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Graded $n-$absorbing $I-$ideals

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Abstract *Let G be an arbitrary group with an identity e, let R be a commutative G-graded ring and let I be a proper graded ideal of R. In this article, we introduce the concept of graded 2-absorbing I-ideal and graded n-absorbing I-ideal in a commutative G-graded rings which is a generalization of graded 2-absorbing ideal and graded n-absorbing ideal. A proper graded ideal P of a G-graded ring R is called a graded 2-absorbing I-ideal if $a, b, c \in h(R)$ with $abc \in P-IP$, then $ab \in P$ or $ac \in P$ or $bc \in P$. Also a proper graded ideal P of a G-graded ring R is called a graded n-absorbing I-ideal if $a_1, a_2, \ldots, a_{n+1} \in h(R)$ with $a_1 \ldots a_{n+1} \in P - IP$, then $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n+1} \in P$ for some $i \in \{1, 2, \ldots, n+1\}$ and $n \geq 1$. A number of results and characterizations concerning these classes of graded ideals and their homogeneous components are given. Furthermore, among many results we prove that every proper graded ideal of a G-graded ring R is a graded n-absorbing I -ideal if and only if every quotient of R is a product of $(n+1)$-fields and also we give a condition under which the intersection of m graded ideals of R is a graded n-absorbing I-ideal.*

1 Introduction

Prime and primary ideals have key roles in commutative ring theory, many authors have studied generalizations of prime and primary ideals (see [4], [5], [6], [8]). Later, A. Badawi in [8] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal P of R to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in P$, then $ab \in P$ or $ac \in P$ or $bc \in P$. Anderson and Badawi in [4] generalized the concept of 2-absorbing ideals to n-absorbing ideals. Take $n \in \mathbb{N}$, R a commutative ring with unity. An ideal I of R is said to be an n-absorbing ideal of a ring R if for any $x_1, x_2, \ldots, x_{n+1} \in R$ such that $x_1 \ldots x_{n+1} \in I$, there are n of the x_i's whose product is in I. Furthermore, the concept of graded 2-absorbing ideal was introduced and studied by Al-Zoubi, Abu-Dawwas and Ceken in [3]. Akray in [1] introduced I-prime ideal. An ideal of a ring R is I-prime if for $a, b \in R$ with $ab \in P - IP$, then $a \in P$ or $b \in P$ for a fixed ideal I of R. Then he defined the concept of n-absorbing I-ideals in [2]. For a fixed proper ideal I, a proper ideal P of R is called an n-absorbing I-ideal if $a_1, a_2, \ldots, a_{n+1} \in R$ with $a_1 \ldots a_{n+1} \in P - IP$, then $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n+1} \in P$ for some $i \in \{1, 2, \ldots, n+1\}$ and $n \geq 1$. In this paper, we introduce the notion of graded 2-absorbing and graded n-absorbing I-ideals in commutative G-graded rings which are the graded versions of 2-absorbing and n-absorbing ideals on the one hand and generalizations of graded prime ideals on the other.

Before we state our results let us recall some notation and terminology. Throughout this work all rings are assumed to be commutative with nonzero identity. Let G be an abelian group with identity e. By a G-graded ring we mean a ring R which is a direct sum of a family of additive subgroups $\{R_g\}_{g\in G}$ of R with the property that $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. Throughout,

$R = \bigoplus_{g\in G} R_g$ denotes graded ring and we call $r_g \in R_g$ a homogeneous element of R of degree g and also the set of all homogeneous elements of R is denoted by $h(R) = \cup_{g\in G} R_g$. Let P be an ideal of R. Then P is called a graded ideal of R if one of the equivalent conditions hold: (i) $P = \bigoplus_{g\in G} P_g$, where $P_g = P \cap R_g$ for all $g \in G$ and (ii) $a = a_{g_1} + a_{g_2} + \ldots + a_{g_n} \in P$ implies that $a_{g_i} \in P$, where $a_{g_i} \in R_{g_i}$. Let R be a G-graded ring and P be a graded ideal of R. The quotient ring R/P is a G-graded ring. Indeed, $R/P = \oplus_{g\in G}(R/P)_g$ where $R/P = \oplus_{g\in G}(R/P)_g = \{a + P | a \in R_g\}$.

Graded rings have been studied since 1955, (see for instance [14], [17]), then various researchers interested in these rings and made several important studies in them and construct a new branch in ring theory. Grading appear in many circumstances, both in elementary and advanced levels. Particularly, there is a wide variety of applications of graded algebras in geometry and physics, for more information on the application of graded rings, see [12].

Let I be a fixed proper ideal of R_e. In this article, we introduce the notion of graded 2-absorbing I-ideal and graded n-absorbing I-ideal in commutative G-graded rings which is a generalization of graded 2-absorbing and graded n-absorbing ideals. A proper graded ideal P of a G-graded ring R is called a graded 2-absorbing I-ideal if for $a, b, c \in h(R)$ with $abc \in P - IP$, then $ab \in P$ or $ac \in P$ or $bc \in P$. Also a proper graded ideal P of a G-graded ring R is called an n-absorbing I-ideal if $a_1, a_2, \ldots, a_{n+1} \in h(R)$ with $a_1 \ldots a_{n+1} \in P - IP$, then $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n+1} \in P$ for some $i \in \{1, 2, \ldots, n+1\}$ and $n \geq 1$. Among many results in this paper. We give an example (Example 2.16) of a graded 2-absorbing I-ideal of R that is not a graded 2-absorbing ideal of R. We show that if P is a proper graded ideal of R such that $Gr(P)$ is a graded prime ideal of R and $Gr(P) \neq P$ and $(P : a)$ is a graded I-prime ideal of R for all $a \in h(Gr(P)) - h(P)$, then P is a graded 2-absorbing I-ideal of R (Theorem 2.8). It is shown that if P_1 and P_2 are graded primary ideals of R, then $Gr(P_1P_2)$ is a graded 2-absorbing I-ideal of R (Theorem 2.14). If P_j is a non zero graded I-prime ideal of a G-graded ring R for $1 \leq j \leq m$, then we show that $\cap_{j=1}^{m} P_j$ is a graded n-absorbing I-ideal (Theorem 3.5). In (Theorem 3.16) we characterize G-graded rings in which every proper graded ideal is a graded n-absorbing I-ideal.

2 Graded 2-absorbing I-ideal

Definition 2.1. Let R be a G-graded ring and I be a fixed proper ideal of R_e. A proper graded ideal P of R is called a graded 2-absorbing I-ideal if $a, b, c \in h(R)$ with $abc \in P - IP$, then $ab \in P$ or $ac \in P$ or $bc \in P$.

From the definition, one can see that any graded 2-absorbing ideal of R is a graded 2-absorbing I-ideal of R. But, the following example illustrates that the converse need not be true.

Example 2.2. Assume that $R = K[X, Y, Z]$ is a $\mathbb{Z}$-graded ring with $deg(X) = deg(Y) = deg(Z) = 1$ and K is any field and $P = \langle XY \rangle$ is a graded ideal of R generated by homogeneous elements XY. Then P is a graded 2-absorbing ideal of R. Thus for any graded ideal I of R, P is a graded 2-absorbing I-ideal of R. Furthermore, take $P = \langle XYZ, X^2Y^2 \rangle$ and $I = \langle XYZ, X^2Y^2 \rangle$. Therefore P is a graded 2-absorbing I-ideal, since $P - IP = \phi$. However, P is not a graded 2-absorbing ideal, since $XYZ \in P$ but $XY \notin P$, $XZ \notin P$ and $YZ \notin P$.

