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TFU extensions in LCA groups

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Abstract Let $\pounds$ be the category of all locally compact abelian (LCA) groups. Let $G \in \pounds$ and $H \subseteq G$ be a subgroup. The first Ulm subgroup of G is denoted by $G^{(1)}$ and the closure of H by $\overline{H}$. A proper short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be a TFU extension if $0 \to \overline{A^{(1)}} \xrightarrow{\overline{\phi}} \overline{B^{(1)}} \xrightarrow{\overline{\psi}} \overline{C^{(1)}} \to 0$ is a proper short exact sequence where $\overline{\phi} = \phi \mid_{\overline{A^{(1)}}}$ and $\overline{\psi} = \psi \mid_{\overline{B^{(1)}}}$. We introduce some results on TFU extensions. Also, we establish conditions under which the TFU extensions are split.

1 Introduction

Throughout, all groups are Hausdorff topological abelian groups and will be written additively. Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called proper if it is open onto its image, and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be proper exact if ϕ and ψ are proper morphisms. In this case the sequence is called an extension of A by C (in $\pounds$). Following [6], we let $Ext(C, A)$ denote the group of extensions of A by C. A subgroup H of a group G is called pure if $nH = H \cap nG$ for all positive integers n. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a pure extension if $\phi(A)$ is pure in B. The elements represented by pure extensions of A by C form a subgroup of $Ext(C, A)$ which is denoted by $Pext(C, A)$ (For more on $Pext$, see [5] and [8]). In this paper, we introduce a new concept on extensions. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is called a TFU extension if $0 \to \overline{A^{(1)}} \xrightarrow{\overline{\phi}} \overline{B^{(1)}} \xrightarrow{\overline{\psi}} \overline{C^{(1)}} \to 0$ be an extension where $\overline{\phi} = \phi \mid_{\overline{A^{(1)}}}$ and $\overline{\psi} = \psi \mid_{\overline{B^{(1)}}}$. In Section 1, we show that every extension of a divisible group by an LCA group is a TFU extension (Corollary 2.6). We show that every pure extension of an LCA group by a compact totally disconnected group is a TFU extension (Corollary 2.8). In Section 2, we establish some results on splitting of TFU extensions (see Lemma 3.3,3.4,3.5,3.7,3.8,).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals with the discrete topology, $\mathbb{Z}$ is the group of integers and $\mathbb{Z}(n)$ is the cyclic group of order n. For any group G, G_0 is the identity component of G and tG, the maximal torsion subgroup of G. For groups G and H, $Hom(G, H)$ is the group of all continuous homomorphisms from G to H, endowed with the compact-open topology. The dual group of G is $\hat{G} = Hom(G, \mathbb{R}/\mathbb{Z})$. For more on locally compact abelian groups, see [9].

2 TFU extensions

Let A and C be groups in $\pounds$. In this section, we define the concept of a TFU extension of A by C.

Definition 2.1. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is called a TFU extension if $0 \to \overline{A^{(1)}} \xrightarrow{\overline{\phi}} \overline{B^{(1)}} \xrightarrow{\overline{\psi}} \overline{C^{(1)}} \to 0$ is an extension.

Remark 2.2. Let G_1 and G_2 be two groups. An easy calculation shows that $\overline{(G_1 \oplus G_2)^{(1)}} = \overline{G_1^{(1)}} \oplus \overline{G_2^{(1)}}$.

Definition 2.3. The extension $0 \to A \to A \oplus C \to C \to 0$ is called the trivial extension.

