let $f(a_0, a_1, ldots, a_{t-1}, b_1, \ldots, b_s, b, c)$ be an $(t+s+2)$-variable analytic function in a neighborhood of $(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, c) = (0, 0, \ldots, 0) \in \mathbb{C}^{t+s+2}$ satisfying the q-difference equation*

$$\begin{aligned}(-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, cq^{j+t-s-1}) \\ = c \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \left\{ f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, cq^j) \right\},\end{aligned} \tag{2.1}$$

where

$$b_0 = q, \quad B_0 = A_0 = 1, \quad B_1 = \sum_{j=0}^{s} b_j, \quad B_2 = \sum_{0 \le i < j \le s} b_i b_j$$

$$B_3 = \sum_{0\le i<j<k\le s} b_i b_j b_k \quad ,\ldots, \quad B_{s+1} = b_0 b_1 \ldots b_s, \quad A_1 = \sum_{i=0}^{t-1} a_i$$
$$A_2 = \sum_{0\le i<j\le t-1} a_i a_j, \quad A_3 = \sum_{0\le i<j<k\le t-1} a_i a_j a_k \quad ,\ldots, \quad A_t = a_0 a_1 \ldots a_{t-1}.$$

Then

$$f(a_0,\ldots,a_{t-1},b_1,\ldots,b_s,b,c) \\ = F(a_0,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,b})\{f(a_0,\ldots,a_{t-1},b_1,\ldots,b_s,b,0)\}.$$

Proof. By using Lemma 2.1, we suppose that

$$f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,c) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\omega_{m,n}b^m c^n. \tag{2.2}$$

$$(-q)^{1+s-t}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}\sum_{m,n=0}^{\infty}\omega_{m,n}b^m c^n q^{n(j+t-s-1)} = c\sum_{j=0}^{t}(-1)^j A_j D_{q,b}\left\{\sum_{m,n=0}^{\infty}\omega_{m,n}b^m c^n q^{jn}\right\}.$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{s+1}(-1)^j B_j(-1)^{1+s-t}q^{j(n-1)+(n-1)(t-s-1)}\sum_{m=0}^{\infty}\omega_{m,n}b^m c^n \\ = \sum_{n=0}^{\infty}\sum_{j=0}^{t}(-1)^j A_j q^{jn} D_{q,b}\left\{\sum_{m=0}^{\infty}\omega_{m,n}b^m c^{n+1}\right\} \\ = \sum_{n=1}^{\infty}\sum_{j=0}^{t}(-1)^j A_j q^{j(n-1)} D_{q,b}\left\{\sum_{m=0}^{\infty}\omega_{m,n-1}b^m\right\}c^n.$$

Equating the coefficients of c^n, we obtain

$$\sum_{j=0}^{s+1}(-1)^j B_j(-1)^{1+s-t}q^{j(n-1)+(n-1)(t-s-1)}\sum_{m=0}^{\infty}\omega_{m,n}b^m \\ = \sum_{j=0}^{t}(-1)^j A_j q^{j(n-1)} D_{q,b}\left\{\sum_{m=0}^{\infty}\omega_{m,n-1}b^m\right\}$$

$$\sum_{m=0}^{\infty}\omega_{m,n}b^m = \frac{(-1)^{1+s-t}q^{(n-1)(1+s-t)}\sum_{j=0}^{t}(-1)^j A_j q^{j(n-1)}}{\sum_{j=0}^{s+1}(-1)^j B_j q^{j(n-1)}} D_{q,b}\left\{\sum_{m=0}^{\infty}\omega_{m,n-1}b^m\right\}$$
$$= \frac{(-1)^{1+s-t}q^{(n-1)(1+s-t)}\prod_{j=0}^{t-1}(1-a_j q^{n-1})}{\prod_{j=0}^{s}(1-b_j q^{n-1})} D_{q,b}\left\{\sum_{m=0}^{\infty}\omega_{m,n-1}b^m\right\}.$$

We discover through repetition that

$$\sum_{m=0}^{\infty}\omega_{m,n}b^m = \frac{(a_0,a_1,\ldots,a_{t-1};q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} D_{q,b}^n\left\{\sum_{m=0}^{\infty}\omega_{m,0}b^m\right\}. \tag{2.3}$$

Setting $c = 0$ in (2.2), we get

$$f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,0) = \sum_{m=0}^{\infty}\omega_{m,0}b^m. \tag{2.4}$$

From equations (2.2), (2.3) and (2.4), we get the required result. □

- For $(t, s, a_0, b, c) = (1, 0, a, c, b)$, Theorem 2.2 reduce to Theorem 1.1 given by Lu [14].
- By choosing $(t, s, a_0, a_1, b_1) = (2, 1, r, w, v)$ in Theorem 2.2, we recover Theorem 1.2 obtained by Cao [4].
- When $t = s + 1$ in Theorem 2.2, we retain Theorem 1.3 provided by Fang [9].
- Letting $(t, s, a_0, b_1, b, c) = (1, 1, a, 0, x, yq)$ in Theorem 2.2, we regain Theorem 1.6 produced by Arjika [3].
- Setting $(t, s) = (3, 2)$ and then $(a_0, a_1, a_2, b_1, b_2, b, c) = (a, b, c, d, e, x, y)$ in Theorem 2.2, we revive Theorem 1.7 gained by Cao and et al. [5].

3 Identities for the q-Operator $F(a_0, \dots, a_{t-1}, b_1 \dots, b_s, cD_b)$

In this section, we give some identities for the q-operator $F(a_0, \dots, a_{t-1}, b_1 \dots, b_s, cD_b)$, which are a generalization of the results given in [9].

Lemma 3.1. *Let $D_{q,b}$ be defined as in (1.14), then*

$$D^n_{q,b}\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\} = \frac{1}{b^n}\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw \end{matrix}; q, q\right), \tag{3.1}$$

where $\max\{|bu|, |bv|\} < 1$.