Clearly, every 2-absorbing I-ideal of a graded ring R is also a graded 2-absorbing I-ideal. However, the next example shows that the converse is not true in general.

Example 2.3. Let $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Let $I = 2R$ and $P = 10R$. Then P is not a 2-absorbing I-ideal of R, since

$$10 = (1 + i)(1 - i)5 \in P - IP$$

while $(1 + i)(1 - i) = 2 \notin P$, $(1 + i)5 \notin P$ and $(1 - i)5 \notin P$. To show that P is a graded 2-absorbing I-ideal of R, take $a, b, c \in h(R)$ with $abc \in P - IP$. So $10|abc$. Suppose $2|a$ and $5 \nmid a$. Then 5 divides b or c and P is a graded 2-absorbing I-ideal of R.

Theorem 2.4. *If P and Q are non zero graded I-prime ideals of a G-graded ring R, then $P \cap Q$ is a graded 2-absorbing I-ideal.*

Proof. Let $a, b, c \in h(R)$ with $(ab)c \in P \cap Q - I(P \cap Q)$. Then $(ab)c \in P - IP$ and $(ab)c \in Q - IQ$. Since P is a graded I-prime ideal, so either $ab \in P$ or $c \in P$. If $ab \in P$, then either $a \in P$ or $b \in P$. Similarly, $a \in Q$ or $b \in Q$ or $c \in Q$. Suppose $a \in P \cap Q$. Then $ab \in P \cap Q$ and $ac \in P \cap Q$, since $P \cap Q$ is an ideal. Therefore $P \cap Q$ is a graded 2-absorbing I-ideal. □

Theorem 2.5. *Let R be a G-graded ring and P be a graded ideal of R. If P is a graded 2-absorbing I-ideal of R, then $P \cap R_e$ is a graded 2-absorbing I-ideal of R_e.*

Proof. Let $a, b, c \in R_e$ with $abc \in P \cap R_e - I(P \cap R_e)$. As P is a graded 2-absorbing I-ideal of R, $ab \in P$ or $bc \in P$ or $ac \in P$. Thus $ab \in P \cap R_e$ or $ac \in P \cap R_e$ or $bc \in P \cap R_e$, since R_e is a subring of R. Hence $P \cap R_e$ is a graded 2-absorbing I-ideal of R_e. □

Theorem 2.6. *Let P and Q be graded ideals of a G-graded ring R with $Q \subseteq P$. Then the following hold:*

(i) P is a graded 2-absorbing I-ideal of R if and only if $\frac{P}{Q}$ is a graded 2-absorbing I-ideal of $\frac{R}{Q}$.

(ii) If Q and $\frac{P}{Q}$ are graded 2-absorbing I-ideals of R and $\frac{R}{Q}$ respectively, then P is a graded 2-absorbing I-ideals of R.

Proof. (i) Assume that

$$xyz + Q = (x + Q)(y + Q)(z + Q) \in \tfrac{P}{Q} - I\tfrac{P}{Q} = \tfrac{P}{Q} - \tfrac{IP+Q}{Q}$$

for some $x, y, z \in h(R)$. Then

$$xyz \in P - (IP + Q),$$

so $xyz \in P - IP$. Since P is a graded 2-absorbing I-ideal, we get $xy + Q \in \frac{P}{Q}$ or $xz + Q \in \frac{P}{Q}$ or $yz + Q \in \frac{P}{Q}$. Therefore $\frac{P}{Q}$ is a graded 2-absorbing I-ideal of $\frac{R}{Q}$. For the converse, assume $abc \in P - IP$ with $a, b, c \in h(R)$. Thus $abc \in P - (IP + Q)$. Hence

$$(a + Q)(b + Q)(c + Q) \in \tfrac{P}{Q} - \tfrac{IP+Q}{Q} = \tfrac{P}{Q} - I\tfrac{P}{Q}.$$

As $\frac{P}{Q}$ is a graded 2-absorbing I-ideal of $\frac{R}{Q}$, we can conclude that $(a + Q)(b + Q) \in \frac{P}{Q}$ or $(a + Q)(c + Q) \in \frac{P}{Q}$ or $(b + Q)(c + Q) \in \frac{P}{Q}$. Hence $ab \in P$ or $bc \in P$ or $ac \in P$, which implies that P is a graded 2-absorbing I-ideal

(ii) Let $xyz \in P - IP$ where $x, y, z \in h(R)$. Then

$$(x + Q)(y + Q)(z + Q) = xyz + Q \in \tfrac{P}{Q}.$$

If $xyz \in Q$, since $xyz \notin IP$ and $Q \subseteq P$, then $IQ \subseteq IP$. Thus $xyz \notin IQ$. Hence $xyz \in Q - IQ$. Since Q is a graded 2-absorbing I-ideal, then we conclude either $xy \in Q \subseteq P$ or $xz \in Q \subseteq P$ or $yz \in Q \subseteq P$. Now, for the case $xyz \notin Q$, we have $\overline{xyz} \notin \frac{IP+Q}{Q}$ and

$$(x + Q)(y + Q)(z + Q) = \overline{xyz} \in \tfrac{P}{Q} - I\tfrac{P}{Q}.$$

Since $\frac{P}{Q}$ are graded 2-absorbing I-ideals of $\frac{R}{Q}$, we obtain that either $xy+Q \in \frac{P}{Q}$ or $xz+Q \in \frac{P}{Q}$ or $yz + Q \in \frac{P}{Q}$. Hence $xy \in P$ or $xz \in P$ or $yz \in P$.

□

The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{Z}^+$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{Z}^+$ [16]. The following lemma is useful in the proof of our next result.

Lemma 2.7. *Let P and I be a proper graded ideals of R. Then $I(P:a) \subset (IP:a)$ for all $a \in h(R)$.*

Proof. Let $x \in I(P:a)$. Then $x = ir$ where $ra \in P$. So $xa = ira \in IP$, this implies that $x \in (IP:a)$. Hence $I(P:a) \subset (IP:a)$. □

Theorem 2.8. *Let P be a proper graded ideal of R such that $Gr(P)$ is a graded prime ideal of R and $Gr(P) \neq P$. If $(P:a)$ is a graded I-prime ideal of R for all $a \in h(Gr(P)) - h(P)$, then P is a graded 2-absorbing I-ideal of R.*

Proof. Suppose that $a, b, c \in h(R)$ such that $abc \in P - IP$. We have $a \in Gr(P)$ or $b \in Gr(P)$ or $c \in Gr(P)$, since $P \subset Gr(P)$ and $Gr(P)$ is a graded prime ideal. Assume that $a \in Gr(P)$. If $a \in P$, then $ab \in P$ and we are done. So, let $a \in Gr(P) - P$. Now, $bc \in (P:a)$. By Lemma 2.7, $I(P:a) \subset (IP:a)$. So $bc \notin (IP:a)$. If $bc \in I(P:a)$, then $abc \in IP$ which is a contradiction. As $(P:a)$ is a graded I-prime ideal, we coclude that either $b \in (P:a)$ or $c \in (P:a)$. Hence $ab \in P$ or $ac \in P$. Therefore P is a graded 2-absorbing I-ideal of R. □

Recall that a proper graded ideal I of a graded ring R is said to be a graded irreducible ideal if whenever J_1 and J_2 are graded ideals of R with $I = J_1 \cap J_2$, then either $I = J_1$ or $I = J_2$ [16]. The following theorem shows the relationship between the graded irreducible ideals and the 2-absorbing I-ideals of R.