Lemma 2.4. *Let A and C be groups in $\pounds$. The trivial extension of A by C is a TFU extension.*

Proof. It is clear by Remark 2.2. □

Lemma 2.5. *Let $G \in \pounds$ and H be a closed, divisible subgroup of G. Then $\overline{(G/H)^{(1)}} = \overline{G^{(1)}}/H$.*

Proof. Let $\pi : G \to G/H$ be the natural mapping. Then, $\pi(\overline{G^1}) \subseteq \overline{\pi(G^{(1)})} = \overline{(G/H)^{(1)}}$. On the other hand, $\pi(\overline{G^{(1)}}) = (\overline{G^{(1)}})/H$. So, $\overline{G^{(1)}}/H \subseteq \overline{(G/H)^{(1)}}$. Let $x + H \in \overline{(G/H)^{(1)}}$. We show that $x \in \overline{G^{(1)}}$ (and hence $\overline{(G/H)^{(1)}} \subseteq \overline{G^{(1)}}/H$). Let V be an open subset of G containing x and n, an arbitrary positive integer. Then $y + H \in (V + H)/H \bigcap (G/H)^{(1)} \neq \phi$ for some $y \in G$. From $y + H \in (V + H)/H$, deduce that $y + H = z + H$ for some $z \in V$. Since H is divisible, it follows that $z = y + nh$ for some $h \in H$. On the other hand, $y + H \in (G/H)^{(1)}$. So $y + H = ng + H$ for some $g \in G$. Hence $z \in nG$. This shows that $z \in V \bigcap G^{(1)}$ and hence, $x \in \overline{G^{(1)}}$. □

Corollary 2.6. *Every extension of a divisible group by an LCA group is a TFU extension.*

Proof. It is clear by Lemma 2.5. □

Lemma 2.7. *Let $G \in \pounds$ and H be a closed, pure subgroup of G such that $(G/H)^{(1)} = 0$. Then $G^{(1)} = H^{(1)}$.*

Proof. Let $x \in G^{(1)}$ and n, an arbitrary positive integer. Then $x = ng$ for some $g \in G$. So $x + H \in n(G/H)$. Since n be arbitrary, it follows that $x + H \in (G/H)^{(1)} = 0$. This shows that $G^{(1)} \subseteq H$. Since H is pure, $G^{(1)} = H^{(1)}$. □

Corollary 2.8. *Every pure extension of an LCA group by a compact totally disconnected group is a TFU extension.*

Proof. Let E be a pure extension of an LCA group by a compact totally disconnected group G. By [9, Theorem 24.24], $G^{(1)} = 0$. Hence, by Lemma 2.7, E is TFU. □

Corollary 2.9. *Let $G \in \pounds$ such that nG is closed in G for all positive integers n. Then, every pure extension of an LCA group by $G/G^{(1)}$ is a TFU extension.*

Proof. Let $x + G^{(1)} \in (G/G^{(1)})^{(1)}$ and n be an arbitrary positive integer. Then $x + G^{(1)} = ng + G^{(1)}$ for some $g \in G$. This shows that $x \in G^{(1)}$. Hence $(G/G^{(1)})^{(1)} = 0$. □

The dual of an extension $E : 0 \to A \to B \to C \to 0$ is defined by $\hat{E} : 0 \to \hat{C} \to \hat{B} \to \hat{A}$. The following example shows that the dual of a TFU extension need not be TFU .

Example 2.10. Consider the extension $E : 0 \to \mathbb{Z} \overset{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. Clearly, E is a TFU extension. But, $\hat{E} : 0 \to \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z} \overset{\times 2}{\to} \mathbb{R}/\mathbb{Z} \to 0$ is not a TFU extension.

Recall that two extensions $0 \to A \overset{\phi_1}{\to} B \overset{\psi_1}{\to} C \to 0$ and $0 \to A \overset{\phi_2}{\to} X \overset{\psi_2}{\to} C \to 0$ are said to be equivalent if there is a topological isomorphism $\beta : B \to X$ such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi_1} & B & \xrightarrow{\psi_1} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{\phi_2} & X & \xrightarrow{\psi_2} & C & \longrightarrow & 0 \end{array}$$

is commutative.

