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Spherical Fourier transform on $(4n+3)$-dimensional quaternionic Heisenberg group

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Abstract In this paper, we determine the bounded spherical functions associated with the Gelfand pair formed by the $(4n+3)$-dimensional quaternionic Heisenberg group $\mathcal{H}_q^n$ and the quaternionic unitary group $Sp(n)$. We define the spherical Fourier transform and prove the Plancherel and inversion formulas. Additionally, we determine the heat kernel associated with the sub-Laplacian on $\mathcal{H}_q^n$.

1 Introduction

The quaternionic Heisenberg group $\mathbb{H}^n \times \mathbb{R}^3$ was first introduced by Barker and Salamon in their paper [2], is a two-step nilpotent Lie group with a dimension of $4n+3$. This group is an H-type group; we can see [11] for the properties of the H-type group, and it can be regarded as the nilpotent part in Iwasawa decomposition of Lorentz group $Sp(1, n+1)$ (see [7] p. 375).

The quaternionic Heisenberg group plays an essential role in several branches of mathematics and physics, including harmonic analysis, representations theory, partial differential equations, and quantum mechanics.

Several authors [1, 9] have studied some properties of the quaternionic Heisenberg group of dimension 7. This paper aims to determine the expressions of the bounded spherical functions associated with the Gelfand pair formed by the $(4n+3)$-dimensional quaternionic Heisenberg group and the quaternionic unitary group $Sp(n)$ and using the spherical Fourier transform to determine an integral representation of the heat kernel associated with the sub-Laplacian on $\mathcal{H}_q^n$.

The remaining part of the paper is organized as follows. In Section 2, we give the necessary definitions and properties of the $(4n+3)$-dimensional quaternionic Heisenberg group, the Gelfand pair, the spherical functions, and their relation to representations. In section 3, we construct the bounded spherical functions associated with the Gelfand pair formed by the $(4n+3)$-dimensional quaternionic Heisenberg group $\mathcal{H}_q^n$ and the quaternionic unitary group $Sp(n)$. In Section 4, we define the spherical Fourier transform and we give the Plancherel and the inversion formulas. Finally, we determine the heat kernel of the $(4n+3)$-dimensional quaternionic Heisenberg group.

2 Preliminaries

2.1 The quaternionic Heisenberg group and its representations

Quaternion algebra $\mathbb{H}$, initially introduced by Hamilton in 1843, is an associative, non-commutative, and division algebra. It is constructed with a basis consisting of elements $(1, i, j, k)$ and possesses the following properties:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Let $q = q_0 + q_1 i + q_2 j + q_3 k$ a quaternion where q_0, q_1, q_2, q_3 are real numbers.

- The real and imaginary parts of q are $Re\ q = q_0$, and $Im\ q = q_1 i + q_2 j + q_3 k$.
- We will identify $Im\ q$ with the triplet (q_1, q_2, q_3) of $\mathbb{R}^3$.
- The conjugate and module (norm) of q are defined by

$$\overline{q} = q_0 - q_1 i - q_2 j - q_3 k, \quad |q|^2 = q\overline{q} = \sum_{p=0}^{3} q_p^2$$

satisfying

$$\overline{qq'} = \overline{q'}\,\overline{q}, \quad |qq'| = |q||q'|.$$

For q and w in $\mathbb{H}^n$, let us define the scalar product of q and w by

$$(q, w) = \overline{w_1} q_1 + ... + \overline{w_n} q_n,$$

so that if a and b are two numbers of $\mathbb{H}$ then $(qa, qb) = \overline{b}(q, w)a$.
The set $\mathbb{H}^n$ being considered as right quaternionic vector space, let $Sp(n)$ be the group of transformations of $\mathbb{H}^n$ which are $\mathbb{H}-$linear and $(uq, uw) = (q, w)$ [7].
$Sp(n)$ is called the quaternionic unitary group, also called the compact symplectic group.
By identifying u with the matrix that represents it in the canonical base we have

$$Sp(n) := \{u \in M_n(\mathbb{H}) : \ u^* u = uu^* = I\}.$$

Note that $Sp(n)$ is isomorphic to $Sp(n, \mathbb{C}) \cap U(2n)$.
The inner product of q and w is defined by

$$\langle q, w \rangle = Re(q, w) = Re(\overline{w_1} q_1) + ... + Re(\overline{w_n} q_n).$$

The associated norm to both the scalar product and the inner product is identical and represented by the notation $|| \cdot ||$.