Theorem 2.9. *Let P be a graded irreducible ideal of R. If $Q^2 \subseteq P$ and $(P:r) = (P:r^2)$ for all $r \in h(R) - Q$, then P is a graded 2-absorbing I-ideal of R.*

Proof. Let $abc \in P - IP$ for $a, b, c \in h(R)$ with $ab \notin P$. Then either $a \notin Q$ or $b \notin Q$. Now we can assume that $(P:a) = (P:a^2)$. Take $P_1 = P + Rac$ and $P_2 = P + Rbc$. Then P_1 and P_2 are graded ideals of R containing P. We claim that $P = P_1 \cap P_2$. Let $x \in P_1 \cap P_2$. Then we can write

$$x = m_1 + r_1 ac = m_2 + r_2 bc$$

for some $m_1, m_2 \in P$ and $r_1, r_2 \in R$. Thus

$$ax = am_1 + r_1 a^2 c = am_2 + r_2 abc.$$

Since $abc \in P$, we conclude that $ax \in P$ and $r_1 a^2 c \in P$. Hence by our assumption $r_1 ac \in P$, that is $x \in P$. It means that $P = P_1 \cap P_2$. Since P is a graded irreducible ideal, we have $P = P_1$ or $P = P_2$ and so either $ac \in P$ or $bc \in P$. Therefore P is a graded 2-absorbing I-ideal of R. □

Recall that a proper graded ideal I of a graded ring R is said to be a graded primary ideal if whenever $a, b \in h(R)$ with $ab \in I$, then $a \in I$ or $b \in Gr(I)$ [16].

Lemma 2.10. *[16, Lemma 1.8] Let R be a G-graded ring and I be a graded primary ideal of R. Then $P = Gr(I)$ is a graded prime ideal of R.*

Theorem 2.11. *Let P be a graded primary ideal of R such that $(Gr(P))^2 \subseteq P$. Then P is a graded 2-absorbing I-ideal of R.*

Proof. Assume that $abc \in P - IP$ such that $a, b, c \in h(R)$. Let $ab \notin P$. If $c \in P$, then we are done. Now, suppose that $c \notin P$. Since P is a graded primary ideal, $c \in Gr(P)$ and $ab \in Gr(P)$. As $Gr(P)$ is a graded prime ideal, by Lemma 2.10, we have $a, c \in Gr(P)$ or $b, c \in Gr(P)$. Since $(Gr(P))^2 \subseteq P$, we conclude that $ac \in P$ or $bc \in P$. □

We say that a proper graded ideal P of a G-graded ring R is a graded I-primary ideal if $a, b \in h(R)$ with $ab \in P - IP$, then $a \in P$ or $b \in Gr(P)$.

Theorem 2.12. *If P is a graded I-primary ideal of R. Then $Gr(P)$ is a graded $Gr(I)$-prime ideal.*

Proof. Let $ab \in Gr(P) - Gr(I)Gr(P) = Gr(P) - Gr(IP)$ for $a, b \in h(R)$. Then $(ab)^n = a^n b^n \in P$ for some $n \in \mathbb{N}$ and $(ab)^m \notin IP$ for all $m \in \mathbb{N}$. So $a^n b^n \in P - IP$ and as P is a graded I-primary, $a^n \in P$ or $b^n \in Gr(P)$, this yields $a \in Gr(P)$ or $b \in Gr(P)$ which means $Gr(P)$ is a graded $Gr(I)$-prime ideal of R. □

Remark 2.13. Let P be a graded primary ideal of a G-graded ring R and I be a graded ideal. Then $Gr(P)$ is a graded I-prime ideal of R.

Theorem 2.14. *Let P_1 and P_2 be graded primary ideals of R. Then $Gr(P_1P_2)$ is a graded 2-absorbing I-ideal of R.*

Proof. By [15, Proposition 2.4], $Gr(P_1P_2) = Gr(P_1 \cap P_2) = Gr(P_1) \cap Gr(P_2)$. Hence $Gr(P_1P_2)$ is a graded 2-absorbing I-ideal of R, by Theorem 2.4 and Remark 2.13 □

In the following two results we characterize graded 2-absorbing I-ideals and graded I-prime ideals in decomposition rings.

Theorem 2.15. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_i is a G_i-graded ring for $(i = 1, 2)$. Suppose I_1 and I_2 be two ideals of $(R_1)_e$ and $(R_2)_e$ respectively with $I = I_1 \times I_2$. Then the following statements hold:*

(i) If $I_2R_2 = R_2$, then P_1 is a graded 2-absorbing I_1-ideal of a G_1-graded ring R_1 if and only if $P_1 \times R_2$ is a graded 2-absorbing I-ideal of a G-graded ring R.

(ii) If $I_1R_1 = R_1$, then P_2 is a graded 2-absorbing I_2-ideal of a G_2-graded ring R_2 if and only if $R_1 \times P_2$ is a graded 2-absorbing I-ideal of a G-graded ring R.

Proof. (i) Let P_1 be a graded 2-absorbing I_1-ideal of a G_1-graded ring R_1. Assume that $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in h(R_1) \times h(R_2)$ such that

$$\begin{aligned}(x_1, y_1)(x_2, y_2)(x_3, y_3) &\in P_1 \times R_2 - (I_1 \times I_2)(P_1 \times R_2)\\ &= P_1 \times R_2 - (I_1P_1 \times I_2R_2)\\ &= P_1 \times R_2 - (I_1P_1 \times R_2)\\ &= (P_1 - I_1P_1) \times R_2.\end{aligned}$$

This implies that $x_1x_2x_2 \in P_1 - I_1P_1$ and as P_1 is a graded 2-absorbing I_1-ideal of R_1, we coclude that either $x_1x_2 \in P_1$ or $x_1x_3 \in P_1$ or $x_2x_3 \in P_1$. Therefore, we have $(x_1, y_1)(x_2, y_2) \in P_1 \times R_2$ or $(x_1, y_1)(x_3, y_3) \in P_1 \times R_2$ or $(x_2, y_2)(x_3, y_3) \in P_1 \times R_2$ which implies $P_1 \times R_2$ is a graded 2-absorbing I-ideal of R. Conversely, on contrary we assume that $P_1 \times R_2$ is a graded 2-absorbing I-ideal of a G-graded ring R and P_1 is not a graded 2-absorbing I_1-ideal of R_1. Therefore, there exists $xyz \in P_1$ but neither $xy \in P_1$ nor $xz \in P_1$ nor $yz \in P_1$. Since

$$\begin{aligned}(x, 1)(y, 1)(z, 1) \in (P_1 - I_1P_1) \times R_2 &= P_1 \times R_2 - (I_1P_1 \times R_2)\\ &= P_1 \times R_2 - (I_1 \times I_2)(P_1 \times R_2)\end{aligned}$$

and as $P_1 \times R_2$ is a graded 2-absorbing I-ideal, this yields $(x, 1)(y, 1) \in P_1 \times R_2$ or $(x, 1)(z, 1) \in P_1 \times R_2$ or $(y, 1)(z, 1) \in P_1 \times R_2$. So we have $xy \in P_1$ or $xz \in P_1$ or $yz \in P_1$, which is a contradiction to ourassumption. Consequently, P_1 becomes a graded 2-absorbing I_1-ideal of R_1.

(ii) The proof is similar to part (1) and hence omitted.