Lemma 2.11. *An extension equivalent to a TFU extension is TFU.*

Proof. Let

$$E_1 : 0 \to A \overset{\phi_1}{\to} B \overset{\psi_1}{\to} C \to 0$$

$$E_2 : 0 \to A \overset{\phi_2}{\to} X \overset{\psi_2}{\to} C \to 0$$

be two equivalent extensions such that E_1 is TFU. Then, there is a topological isomorphism $\beta : B \to X$ such that $\beta\phi_1 = \phi_2$ and $\psi_2\beta = \psi_1$. Since $\beta(\overline{B^{(1)}}) = \overline{X^{(1)}}$,

$$\psi_2(\overline{X^{(1)}}) = \psi_2\beta(\overline{B^{(1)}}) = \psi_1(\overline{B^{(1)}}) = \overline{C^{(1)}}$$

Hence $\overline{\psi_2} : \overline{X^{(1)}} \to \overline{C^{(1)}}$ is surjective. Since $\phi_2 = \beta\phi_1$ and E_1 is TFU,

$$\psi_2\phi_2(\overline{A^{(1)}}) = \psi_2(\beta\phi_1(\overline{A^{(1)}})) = \psi_1\phi_1(\overline{A^{(1)}}) = 0$$

Hence $Im\overline{\phi_2} \subseteq Ker\overline{\psi_2}$. Now, we show that $Ker\overline{\psi_2} \subseteq Im\overline{\phi_2}$. Let $x \in \overline{X^{(1)}}$ such that $\psi_2(x) = 0$. Then there exists $b \in \overline{B^{(1)}}$ such that $x = \beta(b)$. Since $\psi_1(b) = \psi_2\beta(b) = \psi_2(x) = 0$ and E_1 is TFU, $b = \phi_1(a)$ for some $a \in \overline{A^{(1)}}$. Hence

$$\phi_2(a) = \beta\phi_1(a) = \beta(b) = x$$

and $0 \to \overline{A^{(1)}} \overset{\overline{\phi_2}}{\to} \overline{X^{(1)}} \overset{\overline{\psi_2}}{\to} \overline{C^{(1)}} \to 0$ is an exact sequence. Since

$$\overline{\psi_2} = \overline{\psi_1}(\overline{\beta})^{-1}, \overline{\phi_2} = \overline{\beta}(\overline{\phi_1})$$

It follows that $\overline{\psi_2}$ and $\overline{\phi_2}$ are open. So E_2 is a TFU extension. □

Lemma 2.12. *Let $C \in £$ be a torsion-free group, $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ a TFU extension and let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{\mu}{\longrightarrow} & X & \overset{\nu}{\longrightarrow} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A & \overset{\phi}{\longrightarrow} & B & \overset{\psi}{\longrightarrow} & C & \longrightarrow & 0
\end{array}$$

is the standard pullback diagram in £ (See [6, Proposition 2.3]) such that $\overline{f} : \overline{Y^{(1)}} \to \overline{C^{(1)}}$ be a proper morphism. Then

$$0 \to A \overset{\mu}{\to} X \overset{\nu}{\to} Y \to 0$$

is TFU.

Proof. We have

$$X = \{(y, b) \in Y \oplus B : f(y) = \psi(b)\}.$$

and

$$\mu : a \mapsto (0, \phi(a)), \nu : (y, b) \mapsto y, \theta : (y, b) \mapsto b.$$

Consider the following standard pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{A^{(1)}} & \overset{\phi'}{\longrightarrow} & N & \overset{\psi'}{\longrightarrow} & \overline{Y^{(1)}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \overline{A^{(1)}} & \overset{\overline{\phi}}{\longrightarrow} & \overline{B^{(1)}} & \overset{\overline{\psi}}{\longrightarrow} & \overline{C^{(1)}} & \longrightarrow & 0
\end{array}$$

Where $N = \{(y, b) \in \overline{Y^{(1)}} \oplus \overline{B^{(1)}} : f(y) = \psi(b)\}$. First, we show that $X^{(1)} = \{(y, b) \in Y^{(1)} \oplus B^{(1)} : f(y) = \psi(b)\}$. Let $(y, b) \in X^{(1)}$ and n an arbitrary positive integer. Then $y = ny_1$ and $b = nb_1$ for some $y_1 \in Y$ and $b_1 \in B$. Also, $f(y_1) = \psi(b_1)$. We have