Let $\mathcal{H}_q^n = \mathbb{H}^n \times \mathbb{R}^3 = \{(q, t), \quad q \in \mathbb{H}^n, \quad t \in \mathbb{R}^3\}$, with the group law is defined as follows:

$$(q, t)(w, s) = (q + w, t + s - 2Im(q, w)).$$

Then $\mathcal{H}_q^n$ is called the quaternionic Heisenberg group and is a two-step nilpotent Lie group with center $\{0\} \times \mathbb{R}^3$.
The Haar measure on $\mathcal{H}_q^n$ coincides with the Lebesgue measure $dqdt$ on $\mathbb{H}^n \times \mathbb{R}^3$.
The inner product in $L^2(\mathcal{H}_q^n, dqdt)$ is defined by

$$\langle f, g \rangle = \int_{\mathcal{H}_q^n} f(q, t)\overline{g(q, t)} dqdt.$$

For $\lambda \in \mathbb{R}^3 \backslash \{0\}$, the map $J_\lambda : q \longmapsto q\widetilde{\lambda}$ define a complexe structure of $\mathbb{H}^n$, where

$$\widetilde{\lambda} = \frac{\lambda}{|\lambda|} \ \text{and} \ q\widetilde{\lambda} = (q_1\widetilde{\lambda}, ..., q_n\widetilde{\lambda}).$$

Let's consider the Fock space $\mathcal{F}_\lambda$ consisting of holomorphic functions F defined on $(\mathbb{H}^n, J_\lambda) \simeq \mathbb{C}^{2n}$, where F satisfies the following condition:

$$||F||_\lambda^2 = \left(\frac{2|\lambda|}{\pi}\right)^{2n} \int_{\mathbb{H}^n} |F(q)|^2 e^{-2|\lambda| ||q||^2} dq < \infty.$$

The corresponding irreducible unitary representation $\pi_\lambda(x,t)$ of $\mathcal{H}_q^n$ on $\mathcal{F}_\lambda$ is realized by

$$\pi_\lambda(x,t)F(q) = e^{i\langle\lambda,t\rangle - |\lambda|(|x|^2 + 2\langle q,x\rangle - 2i\langle q\widetilde{\lambda},x\rangle)} F(q+x), \text{ for } F \in \mathcal{F}_\lambda.$$

All irreducible infinite-dimensional unitary representations of $\mathcal{H}_q^n$ is determined up to equivalence by the condition $\pi_\lambda(0,t) = e^{i\langle\lambda,t\rangle} Id_{\mathcal{F}_\lambda}, \lambda \in \mathbb{R}^3$ (see [11]).

We choose an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{H}$ such that

$$e_0 = 1, e_1 = \widetilde{\lambda}, e_2\widetilde{\lambda} = e_3.$$

For $q = q_0 + q_1\widetilde{\lambda} + q_2 e_2 + q_3 e_3$, we write

$$z_1 = q_0 + iq_1, z_2 = q_2 + iq_3.$$

For $q = (q_1, ..., q_n) \in \mathbb{H}^n$, we associated the element $(z_1, ..., z_{2n})$ of $\mathbb{C}^{2n}$, where

$$z_p = q_{p,0} + iq_{p,1}, z_{p+n} = q_{p,2} + iq_{p,3} \text{ and}$$

$$q_p = q_{p,0} + q_{p,1}\widetilde{\lambda} + q_{p,2}e_2 + q_{p,3}e_3. \tag{2.1}$$

The space $\mathcal{P}_\lambda(\mathbb{H}^n)$ of holomorphic polynomials on $(\mathbb{H}^n, J_\lambda) \simeq \mathbb{C}^{2n}$ is dense in $\mathcal{F}_\lambda$, and contains an orthonormal basis given by $\{u_\alpha^\lambda : \ \alpha \in \mathbb{N}^{2n}\}$(see [8]), where

$$u_\alpha^\lambda(q) = (\alpha!)^{\frac{-1}{2}} (2|\lambda|)^{\frac{|\alpha|}{2}} q^\alpha = (\alpha_1!)^{\frac{-1}{2}} (2|\lambda|)^{\frac{\alpha_1}{2}} z_1^{\alpha_1} ... (\alpha_{2n}!)^{\frac{-1}{2}} (2|\lambda|)^{\frac{\alpha_{2n}}{2}} z_{2n}^{\alpha_{2n}}.$$

Besides the infinite dimensional representations, $\mathcal{H}_q^n$ has the one-dimensional representations $\tau_w(q,t) = e^{i\langle w,q\rangle}$, for $(q,t) \in \mathcal{H}_q^n$ and $w \in \mathbb{H}^n$.
The π_λ $(\lambda \in \mathbb{R}^3)$ and the τ_w fill up the unitary dual of $\mathcal{H}_q^n$.

2.2 Fourier transform on $\mathcal{H}_q^n$

We associate to an integrable and square-integrable function f on $\mathcal{H}_q^n$, the Fourier transform

$$\widehat{f}(\lambda) := \int_{\mathcal{H}_q^n} f(q,t)\pi_\lambda(q,t) dq dt, \quad \lambda \in \mathbb{R}^3 \backslash \{0\}.$$

Then $\widehat{f}(\lambda)$ is a bounded operator on $\mathcal{F}_\lambda$. Furthermore, if $\phi, \psi \in \mathcal{F}_\lambda$, then

$$\langle \widehat{f}(\lambda)\phi, \psi\rangle = \int_{\mathcal{H}_q^n} f(q,t) \langle \pi_\lambda(q,t)\phi, \psi\rangle dq dt.$$

