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CURVATURE PROPERTIES OF A WARPED PRODUCT METRIC

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Abstract. The purpose of the paper is to investigate curvature restricted geometric properties of a warped product metric with 1-dimensional base and 3-dimensional fibre and found that such a metric is pseudosymmetric and possesses various type of pseudosymmetric structures such as, Ricci generalized pseudosymmetry, Ricci generalized projective pseudosymmetry, Ricci generalized concircular pseudosymmetry ($W \cdot R = f_R Q(S, R)$), pseudosymmetry due to conharmonic curvature tensor ($K \cdot R = f_R Q(g, R)$), semisymmetry due to conharmonic curvature tensor ($R \cdot K = 0$) etc. Later, it is also found that the warped product metric is an Einstein manifold of degree 2 and Ricci tensor has quasi-Einstein nature. Finally, the novelty of the work is that the energy momentum tensor of the metric has also pseudosymmetric nature.

1 Introduction

Let ∇ be the Levi-Civita connection of a connected and smooth manifold M with $dim\ M = n \geq 3$ and let M be furnished with a semi-Riemannian metric g of signature $(\xi, n-\xi)$. Then Lorentzian and Riemannian manifolds form natural subclasses of semi-Riemannian manifolds for $\xi = 1$ or $n-1$ and $\xi = 0$ or n respectively. We also denotes the Riemann-Christoffel curvature (resp., Ricci curvature and the scalar curvature) by R (resp., S and κ). The 4-dimensional connected Lorentzian manifolds are of special interest as these are physically treated as spacetimes in general relativity. During the investigation of the existence of weakly W_2 symmetric manifold, in 2007, Shaikh et al. ([60], Example 4) first exihibited an warped product metric, which can be written in terms of (t, r, θ, ϕ) coordinates system as follows:

$$ds^2 = (\phi)^{4/3}[(dt)^2 + (dr)^2 + (d\theta)^2] + (d\phi)^2. \tag{1.1}$$

The same metric was further considered by Baisya in [4] for the Lorentzian signature as follows:

$$ds^2 = (\phi)^{4/3}[(dt)^2 + (dr)^2 + (d\theta)^2] - (d\phi)^2. \tag{1.2}$$

The curvature of a semi-Riemannian manifold assists to understand the geometry of the manifold as the curvature performs an important role in determination of shape of the manifold. By imposing a particular restriction on the curvature tensor of a semi-Riemannian manifold M, we obtain a specific class of manifolds. For example, the class of locally symmetric manifolds due to Cartan [5] is defined as $\nabla R = 0$; the class of semisymmetric manifolds again due to Cartan [6, 79, 80, 81] is defined as $R \cdot R = 0$; the class of pseudosymmetric manifolds by Adámow and Deszcz [1] is defined as $R \cdot R = L_R Q(g, R)$ etc. For precise definition of the symbols used here we refer the section 2. Again, many authors have generalized the notion of local symmetry in several ways such as recurrent [45, 46, 47, 86] manifolds by Ruse, generalized recurrent [52, 69, 70, 71, 73, 72] manifolds by Shaikh and his coauthors, curvature 2-forms of recurrent manifolds by Besse [3, 36], pseudosymmetric manifolds by Chaki [7, 8] and weakly symmetric manifolds by Tamássy and Binh [83, 84], etc. The curvature restricted geometric structures of a manifold indicate the structures that arise by imposing covariant derivative(s) of 1st order or higher order on several type of curvature tensors of that manifold. Deszcz's notion of pseudosymmetry (see e.g. [2, 10, 26, 32, 35, 49, 50, 56, 77]) is significant in the study of differential geometry because of its application in the theory of general relativity and cosmology. Literature reveals that there are different type of pseudosymmetry, in the sense of Deszcz and Chaki.