Proof. By using Leibniz rule (1.15) and equation (1.16), we get

$$\begin{aligned}
&D^n_{q,b}\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\}\\
&= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D^k_q\left\{\frac{(bw; q)_\infty}{(bu; q)_\infty}\right\} D^{n-k}_q\left\{\frac{(brq^k; q)_\infty}{(bvq^k; q)_\infty}\right\}\\
&= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} \frac{u^k(w/u; q)_k(bwq^k; q)_\infty}{(bu; q)_\infty}\frac{v^{n-k}(r/v; q)_{n-k}(brq^n; q)_\infty}{(bvq^k; q)_\infty}\\
&= \frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\sum_{k=0}^{n}\frac{q^{-n}, w/u, bv; q)_k}{(q, bw, q^{1-n}v/r; q)_k}\frac{(r/v; q)_n}{(br; q)_n}\left(\frac{vq}{r}\right)^k\left(\frac{u}{v}\right)^k v^n\\
&\qquad\qquad\qquad\text{(by using (1.3) and (1.7))}\\
&= \frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\frac{(r/v; q)_n v^n}{(br; q)_n}\sum_{k=0}^{n}\frac{(q^{-n}, w/u, bv; q)_k}{(q, bw, q^{1-n}v/r; q)_k}\left(\frac{uq}{r}\right)^k\\
&= \frac{1}{b^n}\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw \end{matrix}; q, q\right). \quad \text{(by using (1.12))}
\end{aligned}$$

□

In the following theorem, we will demonstrate how to satisfy the q-difference equation (2.1):

Theorem 3.2. *For $a_0 = q^{-G}$, $G \in \mathbb{N}, b, w, r, u, v, a_i, b_j \in \mathbb{C}, i = 0, \dots, t-1, j = 1, \dots, s$, we have*

$$\begin{aligned}
&F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,b})\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\}\\
&= \frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, \dots, a_{t-1}; q)_n}{(q, b_1, \dots, b_s; q)_n}\left(\frac{c}{b}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw \end{matrix}; q, q\right),
\end{aligned} \tag{3.2}$$

provided $\max\{|bu|, |bv|\} < 1$.

Proof. Let $f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, c)$ be the right hand-side of (3.2),

$$\begin{aligned}
&(-q)^{1+s-t}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j} f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, cq^{j+t-s-1}) \\
&= (-q)^{1+s-t}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{cq^{j+t-s-1}}{b}\right)^n \\
&\quad\times\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw\end{matrix}; q, q\right) \\
&= \sum_{n=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-t} c^n\frac{1}{b^n}\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty} \\
&\quad\times{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw\end{matrix}; q, q\right)\sum_{j=0}^{s+1}(-1)^j B_j q^{j(n-1)} \\
&= \sum_{n=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-t} c^n D_{q,b}^n\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\}\prod_{j=0}^{s}(1-b_j q^{n-1}) \\
&= \sum_{n=1}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_{n-1}}{(q, b_1, \ldots, b_s; q)_{n-1}}\left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-t} c^n D_{q,b}^n\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\}\prod_{j=0}^{t-1}(1-a_j q^{n-1}) \\
&= \sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left[(-1)^{n+2} q^{\binom{n+1}{2}-n}\right]^{1+s-t} c^{n+1} D_{q,b}D_{q,b}^n\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\}\prod_{j=0}^{t-1}(1-a_j q^{n}) \\
&= c\sum_{n=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} c^n \\
&\quad\times D_{q,b}\left\{\frac{1}{b^n}\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw\end{matrix}; q, q\right)\right\}\sum_{j=0}^{t}(-1)^j A_j q^{jn} \\
&= c\sum_{j=0}^{t}(-1)^j A_j D_{q,b}\Bigg\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{cq^j}{b}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} \\
&\quad\times{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw\end{matrix}; q, q\right)\Bigg\} \\
&= c\sum_{j=0}^{t}(-1)^j A_j D_{q,b}\left\{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, cq^j)\right\}.
\end{aligned}$$

So, $f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, c)$ satisfies q-difference equation (2.1) and we get

$$f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, 0) = \frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}.$$

From Theorem 2.2, we get the required result. □

- Setting $r = 0$ in equation (3.2), we obtain

Corollary 3.3. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}, b, w, u, v, a_i, b_j \in \mathbb{C}, i = 0, 1, \ldots, t-1, j = 1, 2, \ldots, s$, *we have*

$$\begin{aligned}
&F(a_0, a_1, \ldots, a_{t-1}, b_1 \ldots, , b_s, cD_{q,b})\left\{\frac{(bw; q)_\infty}{(bu, bv; q)_\infty}\right\} \\
&= \frac{(bw; q)_\infty}{(bu, bv; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{c}{b}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}{}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ 0, bw\end{matrix}; q, q\right), \quad (3.3)
\end{aligned}$$

where $\max\{|bu|, |bv|\} < 1$.

- Setting $w = v = 0$ in (3.3), we get

Corollary 3.4. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}, b, u, a_i, b_j \in \mathbb{C}, i = 0, 1, \dots, t-1, j = 1, 2, \dots, s,$ *we have*

$$F(a_0, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,b}) \left\{ \frac{1}{(bu;q)_\infty} \right\} = \frac{1}{(bu;q)_\infty} {}_t\phi_s \left(\begin{array}{c} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s \end{array} ; q, cu \right), \quad (3.4)$$

provided $|bu| < 1$.

- Letting $t = s + 1$ in equation (3.3), we obtain Theorem 1.4 obtained by Fang [9] (equation (1.19)).
- Letting $t = s + 1$ in equation (3.4), we obtain Corollary 1.5 obtained by Fang [9] (equation (1.20)).

4 The Generalized q-Polynomials $\phi_n^{(A,B)}(b, c|q)$

In this section, we define a generalized q-polynomials $\phi_n^{(A,B)}(b, c|q)$. Using the q-difference equation technique, we establish the generating function for the polynomials $\phi_n^{(A,B)}(b, c|q)$.

We define a generalized q-polynomials $\phi_n^{(A,B)}(b, c|q)$ as follows:

$$\phi_n^{(A,B)}(b, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_0, \dots, a_{t-1}; q)_k}{(b_1, \dots, b_s; q)_k} c^k \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-t} P_{n-k}(b; y), \quad (4.1)$$

where $A = (a_0, \dots, a_{t-1})$, $B = (b_1, \dots, b_s)$.

- When $t = s + 1$ and $y = 0$, the generalized q-polynomials $\phi_n^{(A,B)}(b, c|q)$ achieve to the q-polynomials $H_n = H_n(a_0, \dots, a_s; b_1, \dots, b_s; b, c)$ created by Fang [9] (equation (1.21)).
- If $(t, s) = (1, 1)$ and $(a_0, b_1, b, y, c) = (a, 0, x, 0, y)$ in equation (4.1), we get the generalized Cauchy polynomials $P_n(x, y, a)$ mentioned by Srivastava and Arjika [28] (equation (1.25)).
- Taking $(t, s) = (1, 1)$ and $(a_0, b_1, b, y, c) = (a, 0, y, x, b)$ in equation (4.1), we get the polynomials $h_n(x, y, a, b|q)$ described by Srivastava and Arjika [28] (equation (1.26)).
- When $t = r$ and $(a_0, \dots, a_{t-1}) \to (a_1, \cdots, a_r)$ and $(b, y, c) = (x, 0, y)$, the generalized q-polynomials $\phi_n^{(A,B)}(b, c|q)$ achieve to the q-polynomials $h_n(a_1, \cdots, a_r; b_1, \cdots, b_s; x, y|q)$ created by Saad and Hassan [21] (equation (1.29)).