Let S_2 denote the Hilbert-Schmidt operators on $L^2(\mathcal{H}_q^n)$ with the inner product
$\langle S, T\rangle = trace(ST^*)$ (where T^* is the adjoint operator of S) and $d\sigma(\lambda)$ be the measure, defined on $\mathbb{R}^3 \backslash \{0\}$ by

$$d\sigma(\lambda) = \frac{2^{2n-3}}{\pi^{2n+3}} |\lambda|^{2n} d\lambda.$$

The Plancherel and inversion formulas for $\mathcal{H}_q^n$ can be proven in the same manner as [6]:

Theorem 2.1. *(Plancherel formula).*
Let $f \in L^1 \cap L^2(\mathcal{H}_q^n)$, then $\widehat{f}(\lambda)$ is a Hilbert-Schmidt operator, and we have

$$\begin{aligned} \int_{\mathcal{H}_q^n} |f(q,t)|^2 dq dt &= \int_{\mathbb{R}^3 \backslash \{0\}} ||\widehat{f}(\lambda)||_{HS}^2 d\sigma(\lambda) \\ &= \sum_{\alpha \in \mathbb{N}^{2n}} \sum_{\beta \in \mathbb{N}^{2n}} \int_{\mathbb{R}^3 \backslash \{0\}} |\langle \widehat{f}(\lambda)u_\alpha^\lambda, u_\beta^\lambda\rangle|^2 d\sigma(\lambda). \end{aligned}$$

Furthermore, the Fourier transform can be uniquely extended to $L^2(\mathcal{H}_q^n)$.

Theorem 2.2. *(Inversion formula).*
Let f be a function in the Schwartz space $\mathcal{S}(\mathcal{H}_q^n)$ on $\mathcal{H}_q^n$. Then for all $(q,t) \in \mathcal{H}_q^n$,

$$f(q,t) = \int_{\mathbb{R}^3\setminus\{0\}} trace(\pi_\lambda^*(q,t)\widehat{f}(\lambda))d\sigma(\lambda).$$

2.3 Spherical Functions and Representations

In this subsection, we recall some properties of the Gelfand pairs, the representations, and the spherical functions, which will be used in the proof of Theorem (3.2). For details, we refer the reader to [3, 4, 10].

Let N be a connected and simply connected nilpotent Lie group and K a compact group acting continuously on N by automorphism. The semi-direct product $K \ltimes N$ is defined by the law:

$$(u,x)(v,y) = (uv, x\ u.y).$$

Let dx be the Haar measure on N and dk the normalized Haar measure on K.
We denote by $\widehat{N}$ the set of equivalence classes of irreducible unitary representations of N.

Remark 2.3. A bi-K-invariant function on $K \ltimes N$ can be identified with a K-invariant function on N.

Definition 2.4. We say that (K,N) is a Gelfand pair when the algebra $L^1_K(N)$ of $K-$invariant integrable functions on N is commutative under convolution. Equivalently, the algebra $L^1(K \ltimes N//K)$ of integrable bi-K-invariant functions on the semi-direct product $K \ltimes N$ is commutative.

There are several equivalent ways of defining K-spherical functions associated with a Gelfand pair.

Definition 2.5. A K-spherical function associated with the Gelfand pair (K,N) is a continuous K-invariant function ϕ on N with complex-valued, such that

$$\phi(e) = 1, \int_K \phi(x\ u.y)du = \phi(x)\phi(y).$$

Notation:

- Let π be a unitary irreducible representation of N on a Hilbert space $\mathcal{H}_\pi$.
- Let $u \in K$, define π_u on N by $\pi_u(x) = \pi(u.x)$ for all $x \in N$.
- $K_\pi = \{u \in K \text{ such that } \pi_u \text{ is unitary equivalent to } \pi\}$.
- For $u \in K_\pi$, we choose an intertwining operator $\mu_\pi(u)$ such that

$$\pi_u(x) = \mu_\pi(u)\pi(x)\mu_\pi(u)^*.$$

Theorem 2.6. *[4, 5] (K,N) is a Gelfand pair if and only if μ_π decomposes into irreducibles components without multiplicities , for all $\pi \in \widehat{N}$.*

For $(\pi, \mathcal{H}_\pi) \in \widehat{N}$, let $\mathcal{H}_\pi = \sum_\alpha V_\alpha$ be the decomposition of $\mathcal{H}_\pi$ into irreducible subspaces invariant under the action of μ_π. Then we have the following theorem:

Theorem 2.7. *[4] Let (K,N) be a Gelfand pair, then*

(i) ϕ is a bounded K-spherical function if and only if there exists a unitary vector ξ in V_α and $\pi \in \widehat{N}$, such that for all $x \in N$,

$$\phi(x) = \phi_{\pi,\xi}(x) := \int_K \langle \pi(u.x)\xi, \xi\rangle du.$$

(ii) $\phi_{\pi,\xi} = \phi_{\pi',\eta}$ *if and only if there exists* $u \in K$ *such that* $\pi' \sim \pi_u$ *and* ξ, η *belongs to the same* V_α.