The concept of warped product metric is a generalization of Riemannian product metric, and it is applicable in the theory of general relativity and cosmology as FLRW-model of the universe is an warped product metric. Let $M = \bar{M} \times \tilde{M}$, where $(\bar{M}, \bar{g})$ is ρ dimensional and $(\tilde{M}, \tilde{g})$ is $(n-\rho)$ dimensional semi-Riemannian manifolds, $(1 \leq \rho \leq n-1)$. Then the warped product metric on a semi-Riemannian manifolds M is given as follows:

$$g = \pi^*(\bar{g}) + (f \circ \pi)^2 \sigma^*(\tilde{g})$$

where $\pi : M \to \bar{M}$ and $\sigma : M \to \tilde{M}$ are two cannonical projections on $\bar{M}$ and $\tilde{M}$ respectively and $f \in C^\infty(M)$, the ring of smooth functions on M. The manifold $\bar{M}$, $\tilde{M}$ are respectively known as the base and fibre and the positive smooth function f is called the warping function on M.

Our main aim is to investigate the geometric structures arising out from various curvature tensors of spacetimes within the frame of the metrics (1.1) and (1.2). We found that (1.1) and (1.2) modeled various pseudosymmetric type curvature conditions such as: Ricci generalized pseudosymmetric, Ricci generalized projectively pseudosymmetric and Ricci generalized conharmonicly pseudosymmetric conditions. Also, it has semisymmetric conharmonic curvature tensor. Moreover, it is an Einstein manifold of degree 2 and Ricci tensor has quasi-Einstein nature.

The present article is oriented as follows: section 2 contains some introductory definitions of different structures of geometry. In section 3, the curvature restricted geometric structures of the warped product metrics (1.1) and (1.2) are calculated. In section 4, we investigate some geometric structures of energy momentum tensor. Finally, the paper is concluded with some discussions on the respective topic.

2 Preliminaries: some introductory definitions of different geometric structures

Let the second order symmetric covariant tensor be μ and ζ. Now, we define the type of $(0,4)$ tensor as follows:

$$\begin{aligned}(\mu \wedge \zeta)(\vartheta_1, \vartheta_2, \lambda_1, \lambda_2) &= \mu(\vartheta_1, \lambda_2)\zeta(\vartheta_2, \lambda_1) - \mu(\vartheta_2, \lambda_1)\zeta(\vartheta_1, \lambda_2) \\ &\quad + \mu(\vartheta_1, \lambda_1)\zeta(\vartheta_2, \lambda_2) - \mu(\vartheta_2, \lambda_2)\zeta(\vartheta_1, \lambda_1),\end{aligned}$$

which is called as Kulkarni-Nomizu product (see, [17, 33, 68]).
The endomorphisms on M (see, [17, 25, 33, 35, 54, 67]) are represented as:

$$\begin{aligned}
(\vartheta_1 \wedge_\mu \vartheta_2)\lambda &= \mu(\vartheta_2, \lambda)\vartheta_1 - \mu(\vartheta_1, \lambda)\vartheta_2, \\
\mathscr{B}_{\mathscr{R}}(\vartheta_1, \vartheta_2) &= [\nabla_{\vartheta_1}, \nabla_{\vartheta_2}] - \nabla_{[\vartheta_1, \vartheta_2]}, \\
\mathscr{B}_{\mathscr{C}}(\vartheta_1, \vartheta_2) &= \mathscr{B}_{\mathscr{R}}(\vartheta_1, \vartheta_2) - \frac{1}{(n-2)}\left(\mathscr{S}\vartheta_1 \wedge_g \vartheta_2 + \vartheta_1 \wedge_g \mathscr{S}\vartheta_2 - \frac{\kappa}{n-1}\vartheta_1 \wedge_g \vartheta_2\right), \\
\mathscr{B}_{\mathscr{P}}(\vartheta_1, \vartheta_2) &= \mathscr{B}_{\mathscr{R}}(\vartheta_1, \vartheta_2) - \frac{1}{(n-1)}\left(\vartheta_1 \wedge_S \vartheta_2\right), \\
\mathscr{B}_{\mathscr{W}}(\vartheta_1, \vartheta_2) &= \mathscr{B}_{\mathscr{R}}(\vartheta_1, \vartheta_2) - \frac{\kappa}{n(n-1)}\left(\vartheta_1 \wedge_g \vartheta_2\right), \\
\mathscr{B}_{\mathscr{K}}(\vartheta_1, \vartheta_2) &= \mathscr{B}_{\mathscr{R}}(\vartheta_1, \vartheta_2) - \frac{1}{(n-2)}\left(\vartheta_1 \wedge_g \mathscr{S}\vartheta_2 + \mathscr{S}\vartheta_1 \wedge_g \vartheta_2\right),
\end{aligned}$$