□

Lemma 2.16. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_i is a G_i-graded ring for $(i = 1, 2)$. Suppose I_1 and I_2 be two ideals of $(R_1)_e$ and $(R_2)_e$ respectively with $I = I_1 \times I_2$. Then the following statements hold:*

(i) If $I_2R_2 = R_2$, then P_1 is a graded I_1-prime ideal of a G_1-graded ring R_1 if and only if $P_1 \times R_2$ is a graded I-prime ideal of a G-graded ring R.

(ii) If $I_1R_1 = R_1$, then P_2 is a graded I_2-prime ideal of a G_2-graded ring R_2 if and only if $R_1 \times P_2$ is a graded I-prime ideal of a G-graded ring R.

Theorem 2.17. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_1, R_2 are G_1-graded ring and G_2-graded ring respectively. Suppose I_1 and I_2 be two ideals of $(R_1)_e$ and $(R_2)_e$ respectively with $I = I_1 \times I_2$ and P be a proper graded ideal of R. Then the following statements are equivalent:*

(i) *P is a graded 2-absorbing I-ideal of a G-graded ring R;*

(ii) *Either $P = P_1 \times R_2$ for some graded 2-absorbing I_1-ideal P_1 of a G_1-graded ring R_1 with $I_2R_2 = R_2$ or $P = R_1 \times P_2$ for some graded 2-absorbing I_2-ideal P_2 of a G_2-graded ring R_2 with $I_1R_1 = R_1$ or $P = P_1 \times P_2$ for some graded I_1-prime ideal P_1 of a G_1-graded ring R_1 and some graded I_2-prime ideal P_2 of a G_2-graded ring R_2.*

Proof. (1) ⇒ (2) Let P be a graded 2-absorbing I-ideal of a G-graded ring R. Then $P = P_1 \times P_2$ for some graded ideal P_1 of R_1 and some graded ideal P_2 of R_2. Assume that $P_1 = R_1$. Since P is a proper graded ideal of R, $P_2 \neq R_2$. Let $S = \frac{R}{R_1 \times \{0\}}$. Then $Q = \frac{P}{R_1 \times \{0\}}$ is a graded 2-absorbing I-ideal of a G-graded ring S by Theorem 2.6 Since S is isomorphic to R_2 and $P_2 = Q$, P_2 is a graded 2-absorbing I_2-ideal of a G_2-graded ring R_2. Likewise, we can assume that $P_2 = R_2$. Since P is a proper graded ideal of R, $P_1 \neq R_1$. Now by a similar argument as in the previous case we can conclude that P_1 is a graded 2-absorbing I_1-ideal of a G_1-graded ring R_1. Suppose that $P = P_1 \times P_2$ and neither $P_1 = R_1$ nor $P_2 = R_2$. To establish the claim, suppose that P_1 is not a I_1-prime ideal P_1 of a G_1-graded ring R_1.Therefore, there exist $a, b \in h(R_1)$ with $ab \in P_1 - I_1P_1$ but neither $a \in P_1$ nor $b \in P_1$. Assume that $x, y, z \in h(R_1)$ such that $x = (a, 1), y = (b, 1)$ and $z = (1, 0)$. Hence $xyz = (ab, 0) \in P$ but neither $xy = (ab, 1) \in P$ nor $xz = (a, 0) \in P$ nor $yz = (b, 0) \in P$, which is a contradiction to our assumption. Therefore, P_1 is a I_1-prime ideal of a G_1-graded ring R_1. Likewise, by similar argument we can conclude that P_2 is a I_2-prime ideal of a G_2-graded ring R_2.

(2) ⇒ (1) If $P = P_1 \times R_2$ for some graded 2-absorbing I_1-ideal P_1 of a G_1-graded ring R_1 or $P = R_1 \times P_2$ for some graded 2-absorbing I_2-ideal P_2 of a G_2-graded ring R_2, then by Theorem 2.15 P is a graded 2-absorbing I-ideal of a G-graded ring R. Now, suppose $P = P_1 \times P_2$ for some graded I_1-prime ideal P_1 of a G_1-graded ring R_1 and some graded I_2-prime ideal P_2 of a G_2-graded ring R_2. Thus $Q_1 = P_1 \times R_2$ and $Q_2 = R_1 \times P_2$ are graded I-prime ideals of a G-graded ring R by Theorem 2.16. Therefore, $Q_1 \cap Q_2 = P_1 \times P_2 = P$ is a graded 2-absorbing I-ideal of a G-graded ring R by Theorem 2.4.

□

Lemma 2.18. *[10, Theorem 2.1, p.2] Let $I \subseteq P$ be ideals of a ring R, where P is a prime ideal. Then the following statements are equivalent:*

(i) *P is a minimal prime ideal of I;*

(ii) *For each $x \in P$, there is a $y \in R - P$ and a nonnegative integer n such that $yx^n \in I$.*

Theorem 2.19. *Let P be a graded 2-absorbing I-ideal of a G-graded ring R. Then there are at most two graded prime ideals of R that are minimal over P.*

Proof. Assume that $F = \{P_i \mid P_i$ is a graded prime ideal of R that is minimal over $P\}$ and suppose that F has at least three elements. Let $P_1, P_2 \in F$ be two distinct graded prime ideals. Hence there is an $x_1 \in P_1 - P_2$ and there is an $x_2 \in P_2 - P_1$ where $x_1, x_2 \in h(R)$. First we show that $x_1x_2 \in P$. By Lemma 2.18, there is a $c_2 \notin P_1$ and a $c_1 \notin P_2$ such that $c_2x_1^n \in P$ and $c_1x_2^m \in P$ for some $n, m \geq 1$. Now we prove that $c_2x_1^n \notin IP$ and $c_1x_2^m \notin IP$. So $c_1, c_2 \notin IP$, if

$$c_2x_1^n \in IP \subseteq P \subseteq P_1 \cap P_2 \subseteq P_2,$$

then $c_2x_1^n \in P_2$ we conclude that either $c_2 \in P_2$ or $x_1 \in P_2$ which is a contradiction. Also by similar way we get $c_1x_2^m \notin IP$. Since $x_1, x_2 \notin P_1 \cap P_2$ and being P graded 2-absorbing I-ideal of R, we conclude that $c_1x_2 \in P$ and $c_2x_1 \in P$. Since $x_1, x_2 \notin P_1 \cap P_2$ and $c_1x_2, c_2x_1 \in P \subseteq P_1 \cap P_2$, we obtain $c_1 \in P_1 - P_2$ and $c_2 \in P_2 - P_1$, and thus $c_1, c_2 \notin P_1 \cap P_2$. So $(c_1 + c_2)x_1x_2 \in P$, since $c_1x_2 \in P$ and $c_2x_1 \in P$. We have $c_1 + c_2 \notin P_1$ and $c_1 + c_2 \notin P_2$. Since $(c_1 + c_2)x_2 \notin P_1$ and $(c_1 + c_2)x_1 \notin P_2$, we conclude that neither $(c_1 + c_2)x_2 \in P$ and $(c_1 + c_2)x_1 \in P$, and hence $x_1x_2 \in P$. Now suppose $P_3 \in F$ such that P_3 is neither P_1 nor P_2. Then take $y_1 \in P_1 - (P_2 \cup P_3)$, $y_2 \in P_2 - (P_1 \cup P_3)$, and $y_3 \in P_3 - (P_1 \cup P_2)$. By previous argument $y_1y_2 \in P$. Since $P \subseteq P_1 \cap P_2 \cap P_3$ and $y_1y_2 \in P$, we get that $y_1 \in P_3$ or $y_2 \in P_3$ which is a contradiction. Therefore F has at most two graded prime ideals that are minimal over P. □

3 Graded n-absorbing I-ideals

Definition 3.1. Let R be a G-graded ring and I be a fixed proper ideal of R_e and let $n \geq 1$ be a positive integer. A proper graded ideal P of R is called an n-absorbing I-ideal if $a_1, a_2, \ldots, a_{n+1} \in h(R)$ with $a_1 \ldots a_{n+1} \in P - IP$, then $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n+1} \in P$ for some $i \in \{1, 2, \ldots, n+1\}$.