Lemma 4.1. *Let* $|bq/y| < 1$, *we have*

$$D_{q,b}^k \left\{ \frac{(bq^{1-n}/y; q)_\infty}{(bq/y; q)_\infty} \right\} = (-y)^{-n} q^{-\binom{n}{2}} \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(b; y), \quad (4.2)$$

where $|bq/y| < 1$.

Proof. Using equation (3.1), we get

$$\begin{aligned} D_{q,b}^k & \left\{ \frac{(bq^{1-n}/y; q)_\infty}{(bq/y; q)_\infty} \right\} \\ &= \frac{1}{b^k} \frac{(bq^{1-n}/y; q)_\infty}{(bq/y; q)_\infty} {}_2\phi_1 \left(\begin{array}{c} q^{-k}, bq/y \\ bq^{1-n}/y \end{array} ; q, q \right) \\ &= \frac{1}{(bq/y; q)_{-n}} \frac{(q^{-n}; q)_k}{(bq^{1-n}/y; q)_k} (q/y)^k \quad \text{(by using (1.8))} \end{aligned}$$

$$= \frac{(y/b;q)_n}{(-y/b)^n q^{\binom{n}{2}}} \frac{(q;q)_n(-1)^k q^{\binom{k}{2}-nk}}{(q;q)_{n-k}(bq^{1-n}/y;q)_k} (q/y)^k \quad \text{(by using (1.7) and (1.4))}$$

$$= \frac{(q;q)_n}{(q;q)_{n-k}} \frac{(y/b;q)_{n-k} b^{n-k}}{(-y)^n q^{\binom{n}{2}}} \quad \text{(by using (1.3))}$$

$$= (-y)^{-n} q^{-\binom{n}{2}} \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(b;y).$$

□

Theorem 4.2. *Let $\phi_n^{(A,B)}(b,c|q)$ be defined as in (4.1), then*

$$F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,b})\{P_n(b;y)\} = \phi_n^{(A,B)}(b,c|q).$$

Proof. Let $f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,c) = \phi_n^{(A,B)}(b,c|q)$

$$(-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,cq^{j+t-s-1})$$

$$= (-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_0,\ldots,a_{t-1};q)_k}{(b_1,\ldots,b_s;q)_k} \left(cq^{j+t-s-1}\right)^k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-t} P_{n-k}(b;y)$$

$$= \sum_{k=0}^{\infty} \frac{(a_0,a_1,\ldots,a_{t-1};q)_k}{(q,b_1,\ldots,b_s;q)_k} c^k \left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-t} \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(b;y) \sum_{j=0}^{s+1} (-1)^j B_j q^{j(k-1)}$$

$$= \sum_{k=0}^{\infty} \frac{(a_0,a_1,\ldots,a_{t-1};q)_k}{(q,b_1,\ldots,b_s;q)_k} c^k \left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-t} (-y)^n q^{\binom{n}{2}} D_{q,b}^k \left\{ \frac{(bq^{1-n}/y;q)_\infty}{(bq/y;q)_\infty} \right\}$$
$$\times \prod_{j=0}^{s} (1-b_j q^{k-1})$$

$$= \sum_{k=1}^{\infty} \frac{(a_0,a_1,\ldots,a_{t-1};q)_{k-1}}{(q,b_1,\ldots,b_s;q)_{k-1}} c^k \left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-t} (-y)^n q^{\binom{n}{2}} D_{q,b}^k \left\{ \frac{(bq^{1-n}/y;q)_\infty}{(bq/y;q)_\infty} \right\}$$
$$\times \prod_{j=0}^{t-1} (1-a_j q^{k-1})$$

$$= \sum_{k=0}^{\infty} \frac{(a_0,a_1,\ldots,a_{t-1};q)_k}{(q,b_1,\ldots,b_s;q)_k} c^{k+1} \left[(-1)^{k+2} q^{\binom{k+1}{2}-k}\right]^{1+s-t} (-y)^n q^{\binom{n}{2}} D_{q,b} D_{q,b}^k \left\{ \frac{(bq^{1-n}/y;q)_\infty}{(bq/y;q)_\infty} \right\}$$
$$\times \prod_{j=0}^{t-1} (1-a_j q^{k})$$

$$= c \sum_{k=0}^{\infty} \frac{(a_0,a_1,\ldots,a_{t-1};q)_k}{(q,b_1,\ldots,b_s;q)_k} c^k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-t} D_{q,b} \left\{ \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(b;y) \right\} \sum_{j=0}^{t} (-1)^j A_j q^{jk}$$
$$\text{(by using (4.2))}$$

$$= c \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_0,\ldots,a_{t-1};q)_k}{(b_1,\ldots,b_s;q)_k} \left(cq^j\right)^k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-t} P_{n-k}(b;y) \right\}$$

$$= c \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \left\{ f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,cq^j) \right\}.$$

So, $f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,c)$ satisfies the q-difference equation (2.1). Note that

$$f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,0) = P_n(b;y).$$

From Theorem 2.2, we get the desired result. □

Theorem 4.3. (Generating Function for $\phi_n^{(A,B)}(b,c|q)$). *If* $a_0 = q^{-G}$, $G \in \mathbb{N}$, *and* $|bv| < 1$, *then*

$$\sum_{n=0}^{\infty} \phi_n^{(A,B)}(b,c|q) \frac{v^n}{(q;q)_n} = \frac{(yv;q)_\infty}{(bv;q)_\infty} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cv \right). \tag{4.3}$$

Proof. Put

$$f_L = f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s; b; c) = \sum_{n=0}^{\infty} \phi_n^{(A,B)}(b,c|q) \frac{v^n}{(q;q)_n}.$$

We can check f_L satisfies the q-difference equation (2.1) by the same technique used in Theorem 4.2. So

$$\begin{aligned} f_L &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, 0)\} \\ &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \sum_{n=0}^{\infty} P_n(b,y) \frac{v^n}{(q;q)_n} \right\} \\ &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{(yv;q)_\infty}{(bv;q)_\infty} \right\} \quad \text{(by using (1.13))} \\ &= (yv;q)_\infty F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{1}{(bv;q)_\infty} \right\} \\ &= \frac{(yv;q)_\infty}{(bv;q)_\infty} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cv \right). \quad \text{(by using (3.4))} \end{aligned}$$