where $\mathscr{S}$ is the Ricci operator, which is defined by $g(\vartheta_1, \mathscr{S}\vartheta_2) = S(\vartheta_1, \vartheta_2)$.

Through out the paper, we assume that ϑ,ϑ_1,ϑ_2, $\cdots$, λ, λ_1, λ_2, $\cdots \in \chi(M)$, the Lie algebra of all smooth vector fields on M.

Now, for an endomorphism $\mathcal{B}(\lambda_1, \lambda_2)$, we can define the $(0, 4)$ type tensor as

$$B(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = g(\mathcal{B}(\lambda_1, \lambda_2)\lambda_3, \lambda_4).$$

Replacing $\mathcal{B}$ by $\mathcal{B}_{\mathcal{R}}$ (resp., $\mathcal{B}_{\mathcal{W}}$, $\mathcal{B}_{\mathcal{P}}$, $\mathcal{B}_{\mathcal{C}}$, $\mathcal{B}_{\mathcal{K}}$), we can get the type of $(0, 4)$ Riemann curvature tensor R (resp., concircular curvature tensor of W, projective curvature tensor of P, conformal curvature tensor of C, conharmonic curvature tensor of K and Gaussian curvature tensor of G). For a type of $(0, u)$ tensor field on E, $u \geq 1$, we can operate an endomorphism $\mathcal{B}(\lambda_1, \lambda_2)$ to type of $(0, u + 2)$ tensor field $B \cdot E$ as follows ([14, 15, 24, 57, 63])

$$\begin{aligned}(B \cdot E)(\vartheta_1, \vartheta_2, \cdots, \vartheta_u; \lambda_1, \lambda_2) \quad &= \quad (\mathcal{B}(\lambda_1, \lambda_2)E)(\vartheta_1, \vartheta_2, \cdots, \vartheta_u) \\ &= -E(\mathcal{B}(\lambda_1, \lambda_2)\vartheta_1, \vartheta_2, \cdots, \vartheta_u) - \cdots - E(\vartheta_1, \vartheta_2, \cdots, \mathcal{B}(\lambda_1, \lambda_2)\vartheta_u).\end{aligned}$$

Also, if $\mathcal{B}(\lambda_1, \lambda_2)$=$\lambda_1 \wedge_\mu \lambda_2$, then we get the tensor field $Q(\mu, E)$, which is known as Tachibana tensor (see, [22, 57, 62, 82]) given as follows:

$$\begin{aligned}Q(\mu, E)(\vartheta_1, \vartheta_2, \cdots, \vartheta_u; \lambda_1, \lambda_2) \quad &= \quad ((\lambda_1 \wedge_\mu \lambda_2) \cdot E)(\vartheta_1, \vartheta_2, \cdots, \vartheta_u) \\ = -E((\lambda_1 \wedge_\mu \lambda_2)\vartheta_1, \vartheta_2, \cdots, \vartheta_u) &- \cdots - E(\vartheta_1, \vartheta_2, \cdots, (\lambda_1 \wedge_\mu \lambda_2)\vartheta_u) \\ = \mu(\lambda_1, \vartheta_1)E(\lambda_2, \vartheta_2, \cdots, \vartheta_u) &+ \cdots + \mu(\lambda_1, \vartheta_u)E(\vartheta_1, \vartheta_2, \cdots, \lambda_2) \\ -\mu(\lambda_2, \vartheta_1)E(\lambda_1, \vartheta_2, \cdots, \vartheta_u) &- \cdots - \mu(\lambda_2, \vartheta_u)E(\vartheta_1, \vartheta_2, \cdots, \lambda_1).\end{aligned}$$