$$f(y) = f(ny_1) = \psi(nb_1) = \psi(b)$$

Since n be arbitrary, $X^{(1)} \subseteq \{(y,b) \in Y^{(1)} \oplus B^{(1)} : f(y) = \psi(b)\}$. Conversely, let $(y,b) \in Y^{(1)} \oplus B^{(1)}$, $f(y) = \psi(b)$ and n be an arbitrary positive integer. Then $y = ny_1$ and $b = nb_1$ for some $y_1 \in Y$ and $b_1 \in B$. Since C is torsion-free, $n(f(y_1) - \psi(b_1)) = 0$ deduce that $f(y_1) = \psi(b_1)$. Hence $(y,b) \in nX$. Since n be arbitrary, $(y,b) \in X^{(1)}$. An easy calculation shows that $\overline{X^{(1)}} = N$. Clearly, $\phi' = \overline{\mu}$ and $\psi' = \overline{\nu}$. Hence $0 \to \overline{A^{(1)}} \overset{\overline{\mu}}{\to} \overline{X^{(1)}} \overset{\overline{\nu}}{\to} \overline{Y^{(1)}} \to 0$ is an extension. □

Lemma 2.13. *Let $A \in \pounds$ be a divisible group. Then, a pushout of a TFU extension of A by an LCA group is TFU.*

Proof. Let $E : 0 \to A \overset{\phi}{\to} B \to C \to 0$ be a TFU extension and $f : A \to G$ a proper morphism. Then $fE : 0 \to G \to X \to C \to 0$ is a pushout of E, where $X = (G \oplus B)/H$ and $H = \overline{\{(-f(a), \phi(a)); a \in A\}}$ (See [6, Proposition 2.3]). Since E is TFU, $E' : 0 \to A \to \overline{B^{(1)}} \to \overline{C^{(1)}} \to 0$ is an extension. Hence hE' is an extension where $h : A \to \overline{G^{(1)}}$ defined by $h(a) = f(a)$ for every $a \in A$. But, $hE' : 0 \to \overline{G^{(1)}} \to Y \to \overline{C^{(1)}} \to 0$ where $Y = (\overline{G^{(1)}} \oplus \overline{B^{(1)}})/K$ and $K = \overline{\{(-h(a), \phi(a)); a \in A\}}$. Clearly, $K = H$. Since H is a closed, divisible subgroup of $G \oplus B$, by Lemma 2.5 and Remark 2.2, $\overline{X^{(1)}} = (\overline{G^{(1)}} \oplus \overline{B^{(1)}})/H = Y$. It follows that fE is a TFU extension. □

3 Splitting of TFU extensions

An extension is called split if it is equivalent to the trivial extension. Let $Ext_{tfu}(C,A)$ be the class of all equivalence classes of TFU extensions of A by C. Recall that for groups $A, C \in \pounds$, $Ext(C,A) = 0$ (or $Ext_{tfu}(C,A) = 0$) deduce that every extension (or TFU extension) of A by C splits. In this section, we establish some conditions on A and C such that $Ext_{tfu}(C,A) = 0$.

Theorem 3.1. *([11, Theorem 2.1]) Let $G \in \pounds$ and $f : A \to C$ be a proper morphisn .*

(i) $f_* : Ext(G,A) \to Ext(G,C)$ defined by $f_*([E]) = [fE]$ and

(ii) $f^* : Ext(C,G) \to Ext(A,G)$ defined by $f^*([E]) = [Ef]$

are group homomorphisms.

The exact sequences (i) and (ii) of the following proposition establish a closed connection between Hom and Ext in $\pounds$.