Corollary 2.8. *[3] Suppose that* $K_\pi = K$ *and let* $\{v_1, ..., v_{d_\alpha}\}$ *an orthonormal basis of* V_α*, then*

$$\phi_{\pi,\alpha}(x) = \frac{1}{l}\sum_{p=1}^{d_\alpha}\langle \pi(x)v_p, v_p\rangle.$$

2.4 The sub-Laplacian on $\mathcal{H}_q^n$:

Using the coordinates $(q_1, ..., q_n, t_1, t_2, t_3)$ on $\mathcal{H}_q^n$, with $q_r = q_{r,0} + q_{r,1}i + q_{r,2}j + q_{r,3}k$, for $1 \leq r \leq n$. Then the $(4n+3)$-dimensional Lie algebra of $\mathcal{H}_q^n$ is generated by the left-invariant vector fields X_r^0, X_r^1 , X_r^2, X_r^3 , T_1, T_2 and T_3, (see [2]), where

$$T_s = \frac{\partial}{\partial t_s},$$

$$X_r^0 = \frac{\partial}{\partial q_{r,0}} - 2q_{r,1}T_1 - 2q_{r,2}T_2 - 2q_{r,3}T_3,$$

$$X_r^1 = \frac{\partial}{\partial q_{r,1}} + 2q_{r,0}T_1 + 2q_{r,3}T_2 - 2q_{r,2}T_3,$$

$$X_r^2 = \frac{\partial}{\partial q_{r,2}} - 2q_{r,3}T_1 + 2q_{r,0}T_2 + 2q_{r,1}T_3,$$

$$X_r^3 = \frac{\partial}{\partial q_{r,3}} + 2q_{r,2}T_1 - 2q_{r,1}T_2 + 2q_{r,0}T_3.$$

The Lie brackets are given by:

$$[X_r^0, X_s^1] = [X_r^2, X_s^3] = 4\delta_{rs}T_1,$$

$$[X_r^0, X_s^2] = [X_r^3, X_s^1] = 4\delta_{rs}T_2,$$

$$[X_r^0, X_s^3] = [X_r^1, X_s^2] = 4\delta_{rs}T_3,$$

and the other Lie brackets are all null.

Proposition 2.9. *[13] We define the sub-Laplacian* Δ_0 *on* $\mathcal{H}_q^n$ *by*

$$\Delta_0 = \sum_{r=1}^{n}\sum_{s=0}^{3}(X_r^s)^2, \quad \textit{then}$$

$$\begin{aligned}\Delta_0 &= \sum_{r=1}^{n}\sum_{s=0}^{3}\frac{\partial^2}{\partial q_{r,s}^2} + 4\sum_{r=1}^{n}\sum_{s=1}^{3}\left(q_{r,0}\frac{\partial^2}{\partial q_{r,s}\partial t_s} - q_{r,s}\frac{\partial^2}{\partial q_{r,0}\partial t_s}\right)\\ &+ 4\sum_{r=1}^{n}\sum_{s=0}^{3}q_{r,s}^2\sum_{p=1}^{3}\frac{\partial^2}{\partial t_p^2} + 4\sum_{r=1}^{n}\sum_{(s,p,w)}q_{r,s}\left(\frac{\partial^2}{\partial q_{r,p}\partial t_w} - \frac{\partial^2}{\partial q_{r,w}\partial t_s}\right),\end{aligned}$$

where (r, s, w) *means the cyclic permutation of* $(1, 2, 3)$.

3 Bounded spherical functions on $\mathcal{H}_q^n$

3.1 A Gelfand pair associated to $\mathcal{H}_q^n$

Let $u \in Sp(n)$, we define the application ψ_u by $\psi_u(q,t) = (u.q, t)$, for $(q,t) \in \mathcal{H}_q^n$, with $u.q$ is the standard action of $Sp(n)$ on $\mathbb{H}^n$. Then ψ_u is an automorphism of $\mathcal{H}_q^n$.

Indeed for $(q,t),\ (w,s) \in \mathcal{H}_q^n$ and $u \in Sp(n)$

$$\begin{aligned}\psi_u(q,t)\psi_u(w,s) &= (u.q,t)(u.w,s)\\ &= (u.q+u.w, t+s-2Im(u.q,u.w))\\ &= (u.q+u.w, t+s-2Im(q,w))\\ &= (u.(q+w), t+s-2Im(q,w))\\ &= \psi_u((q,t)(w,s))\end{aligned}$$

and it is clear that ψ_k is bijective. Moreover, the map $\psi : u \longmapsto \psi_u$ is a homomorphism of the group $Sp(n)$ into the group $Aut(\mathcal{H}_q^n)$ of the automorphisms of $\mathcal{H}_q^n$.
Then we can define the semi-product $Sp(n) \ltimes \mathcal{H}_q^n$, equipped with the following product

$$(u,q,t)(u^{'},q^{'},t^{'}) = (uu^{'},(q,t)(u.q^{'},t^{'})).$$

Let $\lambda \in \mathbb{R}^3\backslash\{0\}$, then we can define the representation π_λ^u by $\pi_\lambda^u(q,t) = \pi_\lambda(u.q,t)$, which coincide with π_λ at the center. Then π_λ^u is unitarily equivalent to π_λ and we have the unitary intertwining operator μ_λ such that

$$\pi_\lambda(u.q,t) = \mu_\lambda(u)\pi_\lambda(q,t)\mu_\lambda(u)^{-1}.$$

Precisely, we set

$$\mu_\lambda(u)F(q) = F(u^{-1}.q),\ \text{ for }\ F \in \mathcal{F}_\lambda,\ \ u \in Sp(n)\ \text{ and }\ q \in \mathbb{H}^n.$$