Representation of the tensor $B \cdot E$ and $Q(\mu, E)$, in terms of the local coordinates system are given as follows:

$$\begin{aligned}(B \cdot E)_{w_1 w_2 \ldots w_u af} &= -g^{bd}[B_{afw_1d}E_{bw_2\ldots w_u} + \cdots + B_{afw_ud}E_{w_1w_2\ldots b}], \\ Q(\mu, E)_{w_1 w_2 \ldots w_u af} &= \mu_{fw_1}E_{aw_2\ldots w_u} + \cdots + \mu_{fw_u}E_{w_1w_2\ldots a} \\ &- \mu_{aw_1}E_{fw_2\ldots w_u} - \cdots - \mu_{aw_u}E_{w_1w_2\ldots f}.\end{aligned}$$

Definition 2.1. [1, 11, 12, 18, 19, 62, 65, 66] The pseudosymmetric type manifold is defined by the linear dependency of the tensors $B \cdot E$ and $Q(g, E)$ i.e., a manifold M is called E-pseudosymmetric due to the tensor B if $B \cdot E = f_E Q(g, E)$ holds on M, and a Ricci generalized E-pseudosymmetric manifold M due to the tensor B is defined by $E \cdot B = \bar{f}_E Q(S, E)$, f_E and $\bar{f}_E$ being some smooth functions on M. In particular, if $E \cdot B = 0$ holds on M then it is called E-semisymmetric manifold due to B.

For $B = R$ and $E = R$, then a E-pseudosymmetric manifold is called simply a pseudosymmetric manifold and for $B = R$ and $E = C$ (resp., P, K and W) it is called conformal (resp., projective, conharmonic and concircular) pseudosymmetric manifold. Simmilarly Ricci generalized pseudosymmetric manifolds can be defined accordingly. We mention here that Robertson-Walker spacetime [2, 43], Schwarzschild spacetime [32], Reissner-Nordström spacetime [35] are oldest examples of pseudosymmetric manifolds.

Definition 2.2. ([19, 20, 23, 48, 67]) A manifold M is called Einstein (resp., quasi-Einstein [61, 75] and 2-quasi Einstein) manifold if rank of $(S - \lambda g) = 0$ (resp., 1 and 2), for a scalar λ. If $\lambda = 0$, then the quasi-Einstein manifold turns into Ricci simple manifold.

It is to be noted that Morris-Thorne spacetime [29] is a Ricci simple manifold, Robertson-Walker spacetime [2] is quasi-Einstein, Kantowski-Sachs spacetime [54] is 2-quasi Einstein and Kaigorodov spacetime [56] is Einstein.

Definition 2.3. [3, 13, 16, 20, 21, 57, 63, 67] A manifold M corresponds to generalized Roter type if its Riemann curvature tensor gets the following explicit form:

$$R = \mu_{22}(g \wedge g) + (\mu_{11}S + \mu_{12}g) \wedge S + (\mu_{00}S^2 + \mu_{01}S + \mu_{02}g) \wedge S^2$$

where μ_{ij} are some scalars. If the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$ are linearly dependent with R, then we call it a Roter type manifold [13, 14, 23, 27, 34].

We mention here that Melvin magnetic spacetime [50] and Nariai spacetime [51] are Roter type manifold while Vaidya-Bonner spacetime [55] and Lifshitz spacetime [74] are generalized Roter type manifold.