Proposition 3.2. *([7, Corollary 2.10]) Let $G \in \pounds$ and $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ be an extension in $\pounds$. Then the following sequences are exact:*

(i) $0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \to Ext(C,G) \to Ext(B,G) \to Ext(A,G) \to 0$

(ii) $0 \to Hom(G,A) \to Hom(G,B) \to Hom(G,C) \to Ext(G,A) \to Ext(G,B) \to Ext(G,C) \to 0$

Lemma 3.3. *Let A be a discrete group such that $Ext_{tfu}(X,A) = 0$ for all groups $X \in \pounds$. Then A is a reduced group such that A/tA is a divisible group.*

Proof. Let A be a discrete group such that $Ext_{tfu}(X,A) = 0$ for all groups $X \in \pounds$. Let D be a divisible subgroup of A and C a connected group. By Proposition 3.2, we have the following exact sequence

$$... \to Hom(C, A/D) \to Ext(C,D) \overset{i_*}{\to} Ext(C,A) \to ...$$

Since C is a connected group and A/D a discrete group, $Hom(C, A/D) = 0$. Hence i_* is injective. By Lemma 2.13, $i_*(Ext_{tfu}(C,D)) \subseteq Ext_{tfu}(C,A) = 0$. So $Ext_{tcf}(C,D) = 0$. By Corollary 2.6, $Ext(C,D) = 0$. Hence $D = 0$ (see [7, Theorem 3.3]). It follows that A is a reduced group. Now, we show that A/tA is a divisible group. By Corollary 2.8, $Pext(\widehat{\mathbb{Q}/\mathbb{Z}}, A) \subseteq Ext_{tfu}(\widehat{\mathbb{Q}/\mathbb{Z}}, A) = 0$. So $Ext(\widehat{\mathbb{Q}/\mathbb{Z}}, A/tA) = 0$. Hence A/tA is a divisible group (see the proof of [19, Theorem 2]). □

Lemma 3.4. *Let G be a compact group such that $Ext_{tfu}(X,G)=0$ for all groups $X\in \pounds$. Then $G\cong \mathbb{R}^n\oplus(\mathbb{R}/\mathbb{Z})^\sigma\oplus M$, where n is a positive integer, σ a cardinal number and M is a direct product of finite cyclic groups.*

Proof. Let G be a compact group, C a connected group and $Ext_{tfu}(X,G)=0$ for all groups $X\in \pounds$. Then $Ext_{tfu}(C,G)=0$. Consider the exact sequence $0\to G_0\stackrel{i}{\to} G\to G/G_0\to 0$. We have the following exact sequence

$$...\to Hom(C,G/G_0)\to Ext(C,G_0)\stackrel{i_*}{\to} Ext(C,G)\to ...$$

Since C is a connected group and G/G_0 a totally disconnected group, $Hom(C,G/G_0)=0$. So i_* is injective. By Lemma 2.13, $i_*(Ext_{tfu}(C,G_0))\subseteq Ext_{tcf}(C,G)=0$. So $Ext_{tfu}(C,G_0)=0$. By Corollary 2.6, $Ext(C,G_0)=0$ for all connected groups $C\in\pounds$. Hence $G_0\cong(\mathbb{R}/\mathbb{Z})^\sigma\oplus\mathbb{R}^n$ (see [7, Theorem 3.3]). By [7, Corollary 3.4], G_0 splits in G. So $G\cong G_0\oplus G/G_0$. Set $M=G/G_0$. Then M is a compact totally disconnected group. By Corollary 2.8, $Pext(X,G)\subseteq Ext_{tfu}(X,G)=0$ for all compact totally disconnected groups X. Since $\hat{G_0}$ is torsion-free, $0\to\hat{M}\to\hat{G}\to\hat{G_0}\to 0$ is a pure extension. By [4, Theorem 53.7], we have the following exact sequence

$$...\to Pext(\hat{G},\hat{X})\to Pext(\hat{M},\hat{X})\to 0$$

By [10, Lemma 2.3], $Pext(\hat{G},\hat{X})\cong Pext(X,G)=0$. So $Pext(X,M)\cong Pext(\hat{M},\hat{X})=0$ for all compact totally disconnected groups X. Now, let Y be a compact group. By [5, Proposition 2], we have the following exact sequence

$$0=Pext(Y/Y_0,M)\to Pext(Y,M)\to Pext(Y_0,M)\to ...$$

By [3, Theorem 4.2] , $Pext(Y_0,M)=0$. Hence $Pext(Y,M)=0$ for all compact groups $Y\in\pounds$. By [4, Proposition 53.4] and [10, Lemma 2.3], $\hat{M}$ is a direct sum of finite cyclic groups. Hence, M is a direct product of finite cyclic groups. □