Indeed

$$\begin{aligned}\mu_\lambda(u^{-1})[\pi_\lambda(u.q,t)F](w) &= [\pi_\lambda(u.q,t)F](u.w)\\ &= \pi_\lambda(q,t)(\mu_\lambda(u^{-1}))F(w).\end{aligned}$$

Theorem 3.1. *$(Sp(n),\mathcal{H}_q^n)$ is a Gelfand pair.*

Proof. $(Sp(n),\mathcal{H}_q^n)$ is a Gelfand pair if and only the action of $Sp(n)$ on $\mathcal{F}_\lambda$ by μ_λ decomposes into irreducible components of multiplicity one for all $\lambda \in \mathbb{R}^3$ (2.6).
According to the table 1.8 in [3], we find that the action of $Sp(n)$ on $\mathcal{F}_\lambda$ is multiplicity free.
We conclude that $(Sp(n),\mathcal{H}_q^n)$ is a Gelfand pair. □

3.2 Bounded spherical functions on $\mathcal{H}_q^n$

For all the rest, we denote the $Sp(n)$ as K.

Theorem 3.2. *The bounded spherical functions of the Gelfand pair $(K,\mathcal{H}_q^n)$ are giving by*

(i) The functions $\phi_{\lambda,m}$ parametrized by $\lambda \in \mathbb{R}^3\backslash\{0\}$ and $m \in \mathbb{N}$:

$$\phi_{\lambda,m}(q,t) = \frac{(2n-1)!m!}{(m+2n-1)!}e^{i\langle\lambda,t\rangle}e^{-|\lambda|\|q\|^2}L_m^{2n-1}(2|\lambda|\|q\|^2), m \in \mathbb{N},$$

where L_m^{2n-1} is the Laguerre polynomial of order $2n-1$ and of degree m.

(ii) The functions ϕ_r parametrized by $r \in \mathbb{R}^+$:

$$\phi_r(q,t)) = J_{2n-1}(r\|q\|),$$

Where J_{2n-1} is the Bessel function of order 2n-1.

Proof.
(1) For the infinite-dimensional representation, we have π_λ^u is unitarily equivalent to π_λ for every $\lambda \in \mathbb{R}^3\backslash\{0\}$, where the unitary intertwining operator μ_λ is given by

$$\mu_\lambda(u)F(q) = F(u^{-1}.q),\ \text{for}\ F \in \mathcal{F}_\lambda,\ \ u \in K\ \text{and}\ q \in \mathbb{H}^n.$$

As $(K, \mathcal{H}_q^n)$ is a Gelfand pair, then by theorem (2.6) the space $\mathcal{F}_\lambda$ (or equivalently the dense subspace $\mathcal{P}_\lambda(\mathbb{H}^n)$) decomposes into μ_λ-irreducible subspaces without multiplicities. For each $u \in Sp(n)$ and $m \in \mathbb{N}^*$, let $\mathcal{P}_{\lambda,m}$ be the subspace of homogeneous polynomials of degree m, then $\mu_\lambda(u)$ preserves each $\mathcal{P}_{\lambda,m}$ and is irreducible under this action, then the decomposition of $\mathcal{P}_\lambda(\mathbb{H}^n)$ is given by

$$\mathcal{P}_\lambda(\mathbb{H}^n) = \sum_{m\in\mathbb{N}} \mathcal{P}_{\lambda,m}.$$

The space $\mathcal{P}_{\lambda,m}$ has the orthonormal basis $\{u_\alpha^\lambda : \ |\alpha| = m\}$. The dimension of this space is equal to $\frac{(m+2n-1)!}{(2n-1)!m!}$ and we note it by d_m [8].

Then by corollary (2.8), the K-spherical bounded functions of $\mathcal{H}_q$ associated to π_λ is characterized by :

$$\begin{aligned}\phi_{\lambda,m}(q,t) &= \frac{1}{d_m}\sum_{|\alpha|=m}\langle \pi_\lambda(q,t)u_\alpha^\lambda, u_\alpha^\lambda\rangle \\ &= \frac{(2n-1)!m!}{(m+2n-1)!}e^{i\langle\lambda,t\rangle}\sum_{|\alpha|=m}\langle \pi_\lambda(q,0)u_\alpha^\lambda, u_\alpha^\lambda\rangle.\end{aligned}$$