Corollary 3.2. *Let P be a graded n-absorbing I-ideal of R. Then P is a graded m-absorbing I-ideal of R for all $m \geq n$.*

Lemma 3.3. *A proper graded ideal P of a G-graded ring R is a graded n-absorbing I-ideal if and only if $\frac{P}{IP}$ is a graded n-absorbing 0-ideal.*

Theorem 3.4. *If P is a proper graded ideal of a G-graded ring R with $IGr(P) = Gr(IP)$, then $Gr(P)$ is a graded n-absorbing I-ideal.*

Proof. Suppose $x_1x_2 \ldots x_{n+1} \in Gr(P) - IGr(P)$ with $x_1, x_2, \ldots, x_{n+1} \in h(R)$. Then

$$(x_1x_2 \ldots x_{n+1})^m = x_1{}^m x_2{}^m \ldots x_{n+1}{}^m \in P - IP.$$

Thus being graded n-absorbing I-ideal gives us

$$x_1{}^m x_2{}^m \ldots x_{i-1}{}^m x_{i+1}{}^m \ldots x_{n+1}{}^m \in P$$

for some $i \in \{1, 2, \ldots, n+1\}$. So $(x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{n+1})^m \in P$ which implies that $x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{n+1} \in Gr(P)$ for some $i \in \{1, 2, \ldots, n+1\}$. Therefore $Gr(P)$ is a graded n-absorbing I-ideal of R. □

Based on the following result, exploring that what occurs in a more generic case, in other terms, what is the intersection structure of m graded I-prime ideals of R.

Theorem 3.5. *If P_j is a non zero graded I-prime ideal of a G-graded ring R for $1 \leq j \leq m$, then $\cap_{j=1}^m P_j$ is a graded n-absorbing I-ideal.*

Proof. Let $a_1a_2 \ldots a_{n+1} \in \cap_{j=1}^m P_j - I\cap_{j=1}^m P_j$ for $a_1, a_2, \ldots, a_{n+1} \in h(R)$. Thus $a_1a_2 \ldots a_{n+1} \in P_j - IP_j$ for all $j = 1, \ldots, m$ and so $a_i(a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_{n+1}) \in P_j - IP_j$ for all $j = 1, \ldots, m$ and for some $i \in \{1, 2, \ldots, n+1\}$. Since P_j is a graded I-prime ideal for all $j = 1, \ldots, m$, we conclude that either $a_i \in P_j$ or $a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_{n+1} \in P_j$ for all $j = 1, \ldots, m$. This yields $a_i \in \cap_{j=1}^m P_j$ or $a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_{n+1} \in \cap_{j=1}^m P_j$. If $a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_{n+1} \in \cap_{j=1}^m P_j$, then there is nothing to prove. Now in the other case we get $a_i(a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_n) \in \cap_{j=1}^m P_j$ for some $i \in \{1, 2, \ldots, n\}$ and $a_i(a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}) \in \cap_{j=1}^m P_j$ for $i = n+1$. Hence $\cap_{j=1}^m P_j$ is a graded n-absorbing I-ideal. □

Theorem 3.6. *Let P_j be a graded primary ideal of R for each $1 \leq j \leq n$. Then $Gr(P_1P_2 \ldots P_n)$ is a graded n-absorbing I-ideal of R.*

Proof. The proof is straightforward by using a similar argument as in the Theorem 2.14. □

Proposition 3.7. *Let P be a graded n-absorbing I-ideal of a G-graded ring R and let $S \subseteq R$ be a multiplicative closed set of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is a graded n-absorbing $S^{-1}I$-ideal of $S^{-1}R$.*

Proof. Assume $\frac{r_1}{x_1}, \ldots, \frac{r_{n+1}}{x_{n+1}} \in h(S^{-1}R)$ such that

$$\frac{r_1r_2 \ldots r_{n+1}}{x_1x_2 \ldots x_{n+1}} \in S^{-1}P - S^{-1}IS^{-1}P = S^{-1}(P - IP).$$

Then we have $yr_1r_2 \ldots r_{n+1} \in P - IP$ for some $y \in S$. By taking yr_1 as one element, either $r_2r_3 \ldots r_{n+1} \in P$ or $yr_1 \ldots r_{i-1}r_{i+1} \ldots r_{n+1} \in P$ for $i = 2, 3, \ldots, n+1$. Hence

$$\frac{r_2 \ldots r_{n+1}}{x_2 \ldots x_{n+1}} = \frac{r_2}{x_2} \ldots \frac{r_{n+1}}{x_{n+1}} \in S^{-1}P$$

or

$$\frac{yr_1 \ldots r_{i-1}r_{i+1} \ldots r_{n+1}}{yx_1 \ldots x_{i-1}x_{i+1} \ldots x_{n+1}} = \frac{r_1}{x_1} \ldots \frac{r_{i-1}}{x_{i-1}} \frac{r_{i+1}}{x_{i+1}} \ldots \frac{r_{n+1}}{x_{n+1}} \in S^{-1}P,$$

this implies that $S^{-1}P$ is a graded n-absorbing $S^{-1}I$-ideal of $S^{-1}R$. □

Theorem 3.8. *Let P be a proper graded ideal of a G-graded ring R. If P is a graded n-absorbing I-ideal that is not a graded n-absorbing ideal, then $P^{n+1} \subseteq IP$.*

Proof. Suppose that $P^{n+1} \nsubseteq IP$. Now we have to show that P is a graded n-absorbing ideal. Take $x_1, x_2, \dots, x_{n+1} \in h(R)$ such that $x_1x_2\dots x_{n+1} \in P$. If $x_1x_2\dots x_{n+1} \notin IP$ and P is a graded n-absorbing I-ideal, then we are done. Now, for the case $x_1x_2\dots x_{n+1} \in IP$, we have $x_1x_2\dots x_{n+1-k}P^k \subseteq IP$ for $k = 1, 2, \dots, n$, since otherwise we conclude that $x_1x_2\dots x_{n+1-k}p_1p_2\dots p_k \notin IP$ for $p_1, p_2, \dots, p_k \in P$ and so

$$x_1x_2\dots x_{n+1-k}(x_{n+2-k} + p_1)\dots(x_{n+1} + p_k) \in P - IP.$$

As P is a graded n-absorbing I-ideal, $x_1\dots x_{i-1}x_{i+1}\dots x_{n+1} \in P$ for some $i \in \{1, 2, \dots, n+1\}$. Similarly, we can assume that for all $i_1, i_2\dots i_{n+1-k} \in \{1, 2, \dots, n+1\}$, $x_{i_1}\dots x_{i_{n+1-k}}P^k \subseteq IP$ with $1 \le k \le n+1$. Since $P^{n+1} \nsubseteq IP$, there exist $r_1, r_2, \dots, r_{n+1} \in P$ with $r_1r_2\dots r_{n+1} \notin IP$. This yields

$$(x_1 + r_1)(x_2 + r_2)\dots(x_{n+1} + r_{n+1}) \in P - IP.$$

Since P is a graded n-absorbing I-ideal, we get

$$(x_1 + r_1)\dots(x_{i-1} + r_{i-1})(x_{i+1} + r_{i+1})\dots(x_{n+1} + r_{n+1}) \in P$$

and so $x_1\dots x_{i-1}x_{i+1}\dots x_{n+1} \in P$ for some $i \in \{1, 2, \dots, n+1\}$. Consequently, P is a graded n-absorbing ideal. □

We conclude from Theorem 3.8 that a graded n-absorbing I-ideal P with $P^{n+1} \nsubseteq IP$ is a graded n-absorbing ideal.