□

- When $t = s + 1$, $y = 0$ and $v = u$ in (4.3), we obtain the generating function for the q-polynomials $H_n = H_n(a_0, \ldots, a_s; b_1, \ldots, b_s; b, c)$ created by Fang [9] (equation (1.22)).
- If $(t,s) = (1,1)$ and $(a_0, b_1, b, y, c, v) = (a, 0, x, 0, y, t)$ in equation (4.3), we get the generating function for the q-polynomials $P_n(x,y,a)$ provided by Srivastava and Arjika [28] (equation (1.25)).
- Taking $(t,s) = (1,1)$ and $(a_0, b_1, b, y, c, v) = (a, 0, y, x, b, t)$ in equation (4.3), we get the generating function for the q-polynomials $h_n(x, y, a, b|q)$ described by Srivastava and Arjika [28] (equation (1.27)).
- When $t = r$ and $(a_0, \ldots, a_{t-1}) \to (a_1, \cdots, a_r)$ and $(b, y, c, v) = (x, 0, y, t)$ in equation (4.3), we get the generating function of the q-polynomials $h_n(a_1, \cdots, a_r; b_1, \cdots, b_s; x, y|q)$ obtained by Saad and Hassan [21] (equation (1.30)).

5 Two Types Generating Functions for the Polynomials $\phi_n^{(A,B)}(b,c|q)$

In this section, we introduce Srivastava-Agarwal type generating function and another type generating function for $\phi_n^{(A,B)}(b,c|q)$.

Lemma 5.1. *If* $a_0 = q^{-G}$, $G \in \mathbb{N}$, $|bv| < 1$, *then*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(bq^{1-n}/y;q)_n (yv\beta)^n q^{2\binom{n}{2}}}{(q, yv;q)_n} \sum_{m=0}^{\infty} \frac{(a_0, \ldots, a_{t-1};q)_m}{(q, b_1, \ldots, b_s;q)_m} \left(\frac{c}{b}\right)^m \left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t} \\ &\times {}_3\phi_2 \left(\begin{matrix} q^{-m}, bv, bq/y \\ bq^{1-n}/y, 0 \end{matrix} ; q, q \right) \\ &= (\beta;q)_\infty \sum_{n=0}^{\infty} \frac{(bv;q)_n}{(yv;q)_n} \frac{\beta^n}{(q;q)_n} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right). \end{aligned} \tag{5.1}$$

Proof. By using (1.2) and (1.5), we get

$$\sum_{n=0}^{\infty}\frac{(bq^{1-n}/y;q)_n(yv\beta)^n q^{2\binom{n}{2}}}{(q,yv;q)_n}\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}$$

$$\times\sum_{k=0}^{m}\frac{(q^{-m},bv,bq/y;q)_k}{(q,bq^{1-n}/y;q)_k}q^k$$

$$=\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m},bv;q)_k}{(q;q)_k}q^k$$

$$\times\sum_{n=0}^{\infty}\frac{(\beta yv)^n q^{2\binom{n}{2}}}{(q,yv;q)_n}\frac{(bq^{1-n}/y;q)_n(bq/y;q)_k}{(bq^{1-n}/y;q)_k}$$

$$=\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m},bv;q)_k}{(q;q)_k}q^k$$

$$\times\sum_{n=0}^{\infty}\frac{(-b\beta v)^n q^{\binom{n}{2}}}{(q,yv;q)_n}\frac{(y/b;q)_n(bq/y;q)_k}{(bq^{1-n}/y;q)_k}$$

$$=\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m},bv;q)_k}{(q;q)_k}q^k$$

$$\times\sum_{n=0}^{\infty}\frac{(yq^{-k}/b;q)_n}{(q,yv;q)_n}(-b\beta vq^k)^n q^{\binom{n}{2}}$$

$$=\frac{(bv;q)_\infty}{(yv;q)_\infty}\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m};q)_k}{(q;q)_k}q^k$$

$$\times\sum_{n=0}^{\infty}\frac{(yq^{-k}/b,\beta;q)_n}{(q;q)_n}(bvq^k)^n\quad\text{(by using (1.11))}$$

$$=(\beta;q)_\infty\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m},bv;q)_k}{(q;q)_k}q^k$$

$$\times\sum_{n=0}^{\infty}\frac{(bvq^k;q)_n}{(q,yv;q)_n}\beta^n\quad\text{(by using (1.9))}$$

$$=(\beta;q)_\infty\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{n=0}^{\infty}\frac{(bv;q)_n}{(q,yv;q)_n}\beta^n$$

$$\times\sum_{k=0}^{m}\frac{(q^{-m},bvq^n;q)_k}{(q;q)_k}q^k\quad\text{(by using (1.6))}$$

$$=(\beta;q)_\infty\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{n=0}^{\infty}\frac{(bv;q)_n}{(q,yv;q)_n}\beta^n(bvq^n)^m$$

$$\text{(by using (1.8))}$$

$$=(\beta;q)_\infty\sum_{n=0}^{\infty}\frac{(bv;q)_n}{(q,yv;q)_n}\beta^n\,{}_t\phi_s\left(\begin{matrix}a_0,a_1,\ldots,a_{t-1}\\ b_1,\ldots,b_s\end{matrix};q,cvq^n\right).$$

□

Theorem 5.2. (Srivastava-Agarwal type generating function for $\phi_n^{(A,B)}(b,c|q)$). *If* $a_0=q^{-G}$, $G\in\mathbb{N}$ *and* $|bv|<1$, *then*

$$\sum_{n=0}^{\infty}\phi_n^{(A,B)}(b,c|q)(\beta;q)_n\frac{v^n}{(q;q)_n}$$

$$= \frac{(yv,\beta;q)_\infty}{(bv;q)_\infty} \sum_{n=0}^{\infty} \frac{(bv;q)_n}{(yv;q)_n} \frac{\beta^n}{(q;q)_n} \, {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right). \tag{5.2}$$

Proof. Put

$$f_L = f_L(a_0, \ldots, a_{t-1}, b_1, \ldots, b_s; b; c) = \sum_{n=0}^{\infty} \phi_n^{(A,B)}(b, c|q)(\beta;q)_n \frac{v^n}{(q;q)_n}.$$

We can check f_L satisfy the q-difference equation (2.1) by the same way used in Theorem 4.2. Hence

$$f_L = F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, 0) \right\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \sum_{n=0}^{\infty} P_n(b,y)(\beta;q)_n \frac{v^n}{(q;q)_n} \right\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \sum_{n=0}^{\infty} \frac{(y/b,\beta;q)_n}{(q;q)_n} (bv)^n \right\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{(yv;q)_\infty}{(bv;q)_\infty} \sum_{n=0}^{\infty} \frac{(y\beta v)^n q^{2\binom{n}{2}} (bq^{1-n}/y;q)_n}{(q,yv;q)_n} \right\}$$