Let $w \in \mathbb{H}^n$, we denote by $\xi_p = w_{p,0} + iw_{p,1}$, $\xi_{p+n} = w_{p,2} + iw_{p,3}$ and $\xi = (\xi_1, ..., \xi_{2n})$ for $w_p = w_{p,0} + w_{p,1}\widetilde{\lambda} + w_{p,2}e_2 + w_{p,3}e_3$. Combining with (2.1), we have

$$\langle w, q\rangle - i\langle w\widetilde{\lambda}, q\rangle = \sum_{p=1}^{2n}\xi_p.\overline{z_p}.$$

Since

$$\begin{aligned}\langle \pi_\lambda(q,0)u_\alpha^\lambda, u_\alpha^\lambda\rangle &= \left(\frac{2|\lambda|}{\pi}\right)^{2n}\int_{\mathbb{H}^n} e^{-|\lambda|(||q||^2+2\langle w,q\rangle-2i\langle w\widetilde{\lambda},q\rangle)}u_\alpha^\lambda(w+q)\overline{u_\alpha^\lambda(w)}e^{-2|\lambda|||w||^2}dw \\ &= \left(\frac{2|\lambda|}{\pi}\right)^{2n}\int_{\mathbb{C}^{2n}} e^{-|\lambda|(||z||^2+2\xi.z^*}) u_\alpha^\lambda(\xi+z)\overline{u_\alpha^\lambda(\xi)}e^{-2|\lambda|||\xi||^2}d\xi,\end{aligned}$$

then (see [3, 8])

$$\sum_{|\alpha|=m}\langle \pi_\lambda(q,0)u_{\alpha,\lambda}, u_{\alpha,\lambda}\rangle = e^{-|\lambda||q|^2}L_m^{2n-1}(2|\lambda||q|^2).$$

Hence $\phi_{\lambda,m}(q,t) = \frac{(2n-1)!m!}{(m+2n-1)!}e^{i\langle\lambda,t\rangle}e^{-|\lambda||q|^2}L_m^{2n-1}(2|\lambda||q|^2)$.

(2) For the one-dimensional representation τ_w, we have

$$\tau_w(u.q,t) = e^{i\langle u^{-1}.w,q\rangle}, \text{ for } (q,t)\in\mathcal{H}_q^n \text{ and } w\in\mathbb{H}^n.$$

Then by theorem (2.7), we have the $K-$spherical function

$$\phi_w(q,t) = \int_K e^{i\langle u^{-1}.w,q\rangle}du.$$

Thus the spherical function $\phi_w(q,t)$ is independent of t and depend only on the $K-$orbit in $\mathbb{H}^n$ (by theorem (2.7)). Let $\mu_{K.w}$ denote the unit measure supported on the $K-$orbit through w. As a distribution, this is given by

$$\langle \mu_{K.w}, f\rangle = \int_K f(k^{-1}.w, 0)du, \quad \forall f \in \mathcal{C}_c^\infty(\mathcal{H}_q^n).$$

We define the Euclidean Fourier transform of f by

$$\widehat{f}(w,s) = \int_{\mathbb{H}^n\times\mathbb{R}^3} f(q,t)e^{i(\langle w,q\rangle+\langle s,t\rangle)}dqdt.$$

Note that

$$\begin{aligned}\int_{\mathcal{H}_q^n}\phi_w(q,t)f(q,t)dqdt &= \int_K\int_{\mathcal{H}_q^n} f(q,t)e^{i\langle u^{-1}.w,q\rangle}dqdtdu\\ &= \int_K \widehat{f}(u^{-1}.w,0)du\\ &= \langle \widehat{\mu}_{K.w}, f\rangle\end{aligned}$$

for all $f\in\mathcal{C}_c^\infty(\mathcal{H}_q^n)$. Therefore we have $\phi_w(q,t)=\widehat{\mu}_{K.w}(q,0)$.
Since the distinct K-orbits are parametrised by real $r\geq 0$, then

(i) for $r=0$, we have the trivial representation, and the K-spherical function associated with that is $\phi_0(q,t)=1$ (du is a normalized Haar measure on K).

(ii) For $r>0$ the sphere S_r of radius r in $\mathbb{H}^n$ is a K-orbit and the associated K-spherical function is $\phi_r=\widehat{\mu}_r$, where μ_r is the normalised surface measure on S_r.
Since the Fourier transform of the unit measure on the $(4n-1)$-sphere is given in terms of the Bessel function (see [8] p.25), then

$$\phi_r(q,t)=J_{2n-1}(r||q||),$$

where J_{2n-1} is the Besel function of order $(2n-1)$.