Corollary 3.9. *Let R be a G-graded ring and P be a proper graded ideal of R. If P is a graded n-absorbing 0-ideal that is not a graded n-absorbing ideal, then $P^{n+1} = 0$.*

Corollary 3.10. *Let P be a graded n-absorbing I-ideal of a G-graded ring R with $IP \subseteq P^{n+2}$. Then P is a graded n-absorbing $\cap_{i=1}^{\infty} P^i$-ideal $(n \ge 1)$.*

Proof. If P is a graded n-absorbing ideal, then P is a graded n-absorbing I-ideal and so is a graded n-absorbing $\cap_{i=1}^{\infty} P^i$-ideal. Suppose that P is not a graded n-absorbing ideal, then Theorem 3.8 gives us $P^{n+1} \subseteq IP \subseteq P^{n+2}$. Hence $IP = P^k$ for each $k \ge n+1$ and hence $\cap_{i=1}^{\infty} P^i = IP$. Therefore P is a graded n-absorbing $\cap_{i=1}^{\infty} P^i$-ideal. □

Let R and S be two graded rings. If P is a graded n-absorbing 0-ideal of R. Then $P \times S$ need not be a graded n-absorbing 0-ideal of $R \times S$. For a particularly case see [6, Theorem 2]. However, $P \times S$ is a graded n-absorbing I-ideal for each $I = I_1 \times I_2$ where I_1 and I_2 be two graded ideals of R and S respectively with $\cap_{i=1}^{\infty} (P \times S)^i \subseteq I(P \times S) \subseteq P \times S$.

Theorem 3.11. *(i) Let R and S be two G-graded rings and let P be a graded n-absorbing 0-ideal of R. Then $J = P \times S$ is a graded n-absorbing I-ideal of $R \times S$, for each I with $\cap_{i=1}^{\infty} (P \times S)^i \subseteq I(P \times S) \subseteq P \times S$.*

(ii) Let R be a G-graded ring and J be a graded finitely generated proper ideal of R. Suppose that J is a graded n-absorbing I-ideal, where $IP \subseteq J^{n+2}$. Then either J is a graded n-absorbing 0-ideal or $J^{n+1} \neq 0$ is idempotent and R decomposes as $T \times S$, where $S = J^{n+1}$ and $J = P \times S$, where P is a graded n-absorbing 0-ideal. Hence J is a graded n-absorbing I-ideal for each I with $\cap_{i=1}^{\infty} J^i \subseteq IJ \subseteq J$.

Proof. (i) Let R and S be two G-graded rings and let P be a graded n-absorbing 0-ideal of R. Then $P \times S$ need not be a graded n-absorbing 0-ideal of $R \times S$. In fact, $P \times S$ is a graded n-absorbing 0-ideal if and only if $P \times S$ is a graded prime ideal. However, $P \times S$ is a graded n-absorbing I-ideal for each

$$\cap_{i=1}^{\infty} (P \times S)^i \subseteq I(P \times S).$$

If P is a graded n-absorbing ideal, then $P \times S$ is a graded n-absorbing ideal and thus is a graded n-absorbing I-ideal. Assume that P is not a graded n-absorbing ideal. Then $P^{n+1} = 0$ and $(P \times S)^{n+1} = 0 \times S$. Therefore

$$\cap_{i=1}^{\infty}(P \times S)^i = \cap_{i=1}^{\infty} P^i \times S = 0 \times S.$$

Hence

$$P \times S - \cap_{i=1}^{\infty}(P \times S)^i = P \times S - 0 \times S = (P - 0) \times S.$$

Since P is a graded n-absorbing 0-ideal, $P \times S$ is a graded n-absorbing $\cap_{i=1}^{\infty}(P \times S)^i$-ideal and as

$$\cap_{i=1}^{\infty}(P \times S)^i \subseteq I(P \times S),$$

$P \times S$ is a graded n-absorbing I-ideal.

(ii) If J is a graded n-absorbing ideal, then J is a graded n-absorbing 0-ideal. So we can assume that J is not a graded n-absorbing ideal. Then $J^{n+1} \subseteq IP$ and hence

$$J^{n+1} \subseteq IP \subseteq J^{n+2},$$

so $J^{n+1} = J^{n+2}$. Hence J^{n+1} is idempotent. Since J^{n+1} is finitely generated, $J^{n+1} = (e)$ for some idempotent $e \in R$. Suppose $J^{n+1} = 0$. Then $IP = 0$, and hence J is a graded n-absorbing 0-ideal. Assume that $J^{n+1} \neq 0$, and put $S = J^{n+1} = Re$ and $T = R(1-e)$, so R decomposes $T \times S$. Let $P = J(1-e)$; so $J = P \times S$, where

$$P^{n+1} = (J(1-e))^{n+1} = J^{n+1}(1-e)^{n+1} = (e)(1-e) = 0.$$

We claim that P is a graded n-absorbing 0-ideal. Let $x_1, x_2, \ldots, x_{n+1} \in h(R)$ with $0 \neq x_1x_2\ldots x_{n+1} \in P$. Then

$$\begin{aligned}(x_1, 0)(x_2, 0)\ldots(x_{n+1}, 0) = (x_1x_2\ldots x_{n+1}, 0) &\in P \times S - (P \times S)^{n+1}\\ &= P \times S - 0 \times S \subseteq P - IP,\end{aligned}$$

since $IP \subseteq J^{n+2}$, which implies that

$$IP \subseteq J^{n+2} = (P \times S)^{n+2} = 0 \times S.$$

Hence $J - J^{n+1} \subseteq J - IP$. As J is a graded n-absorbing I-ideal,

$$(x_1x_2\ldots x_{i-1}x_{i+1}\ldots x_{n+1}, 0) \in P \times S = J$$

for some $i \in \{1, 2, \ldots, n+1\}$. Thus $x_1x_2\ldots x_{i-1}x_{i+1}\ldots x_{n+1} \in P$. Therefore P is a graded n-absorbing 0-ideal of R.

□

Corollary 3.12. *Let R be an indecomposable G-graded ring and let P be a graded finitely generated n-absorbing I-ideal of R, where $IP \subseteq P^{n+2}$. Then P is a graded n-absorbing 0-ideal. Furthermore, if R is a G-graded integral domain, then P is actually a graded n-absorbing ideal.*

Corollary 3.13. *A proper graded ideal P of a G-graded Noetherian integral domain R is a graded n-absorbing ideal if and only if P is a graded n-absorbing P^{n+1}-ideal for $n \geq 2$.*

Theorem 3.14. *Let P be a proper graded ideal of a G-graded ring R. Then the following conditions are equivalent.*

(i) P is a graded n-absorbing I-ideal

(ii) For $x_1, x_2, \ldots, x_n \in h(R) - h(P)$:

$$(P : x_1x_2\ldots x_n) = \cup_{i=1}^{n}(P : x_1x_2\ldots x_{i-1}x_{i+1}\ldots x_n) \cup (IP : x_1x_2\ldots x_n)$$

Proof. (1) $\Rightarrow$ (2) Suppose $x_1, x_2, \ldots, x_n \in h(R) - h(P)$ and $y \in (P : x_1x_2 \ldots x_n)$. Then $x_1x_2 \ldots x_ny \in P$. If $x_1x_2 \ldots x_ny \notin IP$, then $x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_ny \in P$ for some $i \in \{1, 2, \ldots, n\}$, and so $y \in (P : x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_n)$. If $x_1x_2 \ldots x_ny \in IP$, then $y \in (IP : x_1x_2 \ldots x_n)$. Hence

$$(P : x_1x_2 \ldots x_n) \subseteq \cup_{i=1}^{n}(P : x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_n) \cup (IP : x_1x_2 \ldots x_n)$$

The other containment always holds.