(by using (1.10))

$$= (yv;q)_\infty \sum_{n=0}^{\infty} \frac{(y\beta v)^n q^{2\binom{n}{2}}}{(q,yv;q)_n} F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{(bq^{1-n}/y;q)_\infty}{(bv, bq/y;q)_\infty} \right\}$$

$$= \frac{(yv;q)_\infty}{(bv;q)_\infty} \sum_{n=0}^{\infty} \frac{(bq^{1-n}/y;q)_n (y\beta v)^n q^{2\binom{n}{2}}}{(q,yv;q)_n} \sum_{m=0}^{\infty} \frac{(a_0, \ldots, a_{t-1};q)_m}{(q, b_1, \ldots, b_s;q)_m} \left(\frac{c}{b}\right)^m \left[(-1)^m q^{\binom{m}{2}} \right]^{1+s-t}$$

$$\times \, {}_3\phi_2 \left(\begin{matrix} q^{-m}, bv, bq/y \\ 0, bq^{1-n}/y \end{matrix} ; q, q \right) \quad \text{(by using (3.3))}$$

$$= \frac{(yv,\beta;q)_\infty}{(bv;q)_\infty} \sum_{n=0}^{\infty} \frac{(bv;q)_n}{(yv;q)_n} \frac{\beta^n}{(q;q)_n} \, {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right). \quad \text{(by using (5.1))}$$

□

- When $t = s + 1$ and $y = 0$ in equation (5.2), we get Srivastava-Agarwal type generating function for $H_n = H_n(a_0, \ldots, a_s; b_1, \ldots, b_s; b, c)$.

  Corollary 5.3. *If* $\max\{|bv|, |cv|\} < 1$, *then*

$$\sum_{n=0}^{\infty} H_n(\beta;q)_n \frac{v^n}{(q;q)_n} = \frac{(\beta;q)_\infty}{(bv;q)_\infty} \sum_{n=0}^{\infty} (bv;q)_n \frac{\beta^n}{(q;q)_n} \, {}_{s+1}\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right).$$

- If $(t, s) = (1, 1)$ and $(a_0, b_1, b, y, c) = (a, 0, x, 0, y)$ in equation (5.2), we get Srivastava-Agarwal type generating function for $P_n(x, y, a)$.

  Corollary 5.4. *For* $|xv| < 1$, *we have*

$$\sum_{n=0}^{\infty} P_n(x,y,a)(\beta;q)_n \frac{v^n}{(q;q)_n} = \frac{(\beta;q)_\infty}{(xv;q)_\infty} \sum_{n=0}^{\infty} (xv;q)_n \frac{\beta^n}{(q;q)_n} \, {}_1\phi_1 \left(\begin{matrix} a \\ 0 \end{matrix} ; q, yvq^n \right).$$

- Taking $(t, s) = (1, 1)$ and $(a_0, b_1, b, y, c) = (a, 0, y, x, b)$ in equation (5.2), we get Srivastava-Agarwal type generating function for $h_n(x, y, a, b|q)$.

Corollary 5.5. *If* $|yv|<1$, *then*

$$\sum_{n=0}^{\infty} h_n(x,y,a,b|q)(\beta;q)_n \frac{v^n}{(q;q)_n} = \frac{(xv,\beta;q)_\infty}{(yv;q)_\infty}\sum_{n=0}^{\infty}\frac{(yv;q)_n}{(xv;q)_n}\frac{\beta^n}{(q;q)_n}\,{}_1\phi_1\left(\begin{array}{c} a \\ 0 \end{array};q,bvq^n\right).$$

- When $t=r$ and $(a_0,\ldots,a_{t-1})\to(a_1,\cdots,a_r)$ and $(b,y,c)=(x,0,y)$ in equation (5.2), we get Srivastava-Agarwal type generating function for $h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$.

Corollary 5.6. *If* $a_1=q^{-G}$, $G\in\mathbb{N}$ *and* $|xv|<1$, *then*

$$\sum_{n=0}^{\infty} h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)(\beta;q)_n\frac{v^n}{(q;q)_n}$$
$$= \frac{(\beta;q)_\infty}{(xv;q)_\infty}\sum_{n=0}^{\infty}(xv;q)_n\frac{\beta^n}{(q;q)_n}\,{}_r\phi_s\left(\begin{array}{c} a_0,a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,yvq^n\right).$$

Lemma 5.7. *If* $a_0=q^{-G}$, $G\in\mathbb{N}$ *and* $\max\{|bv|,|\lambda|\}<1$, *then*

$$(bv\beta;q)_\infty\sum_{n=0}^{\infty}\frac{(\beta,b\lambda/y;q)_n}{(\lambda,bv\beta;q)_n}\frac{(-1)^n q^{\binom{n}{2}}(yv)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}$$
$$\times\,{}_3\phi_2\left(\begin{array}{c} q^{-m},bv,b\lambda q^n/y \\ b\lambda/y,bv\beta q^n \end{array};q,q\right)$$
$$=\frac{(yv,\beta;q)_\infty}{(\lambda;q)_\infty}\sum_{n=0}^{\infty}\frac{(\lambda/\beta,bv;q)_n}{(yv;q)_n}\frac{\beta^n}{(q;q)_n}\,{}_t\phi_s\left(\begin{array}{c} a_0,a_1,\ldots,a_{t-1} \\ b_1,\ldots,b_s \end{array};q,cvq^n\right). \tag{5.3}$$

Proof. Employing identities (1.6) and (1.11), we get

$$(bv\beta;q)_\infty\sum_{n=0}^{\infty}\frac{(\beta,b\lambda/y;q)_n}{(q,\lambda,bv\beta;q)_n}(-yv)^n q^{\binom{n}{2}}\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}$$
$$\times\sum_{k=0}^{m}\frac{(q^{-m},bv,b\lambda q^n/y;q)_k}{(q,b\lambda/y,bv\beta q^n;q)_k}q^k$$
$$=(bv\beta;q)_\infty\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m},bv;q)_k}{(q,bv\beta;q)_k}q^k$$
$$\times\sum_{n=0}^{\infty}\frac{(\beta,b\lambda q^k/y;q)_n}{(q,\lambda,bv\beta q^k;q)_n}(-yv)^n q^{\binom{n}{2}}$$
$$=(bv\beta;q)_\infty\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m},bv;q)_k}{(q,bv\beta;q)_k}q^k\frac{(bvq^k;q)_\infty}{(bv\beta q^k;q)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{(\beta,yq^{-k}/b;q)_n}{(q,\lambda;q)_n}(bvq^k)^n$$
$$=(bv;q)_\infty\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}\sum_{k=0}^{m}\frac{(q^{-m};q)_k}{(q;q)_k}q^k\frac{(\beta,yv;q)_\infty}{(\lambda,bvq^k;q)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{(\lambda/\beta,bvq^k;q)_n}{(q,yv;q)_n}\beta^n \quad \text{(by using (1.9))}$$
$$=\frac{(\beta,yv;q)_\infty}{(\lambda;q)_\infty}\sum_{n=0}^{\infty}\frac{(\lambda/\beta,bv;q)_n}{(q,yv;q)_n}\beta^n\sum_{m=0}^{\infty}\frac{(a_0,\ldots,a_{t-1};q)_m}{(q,b_1,\ldots,b_s;q)_m}\left(\frac{c}{b}\right)^m\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}$$