□

4 Spherical Fourier transform on $\mathcal{H}_q^n$

Theorem 4.1. *Let f be an integrable $K-$invariant function on $\mathcal{H}_q^n$, then for $\lambda\in\mathbb{R}^3\backslash\{0\}$ and $m\in\mathbb{N}$, the space $\mathcal{P}_{\lambda,m}$ is an eigen-subspace of $\widehat{f}(\lambda)$ with eigenvalue $\widehat{f}(\lambda,m)$, where*

$$\widehat{f}(\lambda,m):=\int_{\mathcal{H}_q^n} f(q,t)\phi_{\lambda,m}(q,t)dqdt.$$

Proof. $\mathcal{P}_{\lambda,m}$ is a subspace of $\mathcal{F}_\lambda$ and is invariant under the action of the compact group K and is irreducible under its action, furthermore for $u\in K$ and $q\in\mathbb{H}^n$, we have

$$\mu_\lambda(u)\pi_\lambda(q)=\pi_\lambda(q)\mu_\lambda(u),$$

then $\mu_\lambda(u)\pi_\lambda(f)=\pi_\lambda(f)\mu_\lambda(u)$.
Therefore, according to Schur's lemma the subspace $\mathcal{P}_{\lambda,m}$ is an eigenspace of $\pi_\lambda(f)$.
Then, for $\varphi\in\mathcal{P}_{\lambda,m}$, we have $\widehat{f}(\lambda)\varphi=\langle\widehat{f}(\lambda)\xi,\xi\rangle\varphi$ with $\xi\in P_{\lambda,m}$ and $||\xi||=1$.
Hence, $\widehat{f}(\lambda)\varphi=\widehat{f}(\lambda,m)\varphi$. □

Definition 4.2. Let f be an integrable K-invariant function on $\mathcal{H}_q^n$. The spherical Fourier transform of f is defined by

$$\widehat{f}(\lambda,m):=\int_{\mathcal{H}_q^n} f(q,t)\phi_{\lambda,m}(q,t)dqdt,$$

for all (λ,m) in $\mathbb{R}^3\backslash\{0\}\times\mathbb{N}$.

Let f and g be two integrable K-invariant functions on $\mathcal{H}_q^n$, then it is easy to prove that:

$\widehat{f*g}(\lambda,m)=\widehat{f}(\lambda,m)\widehat{g}(\lambda,m)$, where $'*'$ is the convolution product defined by

$$f*g(q,t)=\int_{\mathcal{H}_q^n} f(p,r)g((-p,-r)(q,t))dpdr,\quad\text{for } f,g\in L^1(\mathcal{H}_q^n).$$

Theorem 4.3. *Inversion and Plancherel formulas.*

(i) If f is integrable and square-integrable on $\mathcal{H}_q^n$, then

$$\int_{\mathcal{H}_q^n} |f(q,t)|^2 dqdt = \int_{\mathbb{R}^3\setminus\{0\}} \sum_{m=0}^{\infty} \frac{(m+2n-1)!}{(2n-1)!m!} |\widehat{f}(\lambda,m)|^2 d\sigma(\lambda). \qquad (1)$$

(ii) If the quantity $\displaystyle\int_{\mathbb{R}^3\setminus\{0\}} \sum_{m=0}^{\infty} \frac{(m+2n-1)!}{(2n-1)!m!} \phi_{\lambda,m}(-q,-t)\widehat{f}(\lambda,m)d\sigma(\lambda)$ *is finite, then*

$$f(q,t) = \int_{\mathbb{R}^3\setminus\{0\}} \sum_{m=0}^{\infty} \frac{(m+2n-1)!}{(2n-1)!m!} \phi_{\lambda,m}(-q,-t)\widehat{f}(\lambda,m)d\sigma(\lambda). \qquad (2)$$

Proof. (i) The space $\mathcal{P}_{\lambda,m}$ is an eigen-subpace of $\widehat{f}(\lambda)$ for the eigenvalue $\widehat{f}(\lambda,m)$, and since the dimension of $\mathcal{P}_{\lambda,m}$ is $\frac{(m+2n-1)!}{(2n-1)!m!}$, then

$$||\widehat{f}(\lambda)||_{HS}^2 = \sum_{m=0}^{\infty} \frac{(m+2n-1)!}{(2n-1)!m!} |\widehat{f}(\lambda,m)|^2.$$

According to the Plancherel formula (2.1), we obtain the formula (1).

(ii) $\pi_\lambda^*(q,t)$ is a bounded operator and $\widehat{f}(\lambda)$ is a trace class operator, then $\pi_\lambda^*(q,t)\widehat{f}(\lambda)$ is a trace classs operator, so

$$\begin{aligned} trace(\pi_\lambda^*(q,t)\widehat{f}(\lambda)) &= \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \langle \pi_\lambda^*(q,t)\widehat{f}(\lambda)u_\alpha^\lambda, u_\alpha^\lambda\rangle \\ &= \sum_{m=0}^{\infty} \widehat{f}(\lambda,m) \sum_{|\alpha|=m} \langle \pi_\lambda(-q,-t)u_\alpha^\lambda, u_\alpha^\lambda\rangle \\ &= \sum_{m=0}^{\infty} \frac{(m+2n-1)!}{(2n-1)!m!} \phi_{\lambda,m}(-q,-t)\widehat{f}(\lambda,m). \end{aligned}$$

According to the inversion formula (2.2), we obtain the formula (2).

□

5 The heat kernel on $(4n+3)$-dimensional quaternionic Heisenberg group

There are various methods to determine the heat kernel associated with the sub-Laplacian on $\mathcal{H}_q^n$ (see for example [12, 13]). In this paragraph, we will give a method based on the inversion formula of the spherical Fourier transform. We recall the following result:

Proposition 5.1.