(2) $\Rightarrow$ (1) Suppose $x_1x_2 \ldots x_{n+1} \in P - IP$. If $x_1x_2 \ldots x_n \in P$, then we are done. Assume that $x_1x_2 \ldots x_n \notin P$. Thus

$$(P : x_1x_2 \ldots x_n) = \cup_{i=1}^{n}(P : x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_n) \cup (IP : x_1x_2 \ldots x_n).$$

Since $x_1x_2 \ldots x_{n+1} \in P$, $x_{n+1} \in (P : x_1x_2 \ldots x_n)$ and the fact $x_1x_2 \ldots x_{n+1} \notin IP$ gives us $x_{n+1} \notin (IP : x_1x_2 \ldots x_n)$. Hence $x_{n+1} \in (P : x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_n)$, for some $i \in \{1, 2, \ldots, n\}$, that is $x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{n+1} \in P$. Thus P is a graded n-absorbing I-ideal. $\square$

In the following result we show that all components of graded n-absorbing I-ideals in decomposition rings is the product of graded n-absorbing ideals except one of the components is the whole ring.

Proposition 3.15. *Let $R = R_1 \times R_2 \times \ldots \times R_{n+1}$, where R_i is a G-graded ring, for $i \in \{1, 2, \ldots, n+1\}$. If P is a graded n-absorbing I-ideal of R, then either $P = IP$ or $P = P_1 \times P_2 \times \ldots \times P_{i-1} \times R_i \times P_{i+1} \times \ldots \times P_{n+1}$ for some $i \in \{1, 2, \ldots, n+1\}$ and if $P_j \neq R_i$ for $j \neq i$, then P_j is a graded n-absorbing ideal in R_j.*

Proof. Let $P = P_1 \times P_2 \times \ldots \times P_{n+1}$ be a graded n-absorbing I-ideal of R and $P \neq IP$. Then there exists $(x_1, x_2, \ldots, x_{n+1}) \in P - IP$, and so

$$(x_1, 1, \ldots, 1)(1, x_2, \ldots, 1) \ldots (1, 1, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_{n+1}) \in P.$$

As P is a graded n-absorbing I-ideal, we have $(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1}) \in P$ for some $i \in \{1, 2, \ldots, n+1\}$. Thus $(0, 0, \ldots, 0, 1, 0, \ldots, 0) \in P$ and hence

$$P = P_1 \times P_2 \times \ldots \times P_{i-1} \times R_i \times P_{i+1} \times \ldots \times P_{n+1}.$$

If $P_j \neq R_i$ for $j \neq i$, then we have to prove P_j is a graded n-absorbing ideal in R_j. Let $i < j$ and let $y_1y_2 \ldots y_{n+1} \in P_j$. Then

$$\begin{aligned} &(0, 0, \ldots, 0, 1, 0, \ldots, 0, y_1y_2 \ldots y_n, 0, \ldots, 0) \\ = &(0, 0, \ldots, 1, 0, \ldots, y_1, \ldots, 0)(0, 0, \ldots, 1, 0, \ldots, y_2, \ldots, 0) \\ &\ldots (0, 0, \ldots, 1, 0, \ldots, y_{n+1}, \ldots, 0) \in P - IP \end{aligned}$$

and the graded n-absorbing I-ideal P gives that

$$(0, 0, \ldots, 0, 1, 0, \ldots, 0, y_1y_2 \ldots y_{k-1}y_{k+1} \ldots y_{n+1}, 0, \ldots, 0) \in P$$

for some $k \in \{1, 2, \ldots, n+1\}$. Thus $y_1y_2 \ldots y_{k-1}y_{k+1} \ldots y_{n+1} \in P_j$ and hence P_j is a graded n-absorbing ideal in R_j. We can do the same arguments for the case $j < i$. $\square$

We characterize G-graded rings in which every proper graded ideal is a graded n-absorbing I-ideal.

Theorem 3.16. *Let R be a G-graded ring and let $|Max(R)| \geq n + 1 \geq 2$. Every proper graded ideal of R is a graded n-absorbing I-ideal if and only if every quotient of R is a product of $(n+1)$-fields.*

Proof.

$(\Rightarrow)$: Let $m_1, m_2, \ldots, m_{n+1}$ be a distinct graded maximal ideals of R. Then $m = m_1m_2 \ldots m_{n+1}$ is a graded n-absorbing I-ideal of R. We want to show that m is not a graded n-absorbing ideal. First to show that $m_i \nsubseteq \cup_{j \neq i} m_j$ for all $i \in \{1, 2, \ldots, n+1\}$, we suppose by contrary that $m_i \subseteq \cup_{j \neq i} m_j$. Then there exists m_j with $m_i \subseteq m_j$ by prime avoidance lemma, which contradicts the fact that m_i, $i = 1, 2, \ldots, n+1$ are distinct graded maximal ideals. Hence there exists

$$x_i \in m_i - \cup_{i\neq j=1}^{n+1} m_j$$

and so $x_1x_2\dots x_{n+1} \in m$. If there exists $j \in \{1,2,\dots,n+1\}$ with $x_1x_2\dots x_{j-1}x_{j+1}\dots x_{n+1} \in m \subseteq m_j$, then $x_i \in m_j$, for some $i \neq j$, a contradiction. Hence m is not a graded n-absorbing ideal and so $m^{n+1} = Im$. Thus by Chinese remainder theorem,

$$\frac{R}{Im} \cong \frac{R}{m_1^{n+1}} \times \frac{R}{m_2^{n+1}} \times \dots \times \frac{R}{m_{n+1}^{n+1}}.$$

Put $F_i = \frac{R}{m_i^{n+1}}$. If F_i is not field, then it has a nonzero proper graded ideal K and so

$$0 \times 0 \times \dots \times 0 \times K \times 0 \times \dots \times 0$$

is a graded n-absorbing 0-ideal of $\frac{R}{Im}$. Thus by Proposition 3.15, we have $K = F_i$ or $K = 0$, which is impossible. Hence F_i is a field.

$(\Leftarrow)$: Let P be a proper graded ideal of R. Then

$$\frac{R}{IP} \cong F_1 \times F_2 \times \dots \times F_{n+1}$$

and

$$\frac{P}{IP} \cong P_1 \times P_2 \times \dots \times P_{n+1},$$

where P_i is an ideal of F_i, $i = 1,2,\dots,n+1$. If $P = IP$, there is nothing to prove, otherwise we have $P_j = 0$ for at least one $j \in \{1,2,\dots,n+1\}$, since $\frac{P}{IP}$ is proper. Therefore $\frac{P}{IP}$ is a graded n-absorbing 0-ideal of $\frac{R}{IP}$ and P is a graded n-absorbing I-ideal of R.