$$\times \sum_{k=0}^{m} \frac{(q^{-m}, bvq^n; q)_k}{(q;q)_k} q^k \qquad \text{(by using (1.6))}$$

$$= \frac{(\beta, yv; q)_\infty}{(\lambda; q)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda/\beta, bv; q)_n}{(q, yv; q)_n} \beta^n \sum_{m=0}^{\infty} \frac{(a_0, \ldots, a_{t-1}; q)_m}{(q, b_1, \ldots, b_s; q)_m} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t} (cvq^n)^m$$

$$= \frac{(yv, \beta; q)_\infty}{(\lambda; q)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda/\beta, bv; q)_n}{(yv; q)_n} \frac{\beta^n}{(q;q)_n} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right).$$

□

Theorem 5.8. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}$ *and* $\max\{|bv|, |\lambda|\} < 1$, *we have*

$$\sum_{n=0}^{\infty} \phi_n^{(A,B)}(b, c|q) \frac{(\beta; q)_n}{(\lambda; q)_n} \frac{v^n}{(q;q)_n}$$
$$= \frac{(yv, \beta; q)_\infty}{(bv, \lambda; q)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda/\beta, bv; q)_n}{(yv; q)_n} \frac{\beta^n}{(q;q)_n} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right). \tag{5.4}$$

Proof. Let $f_L = f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s; a; c)$ =LHS of equation (5.4). By using the same technique used in Theorem 4.2 to check f_L satisfies q-difference equation (2.1), we have

$$f_L = F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, 0)\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \sum_{n=0}^{\infty} P_n(b, y) \frac{(\beta; q)_n}{(\lambda; q)_n} \frac{v^n}{(q;q)_n} \right\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \sum_{n=0}^{\infty} \frac{(y/b, \beta; q)_n}{(\lambda; q)_n} \frac{(bv)^n}{(q;q)_n} \right\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{(bv\beta; q)_\infty}{(bv; q)_\infty} \sum_{n=0}^{\infty} \frac{(\beta, b\lambda/y; q)_n}{(\lambda, bv\beta; q)_n} \frac{(-1)^n q^{\binom{n}{2}} (yv)^n}{(q;q)_n} \right\}$$
$$\text{(by using (1.11))}$$

$$= \sum_{n=0}^{\infty} \frac{(\beta; q)_n}{(\lambda, ; q)_n} \frac{(-1)^n q^{\binom{n}{2}} (yv)^n}{(q;q)_n} F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{(b\lambda/y, bv\beta q^n; q)_\infty}{(bv, b\lambda q^n/y; q)_\infty} \right\}$$

$$= \frac{(bv\beta; q)_\infty}{(bv; q)_\infty} \sum_{n=0}^{\infty} \frac{(\beta, b\lambda/y; q)_n (-yv)^n q^{\binom{n}{2}}}{(q\lambda, bv\beta; q)_n} \sum_{m=0}^{\infty} \frac{(a_0, \ldots, a_{t-1}; q)_m}{(q, b_1, \ldots, b_s; q)_m} \left(\frac{c}{b}\right)^m \left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t}$$

$$\times {}_3\phi_2 \left(\begin{matrix} q^{-m}, bv, b\lambda q^n/y \\ b\lambda/y, bv\beta q^n \end{matrix} ; q, q \right) \qquad \text{(by using (3.2))}$$

$$= \frac{(yv, \beta; q)_\infty}{(bv, \lambda; q)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda/\beta, bv; q)_n}{(yv; q)_n} \frac{\beta^n}{(q;q)_n} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right). \qquad \text{(by using (5.3))}$$

This patently completes the proof of Theorem 5.8. □

- When $t = s + 1$ and $y = 0$ in equation (5.4), we obtain

  Corollary 5.9. *If* $\max\{|bv|, |\lambda|, |cv|\} < 1$, *then*

$$\sum_{n=0}^{\infty} H_n \frac{(\beta; q)_n}{(\lambda; q)_n} \frac{v^n}{(q;q)_n}$$
$$= \frac{(\beta; q)_\infty}{(bv, \lambda; q)_\infty} \sum_{n=0}^{\infty} (\lambda/\beta, bv; q)_n \frac{\beta^n}{(q;q)_n} {}_{s+1}\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right).$$

- If $(t,s)=(1,1)$ and $(a_0,b_1,b,y,c)=(a,0,x,0,y)$ in equation (5.4), we get

Corollary 5.10. *For* $\max\{|xv|,|\lambda|\}<1$*, we have*

$$\sum_{n=0}^{\infty} P_n(x,y,a)\frac{(\beta;q)_n}{(\lambda;q)_n}\frac{v^n}{(q;q)_n}=\frac{(\beta;q)_\infty}{(xv,\lambda;q)_\infty}\sum_{n=0}^{\infty}(\lambda/\beta,xv;q)_n\frac{\beta^n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} a \\ 0\end{matrix};q,yvq^n\right).$$

- Taking $(t,s)=(1,1)$ and $(a_0,b_1,b,y,c)=(a,0,y,x,b)$ in equation (5.4), we gain

Corollary 5.11. *If* $\max\{|yv|,|\lambda|\}<1$*, then*

$$\sum_{n=0}^{\infty} h_n(x,y,a,b|q)\frac{(\beta;q)_n}{(\lambda;q)_n}\frac{v^n}{(q;q)_n}$$
$$=\frac{(xv,\beta;q)_\infty}{(yv,\lambda;q)_\infty}\sum_{n=0}^{\infty}(\lambda/\beta,yv;q)_n\frac{\beta^n}{(q,xv;q)_n}\,{}_1\phi_1\left(\begin{matrix} a \\ 0\end{matrix};q,bvq^n\right).$$

- When $t=r$ and $(a_0,\ldots,a_{t-1})\to(a_1,\cdots,a_r)$ and $(b,y,c)=(x,0,y)$ in equation (5.4), we acquire

Corollary 5.12. *If* $a_1=q^{-G}$*,* $G\in\mathbb{N}$ *and* $\max\{|xv|,|\lambda|\}<1$*, then*

$$\sum_{n=0}^{\infty} h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)\frac{(\beta;q)_n}{(\lambda;q)_n}\frac{v^n}{(q;q)_n}$$
$$=\frac{(\beta;q)_\infty}{(xv,\lambda;q)_\infty}\sum_{n=0}^{\infty}(\lambda/\beta,xv;q)_n\frac{\beta^n}{(q;q)_n}\,{}_r\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,yvq^n\right).$$

6 Generalizations of the transformational identity involving generating function for $\phi_n^{(A,B)}(b,c|q)$

In this section, we use the method of q-difference equation to deduce a generalization of the transformational identity.