(i) The spherical function $\phi_{\lambda,m}$ is a eigenfunction of the sub-Laplacian Δ_0 :

$$\Delta_0\phi_{\lambda,m} = -8|\lambda|(n+m)\phi_{\lambda,m}.$$

(ii) If $f : \mathcal{H}_q^n \longrightarrow \mathbb{C}$ is a K-invariant $\mathcal{C}^2$ function and compact support, then

$$\widehat{\Delta_0 f}(\lambda,m) = -8|\lambda|(n+m)\widehat{f}(\lambda,m).$$

Proof. See, for example, [8], for proof of this result. □

Let us consider the heat equation associated with the sub-laplacian

$$\frac{\partial g}{\partial s}(s;q,t) = \Delta_0 g(s;q,t)$$

with the initial condition $g(0;q,t) = h(q,t)$, where $h : \mathcal{H}_q^n \longrightarrow \mathbb{C}$ is a known continuous function and $g : [0,+\infty[\times\mathcal{H}_q^n \longrightarrow \mathbb{C}$ is a continuous function on $[0,+\infty[\times\mathcal{H}_q^n$ and class $\mathcal{C}^2$ on $]0,+\infty[\times\mathcal{H}_q^n$ to be determined.
Suppose that h and g are K-invariant functions and taking the spherical Fourier transform of the previous equations, then we obtain

$$\frac{\partial \widehat{g}}{\partial s}(s;\lambda,m) = -8|\lambda|(n+m)\widehat{g}(s;\lambda,m)$$

$$\widehat{g}(0;\lambda,m) = \widehat{h}(\lambda,m),$$

therfore $\widehat{g}(s;\lambda,m) = e^{-8|\lambda|(n+m)s}\,\widehat{h}(\lambda,m)$.
Let $p_s(q,t)$ be an integrable K-invariant function on $\mathcal{H}_q^n$ such that

$$\widehat{p_s}(\lambda,m) = e^{-8|\lambda|(n+m)s},$$

then $g(s,q,t) = (p_s * h)(q,t)$.
Applying the inversion formula (4.3), we have

$$\begin{aligned}
p_s(q,t) &= \int_{\mathbb{R}^3\backslash\{0\}} \sum_{m=0}^{\infty} d_m \phi_{\lambda,m}(-q,-t)\widehat{p_s}(\lambda,m)\, d\sigma(\lambda) \\
&= \int_{\mathbb{R}^3\backslash\{0\}} \sum_{m=0}^{\infty} d_m \phi_{\lambda,m}(-q,-t)\, e^{-8|\lambda|(n+m)s}\, d\sigma(\lambda) \\
&= \int_{\mathbb{R}^3\backslash\{0\}} \sum_{m=0}^{\infty} e^{-i\langle\lambda,t\rangle}\, e^{-|\lambda|||q||^2}\, L_m^{2n-1}(2|\lambda|||q||^2)\, e^{-8|\lambda|(n+m)s}\, d\sigma(\lambda) \\
&= \int_{\mathbb{R}^3\backslash\{0\}} e^{-i\langle\lambda,t\rangle}\, e^{-|\lambda|||q||^2}\, e^{-8|\lambda|ns} \left(\sum_{m=0}^{\infty} L_m^{2n-1}(2|\lambda|||q||^2)\, \left(e^{-8|\lambda|s}\right)^m\right) d\sigma(\lambda),
\end{aligned}$$

for $|z|<1$, we have from [8, p.71] that

$$\sum_{m=0}^{\infty} L_m^{\alpha}(w)\, z^m = (1-z)^{-\alpha-1}\, e^{-\frac{z}{1-z}.w},$$

then

$$\begin{aligned}
p_s(q,t) &= \int_{\mathbb{R}^3\backslash\{0\}} e^{-i\langle\lambda,t\rangle}\, e^{-|\lambda|||q||^2}\, e^{-8|\lambda|ns} \left(1-e^{-8|\lambda|.s}\right)^{-2n} \exp\left(-\frac{e^{-8|\lambda|.s}}{1-e^{-8|\lambda|.s}}\, 2|\lambda|||q||^2\right) d\sigma(\lambda) \\
&= \frac{1}{8\pi^{2n+3}} \int_{\mathbb{R}^3\backslash\{0\}} e^{-i\langle\lambda,t\rangle} \left(\frac{|\lambda|}{\sinh(4|\lambda|s)}\right)^{2n} \exp\left(-|\lambda|||q||^2\coth(4|\lambda|s)\right)\, d\lambda.
\end{aligned}$$

We deduce the following theorem:

Theorem 5.2. *The heat kernel on the quaternionic Heisenberg group $\mathcal{H}_q^n$ is given by*

$$p_s(q,t) = \frac{1}{8\pi^{2n+3}} \int_{\mathbb{R}^3} e^{-i\langle\lambda,t\rangle} \left(\frac{|\lambda|}{\sinh(4|\lambda|s)}\right)^{2n} \exp\left(-|\lambda|||q||^2\coth(4|\lambda|s)\right)\, d\lambda.$$

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