□

Corollary 3.17. *Let R be a G-graded ring and let $|Max(R)| \geq n+1 \geq 2$. Every proper graded ideal of R is a graded n-absorbing 0-ideal if and only if $R \cong F_1 \times F_2 \times ... \times F_{n+1}$, where $F_1, F_2, ..., F_{n+1}$ are fields.*

Theorem 3.18. *Let P be a graded n-absorbing I-ideal of a G-graded ring R. Then there are at most n graded prime ideals of R that are minimal over P.*

Proof. Let $C = \{Q_i \mid Q_i$ is a grade prime ideal of R that is minimal over $P\}$ and let C has at least n elements. Assume $Q_1, Q_2, ..., Q_n \in C$ be distinct elements and $x_i \in Q_i - \cup_{i\neq j} Q_j$ where $x_i \in h(R)$ for $i = 1,2,...,n$. By Lemma 2.18 there is a $y_i \notin Q_i$ where $y_i \in h(R)$ for $i = 1,2,...,n$ such that $y_i x_i^{t_i} \in P$ for some positive integers $t_1, t_2, ..., t_n$. Since $x_i \notin \cap_{j=1}^n Q_j$ for all $i = 1,2,...,n$ and P is graded n-absorbing I-ideal we have $y_i x_i^{n-1} \in P$. As $x_i \notin \cap_{j=1}^n Q_j$ and $y_i x_i^{n-1} \in P \subseteq \cap_{j=1}^n Q_j$ we get that $y_i \in Q_i - \cup_{i\neq j} Q_j$ and so $y_i \notin \cap_{j=1}^n Q_j$ for all $i = 1,2,...,n$. Since $y_i x_i^{n-1} \in P$, $\sum_{j=1}^n y_j \prod_{i=1}^n x_i^{n-1} \in P$ and clearly $\sum_{j=1}^n y_j \notin Q_i$ for all $i = 1,2,...,n$. Also $\sum_{j=1}^n y_j \prod_{i\neq r}^n x_r^{n-1} \notin P$, since $\sum_{j=1}^n y_j \prod_{i\neq r}^n x_r^{n-1} \notin Q_i$, for $i = 1,2,...,n$, and being P graded n-absorbing I-ideal, we obtain $\prod_{i=1}^n x_i^{n-1} \in P$. Now, suppose $Q_{n+1} \in C$ such that $Q_{n+1} \neq Q_i$, for $i = 1,2,...,n$. Take $z_i \in Q_i - \cup_{i\neq j} Q_j$ for $i = 1,2,...,n+1$ and by previous argument, $\prod_{i=1}^n z_i^{n-1} \in P$. Since $P \subseteq \cap_{i=1}^{n+1} Q_i$ and $\prod_{i=1}^n z_i^{n-1} \in P$, we have $z_i^{n+1} \in Q_{n+1}$ for some $i = 1,2,...,n$ and consequently, $z_i \in Q_{n+1}$ for $i = 1,2,...,n$ which is a contraduction. Therefore C has at most n elements. □

Theorem 3.19. *Let P be a graded n-absorbing ideal of a G-graded ring R. Then one of the following statements must hold:*

(i) $Gr(P) = Q$ is a graded prime ideal of R such that $Q^2 \subseteq P$.

(ii) $Gr(P) = \cap_{i=1}^n Q_i, (\prod_{i=1}^n Q_i)^{n-1} \subseteq P$ and $Gr(P) \subseteq P$, where $Q_i, i = 1,2,\dots,n$ are the only distinct graded prime ideals of R that are minimal over P.

Proof. By Theorem 3.18 we conclude that either $Gr(P) = Q$ is a graded prime ideal of R or $Gr(P) = \cap_{i=1}^n Q_i$, where Q_i are the only distinct graded prime ideals of R that are minimal over P. Assume that $Gr(P) = Q$ is a graded prime ideal of R and take $x, y \in Q$. By [8, Theorem 2.1], we get $x^n, y^n \in P$ and so $x(x^{n-1} + y^{n-1})y \in P$. Since $P$is a graded n-absorbing ideal

$$x(x^{n-1} + y^{n-1}) = x^n + xy^{n-1} \in P$$

or

$$(x^{n-1} + y^{n-1})y = x^{n-1}y + y^n \in P \text{ or } xy \in P.$$

Hence in either case, we have $Q^n \subseteq P$. For the second assertions, suppose $Gr(P) = \cap_{i=1}^{n} Q_i$,where Q_i are the only distinct graded prime ideals of R that are minimal over P and take $x, y \in Gr(P)$. By a bove argument $xy \in P$ and so $Gr(P)^2 \subseteq P$. To prove $\prod_{i=1}^{n} Q_i \subseteq P$, take $x_i \in P_i - \cup_{i \neq j}^{n} P_j$ and by the proof of Theorem 3.18, we have $x_1^{n-1} \ldots x_n^{n-1} \in P$. Let $r \in Gr(P)$ and $z_i \in P_i - \cup_{i \neq j}^{n} P_j$. Then $z_1^{n-1} \ldots z_n^{n-1} \in P$ by the proof of Theorem 3.18 and $r + z_1 \in P_1 - \cup_{i \neq j}^{n} P_j$. Hence

$$rz_2^{n-1} \ldots z_n^{n-1} + z_1^{n-1} \ldots z_n^{n-1} = (r + z_1^{n-1})z_2^{n-1} \ldots z_n^{n-1} \in P$$

and so $rz_2^{n-1} \ldots z_n^{n-1} \in P$. □

Theorem 3.20. *Let P be a graded n-absorbing I-ideal of a G-graded ring R and $Q_1 \neq Q_2$ be distinct graded prime ideals of R and $I(P : a) = (IP : a)$ for all $a \in h(R)$. Then*

(i) if $Gr(P) = Q_1$, then $(P :_R a)$ is a graded n-absorbing I-ideal of R with $Gr(P :_R a) = Q_1, \forall a \in h(R) - Q_1$;

(ii) if $Gr(P) = Q_1 \cap Q_2$, then $(P :_R a)$ is a graded n-absorbing I-ideal of R with $Gr(P :_R a) = Q_1 \cap Q_2, \forall a \in h(R) - \{Q_1 \cup Q_2\}$.

Proof. (i) Let $a \in h(R) - Q_1$ and $x_1, \ldots, x_{n+1} \in h(R)$ with

$$x_1 \ldots x_{n+1} \in (P :_R a) - I(P :_R a).$$

Then $x_1 \ldots x_{n+1}a \in P - IP$ since

$$IP \subseteq I(P :_R a) = (IP :_R a).$$

So $x_1 \ldots x_{i-1}x_{i+1} \ldots x_{n+1}a \in P$ or $x_1 \ldots x_{n-1}a \in P$ or $x_1 \ldots x_{n+1} \in P$ for $i = 1, 2, \ldots, n-1$, since P is a graded n-absorbing I-ideal. If one of the first two cases holds, then we are done. If $x_1 \ldots x_{n+1} \in P$, then

$$x_1 \ldots x_{i-1}x_{i+1} \ldots x_{n+1} \in P$$

for $i = 1, 2, \ldots, n+1$ which implies

$$x_1 \ldots x_{i-1}x_{i+1} \ldots x_{n+1}a \in P.$$

Thus $(P :_R a)$ is graded n-absorbing I-ideal of a R and as $P \subseteq (P :_R a) \subseteq Q_1$, we have $Gr(P :_R a) = Q_1$.

(ii) By similar arguments to that of (1), we can prove $(P :_R a)$ is graded n-absorbing I-ideal. □

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