Theorem 6.1. *Let* $K(n)$ *and* $J(n)$ *satisfy*

$$\sum_{n=0}^{\infty} K(n)P_n(b;y)=\frac{(yv;q)_\infty}{(bv;q)_\infty}\sum_{n=0}^{\infty}J(n)\frac{(bv;q)_n}{(yv;q)_n}. \tag{6.1}$$

Then

$$\sum_{n=0}^{\infty} K(n)\phi_n^{(A,B)}(b,c|q)=\frac{(yv;q)_\infty}{(bv;q)_\infty}\sum_{n=0}^{\infty}J(n)\frac{(bv;q)_n}{(yv;q)_n}\,{}_t\phi_s\left(\begin{matrix} a_0,a_1,\ldots,a_{t-1} \\ b_1,\ldots,b_s\end{matrix};q,cvq^n\right),$$

provided that $s\geqslant t-1$ *and* $|bv|<1$.

Proof. Put

$$f_L=f_L(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s;b;c)=\sum_{n=0}^{\infty}K(n)\phi_n^{(A,B)}(b,c|q).$$

We can check f_L satisfy q-difference equation (2.1) by the same way in Theorem 4.2. Hence we have

$$f_L=F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,b})\{f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,b,0)\}$$
$$=F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,eD_{q,b})\left\{\sum_{n=0}^{\infty}K(n)P_n(b;y)\right\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, eD_{q,b}) \left\{ \frac{(yv;q)_\infty}{(bv;q)_\infty} \sum_{n=0}^{\infty} J(n) \frac{(bv;q)_n}{(yv;q)_n} \right\} \quad \text{(by using (6.1))}$$

$$= (yv;q)_\infty \sum_{n=0}^{\infty} J(n) \frac{1}{(yv;q)_n} F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, eD_{q,b}) \left\{ \frac{1}{(bvq^n;q)_\infty} \right\}$$

$$= \frac{(yv;q)_\infty}{(bv;q)_\infty} \sum_{n=0}^{\infty} J(n) \frac{(bv;q)_n}{(yv;q)_n} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right). \quad \text{(by using (3.4))}$$

□

- When $t = s + 1$ and $y = 0$ in Theorem 6.1, we gain

Corollary 6.2. *Let $K(n)$ and $J(n)$ satisfy*

$$\sum_{n=0}^{\infty} K(n) b^n = \frac{1}{(bv;q)_\infty} \sum_{n=0}^{\infty} J(n)(bv;q)_n.$$

Then

$$\sum_{n=0}^{\infty} K(n) H_n = \frac{1}{(bv;q)_\infty} \sum_{n=0}^{\infty} J(n)(bv;q)_n \, {}_{s+1}\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, \ldots, b_s \end{matrix} ; q, cvq^n \right),$$

provided that $\max\{|bv|, |cv|\} < 1$.

- If $(t,s) = (1,1)$ and $(a_0, b_1, b, y, c) = (a, 0, x, 0, y)$ Theorem 6.1, we get

Corollary 6.3. *Let $K(n)$ and $J(n)$ satisfy*

$$\sum_{n=0}^{\infty} K(n) x^n = \frac{1}{(xv;q)_\infty} \sum_{n=0}^{\infty} J(n)(xv;q)_n.$$

Then

$$\sum_{n=0}^{\infty} K(n) P_n(x,y,a) = \frac{1}{(xv;q)_\infty} \sum_{n=0}^{\infty} J(n)(xv;q)_n \, {}_1\phi_1 \left(\begin{matrix} a \\ 0 \end{matrix} ; q, yvq^n \right),$$

where $|xv| < 1$.

- Taking $(t,s) = (1,1)$ and $(a_0, b_1, b, y, c) = (a, 0, y, x, b)$ Theorem 6.1, we obtain

Corollary 6.4. *Let $K(n)$ and $J(n)$ satisfy*

$$\sum_{n=0}^{\infty} K(n) P_n(y,x) = \frac{(xv;q)_\infty}{(yv;q)_\infty} \sum_{n=0}^{\infty} J(n) \frac{(yv;q)_n}{(xv;q)_n}.$$

Then

$$\sum_{n=0}^{\infty} K(n) h_n(x,y,a,b|q) = \frac{(xv;q)_\infty}{(yv;q)_\infty} \sum_{n=0}^{\infty} J(n) \frac{(yv;q)_n}{(xv;q)_n} \, {}_1\phi_1 \left(\begin{matrix} a \\ 0 \end{matrix} ; q, bvq^n \right),$$

provided $|yv| < 1$.

- When $t = r$ and $(a_0, \ldots, a_{t-1}) \to (a_1, \cdots, a_r)$ and $(b, y, c) = (x, 0, y)$ Theorem 6.1, we gain

Corollary 6.5. *Let $K(n)$ and $J(n)$ satisfy*

$$\sum_{n=0}^{\infty} K(n)x^n = \frac{1}{(xv;q)_\infty}\sum_{n=0}^{\infty} J(n)(xv;q)_n.$$

Then

$$\sum_{n=0}^{\infty}K(n)h_n(a_1,\cdots,a_r;b_1,\cdots,b_s;x,y|q)$$
$$=\frac{1}{(xv;q)_\infty}\sum_{n=0}^{\infty}J(n)(xv;q)_n\ {}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,yvq^n\right),$$

where $|xv|<1$.