Lemma 3.5. *Let A be a $\sigma-$compact group such that $A^{(1)}=0$ and C, a divisible group in $\pounds$. Then every TFU extension of A by C splits.*

Proof. Let $E:0\to A\stackrel{\phi}{\to}B\stackrel{\psi}{\to}C\to 0$ be a TFU pure extension. An easy calculation shows that $B=\overline{B^{(1)}}+\phi(A)$. Since $\psi:\overline{B^{(1)}}\to C$ is a topological isomorphism, $\overline{B^{(1)}}\bigcap\phi(A)=0$. Hence, by [6, Corollary 3.2], $B=\overline{B^{(1)}}\oplus\phi(A)$. So E splits. □

Lemma 3.6. *Let A be a discrete, torsion-free group. Then $A\cong A^{(1)}\oplus A/A^{(1)}$.*

Proof. Let A be a discrete, torsion-free group. First, we show that $A/A^{(1)}$ is torsion-free. Let $m(a+A^{(1)})=0$ for some positive integer m. Then $ma\in A^{(1)}$. Let n be an arbitrary positive integer. Then $ma=mna_1$ for some $a_1\in A$. So $a=na_1$ and hence, $a\in A^{(1)}$. An easy calculation shows that $A^{(1)}$ is pure in A. On the other hand, $A^{(1)}\subseteq nA$ for every positive integer n. Hence $A^{(1)}$ is a divisible group. By [4, Theorem 21.1], $A\cong A^{(1)}\oplus A/A^{(1)}$. □

Lemma 3.7. *Let A be a discrete, torsion-free group such that $Ext_{tfu}(A,X)=0$ for all groups $X\in\pounds$. Then $A\cong(\oplus_\sigma\mathbb{Z})\oplus D$ where D is a discrete, torsion-free and divisible group.*

Proof. Let A be a discrete, torsion-free group such that $Ext_{tfu}(A,X)=0$ for all groups $X\in\pounds$. By Lemma 3.6, $A\cong A^{(1)}\oplus A/A^{(1)}$. By Lemma 2.12, $Ext_{tfu}(A/A^{(1)},X)=0$ for all groups $X\in\pounds$. Hence, $Ext(A/A^{(1)},X)=0$ for all groups $X\in\pounds$ (see Corollary 2.9). By [12, Theorem 3.3], $A/A^{(1)}\cong\oplus_\sigma\mathbb{Z}$. □

Lemma 3.8. *Let G be a compact, torsion-free group. Then $Ext_{tfu}(G,X)=0$ for all groups $X\in\pounds$ if and only if $G=0$.*

Proof. Let G be a compact, torsion-free group and $Ext_{tfu}(G,X)=0$ for all groups $X\in\pounds$. Let X be totally disconnected in $\pounds$. Consider the following exact sequence

$$0=Hom(G_0,X)\to Ext(G/G_0,X)\stackrel{\pi^*}{\to}Ext(G,X)\to ...$$

By Lemma 2.12, $\pi^*(Ext_{tfu}(G/G_0,X))\subseteq Ext_{tfu}(G,X)=0$. So $Ext_{tfu}(G/G_0,X)=0$. On the other hand, by [13, Lemma 2.4], G/G_0 is torsion-free. So by Corollary 2.8, $Ext(G/G_0,X)=0$. It follows that $G=G_0$ (see [7, Theorem 3.5]). By Corollary 2.6, $Ext(G,\mathbb{Q})=0$ which is a contradiction by [1, Lemma 2.10]. So $G=0$. □

4 Conclusion remarks

This paper is part of an investigation which answers the following question:
For groups $A, C \in \pounds$, under what conditions on A and C, an extension of A by C splits? In [2, 3, 14, 15, 16, 18, 19] we have been able to answer the above question by defining a new subset or subgroup of $Ext(C, A)$. The concept of splitting of extensions is very important in LCA groups. By this concept, We determined the structure of an LCA group G such that tG is closed [17]. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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