7 Conclusion remarks

We solve a generalized q-difference equation and state the solution in the q-operator form, which is a generalization of Fang's result [9]. We give some identities that generalize the findings found in [9] for the q-operator $F(a_0,a_1\ldots,a_{t-1},b_1\ldots,b_s,cD_{q,b})$. $\phi_n^{(A,B)}(b,c|q)$ are defined as generalized q-polynomials. To get the generating function of the polynomials $\phi_n^{(A,B)}(b,c|q)$, we apply the q-difference equation technique. For $\phi_n^{(A,B)}(b,c|q)$, we introduce the Srivastava-Agarwal type generating function and another type generating function. A generalization of the transformational identity is derived by means of the q-difference equation approach.

References

[1] S.A. Abdul-Ghani and H.L. Saad, *Applications of q-difference equation and homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$ in q-polynomials*, Partial Differ. Equ. Appl. Math., **8**, (2023) https://doi.org/10.1016/j.padiff.2023.100536.

[2] S.A. Abdul-Ghani and H.L. Saad, *The generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$ for q-identities and q-integrals*, Iraqi J. Science, **64(11)**, 5815-5829, (2023).

[3] S. Arjika, *q-Difference equation for homogeneous q-difference operators and their applications*, J. Differ. Equ. Appl., **26(7)**, 987-999, (2020).

[4] J. Cao, *A note on generalized q-difference equations for q-beta and Andrews-Askey integral*, J. Math. Anal. Appl., **412**, 841-851, (2014).

[5] J. Cao, B. Xu and S. Arjika, *A note on generalized q-difference equations for general Al-Salam-Carlitz polynomials*, Adv. Difference Equ., **2020**, 1-17, (2020).

[6] W.Y.C. Chen and N.S.S. Gu, *The Cauchy operator for basic hypergeometric series*, Adv. in Appl. Math., **41**, 177-196, (2008).

[7] W.Y.C. Chen and Z.G. Liu, *Parameter augmenting for basic hypergeometric series*, II, J. Combin. Theory, Ser. A, **80**, 175–195, (1997).

[8] W.Y.C. Chen and H.L. Saad, *On the Gosper–Petkovšek representation of rational functions*, J. Symbolic Computation, **40**, 955–963, (2005).

[9] J.P. Fang, *Applications of a generalized q-difference equation*, Adv. Difference Equ., **2014**, 1-19, (2014).

[10] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2^{nd} ed., Cambridge University Press, Cambridge, MA, (2004).

[11] I.P. Goulden and D.M. Jackson, *Combinatorial Enumeration*, John Wiley, New York, (1983).

[12] R.H. Jaber and H.L. Saad, *Applications of the finite Operator* ${}_3\mathscr{E}_2\left(\begin{matrix}q^{-N},a,b\\ c,d\end{matrix};q,-f\theta\right)$ *for the polynomials* $B_n(a,b,c,d,f,x,y|q)$, Iraqi J. Science, **64(2)**, 783-797, (2023).

[13] N.N. Li and W. Tan, *Two generalized q-exponential operators and their applications*, Adv. Differ. Equ., **2016**, 1-14, (2016).

[14] D.-Q. Lu, *q-Difference equation and the Cauchy operator identities*, J. Math. Anal. Appl., **359**, 265-274, (2009).

[15] B. Malgrange, *Lectures on the Theory of Functions of Several Complex Variables*, Springer, Berlin, (1984).

[16] F. Masood, T.M. Al-shami and H. El-Metwally, *Solving some partial q-differential equations using transformation methods*, Palest. J. Math., **12(1)**, 937-946, (2023).

[17] P. Nandini, M. R. Salestina and S. Latha, *Faber polynomial coefficient estimates for certain certain subclass of bi-univalent functions associated with (p, q)-ruscheweyh derivative*, Palest. J. Math., **11(3)**, 133-146, (2022).

[18] H. Orhan, S. Porwal and N. Magesh, *The Fekete-Szegō problem for a generalized class of analytic functions of complex order associated with q-calculus*, Palest. J. Math., **11(3)**, 39-47, (2022).

[19] S. Roman, *More on the umbral calculus, with emphasis on the q-umbral caculus*, J. Math. Anal. Appl., **107**, 222–254, (1985).

[20] H.L. Saad and M.A. Abdlhusein, *New application of the Cauchy operator on the homogeneous Rogers-Szegö polynomials*, The Ramanujan Journal, **56**, 347-367, (2021).

[21] H.L. Saad and H.J. Hassan, *Applications of the operator ${}_r\Phi_s$ in q-identities*, Basrah J. Science, **39(1)**, 20-36, (2021).

[22] H.L. Saad and H.J. Hassan, *Applications of the operator ${}_r\Phi_s$ in q-polynomials*, TWMS J. App. and Eng. Math., **13(2)**, 696-709, (2023).

[23] H.L. Saad and R.H. Jaber, *Application of the Operator $\phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,fD_q\right)$ for the Polynomials $Y_n(a,b,c;d,e;x,y|q)$*, TWMS J. App. and Eng. Math., **12(2)**, 691-702, (2022).

[24] H.L. Saad and S.M. Khalaf, *The generalized q-operator ${}_r\Phi_s$ and its applications in q-identities*, TWMS J. App. and Eng. Math., **13(3)**, 1137-1149, (2023).

[25] H.L. Saad and F.A. Reshem, *The operator $\mathbb{S}(a,b;\theta_x)$ for the polynomials $Z_n(x,y,a,b|q)$*, Iraqi J. Science, **63(10)**, 4397-4409, (2022).

[26] F.A. Reshem and H.L. Saad, *Generalized q-difference equation of the generalized q-operator ${}_r\Phi_s(\theta)$ and its application in q-integrals*, Partial Differ. Equ. Appl. Math., **7** (2023) https://doi.org/10.1016/j.padiff.2023.100517.

[27] H.L. Saad and A.A. Sukhi, *The q-exponential operators*, Appl. Math. Sci., **128**, 6369-6380, (2013).

[28] H.M. Srivastava, S. Arjika, A.S. Kelil, *Some homogeneous q-difference operators and the associated generalized Hahn polynomials*, Appl. Set-Valued Anal. Optim., **1**, 187–201, (2019).

[29] A. Sudhakar, B. Mallik, B.P. Padhy and U. Misra, *Quantum calculus and norlund integrability*, Palest. J. Math., **12(1)**, 352-360, (2023).

[30] Z.Z. Zhang and J. Wang, *Two operator identities and their applications to terminating basic hypergeometric series and q-integrals*, J. Math. Anal. Appl., **312**, 653–665, (2005).

Author information

Faiz A. Reshem, Department of Mathematics, College of Science, University of Basrah, Iraq.
E-mail: fa7786@yahoo.com

Husam L. Saad, Department of Mathematics, College of Science, University of Basrah, Iraq.
E-mail: hus6274@hotmail.com

Received: 2022-10-20
Accepted: 2024-01-07