Definition 2.4. ([3, 62, 64, 67]) An Einstein manifold of degree 4 is defined by the equation

$$\zeta_1 S^4 + \zeta_2 S^3 + \zeta_3 S^2 + \zeta_4 S + \zeta_5 g = 0$$

where $\zeta_i \in C^\infty(M)$ and $\zeta_1 \neq 0$. Again, for $\zeta_1 = 0$ but $\zeta_2 \neq 0$ (resp., $\zeta_1 = \zeta_2 = 0$ but $\zeta_3 \neq 0$) it is known as Einstein manifold of degree 3 (resp., Einstein manifold of degree 2).

Definition 2.5. The Ricci tensor of a semi-Riemannian manifold M is called cyclic parallel [31, 53, 58, 59] if

$$(\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) + (\nabla_{\vartheta_2} S)(\vartheta_3, \vartheta_1) + (\nabla_{\vartheta_3} S)(\vartheta_1, \vartheta_2) = 0$$

holds, and Codazzi type Ricci tensor [30, 76] is defined by the relation

$$(\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) = (\nabla_{\vartheta_2} S)(\vartheta_1, \vartheta_3).$$

It may be noted that the Gödel spacetime [25] has Ricci tensor of cyclic parallel while the $(t-z)$-type plane wave spacetime [28] has been investigated with Codazzi Ricci tensor.

Definition 2.6. ([9, 15, 16, 22, 37, 38, 39]) Let μ be a symmetric type of $(0, 2)$ tensor on M corresponding to the endomorphism $\mathscr{B}_\mu$ and B be a $(0, 4)$ tensor. Then μ is called B-compatible if

$$B(\mathscr{B}_\mu \vartheta_1, \lambda, \vartheta_2, \vartheta_3) + B(\mathscr{B}_\mu \vartheta_2, \lambda, \vartheta_3, \vartheta_1) + B(\mathscr{B}_\mu \vartheta_3, \lambda, \vartheta_1, \vartheta_2) = 0$$

holds. When $\varphi \otimes \varphi$ is B-compatible then 1-forms φ is said to be B-compatible.

Replacing, μ by S and B by R (resp., K, C, W and P), we can get Ricci and Riemann (resp., conharmonic, conformal, concircular and projective) compatible tensors.

Definition 2.7. A weakly symmetric manifold in the sense of Tamássy and Binh [83, 84] is defined as:

$$\begin{aligned}(\nabla_X R)(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) &= \Pi(X) \otimes R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) + \tilde{A}(\vartheta_4) \otimes R(\vartheta_1, \vartheta_2, \vartheta_3, X) \\ &+ A(\vartheta_3) \otimes R(\vartheta_1, \vartheta_2, X, \vartheta_4) + \tilde{B}(\vartheta_2) \otimes R(\vartheta_1, X, \vartheta_3, \vartheta_4) \\ &+ B(\vartheta_1) \otimes R(X, \vartheta_2, \vartheta_3, \vartheta_4)\end{aligned}$$

where $\Pi, A, \tilde{A}, B, \tilde{B}$ are some 1-forms on a semi-Riemannian manifold M. If $\Pi = \frac{A}{2} = \frac{\tilde{A}}{2} = \frac{B}{2} = \frac{\tilde{B}}{2}$, then it reduces to a Chaki pseudosymmetric manifold [7, 8].

Definition 2.8. Let B be a $(0, 4)$ type tensor field on M. Then the corresponding curvature 2-forms $\Omega^m_{(B)l}$ [78] are recurrent [40, 41, 42] if and only if

$$\underset{\vartheta_1, \vartheta_2, \vartheta_3}{\mathcal{S}} (\nabla_{\vartheta_1} B)(\vartheta_2, \vartheta_3, \lambda, \vartheta) = \underset{\vartheta_1, \vartheta_2, \vartheta_3}{\mathcal{S}} \eta(\vartheta_1) B(\vartheta_2, \vartheta_3, \lambda, \vartheta)$$

where $\mathcal{S}$ being the cyclic sum over ϑ_1, ϑ_2 and ϑ_3. Again, let μ be the symmetric type of $(0, 2)$ tensor field. Then the 1-forms $\Lambda_{(\mu)l}$ [78] are recurrent if

$$(\nabla_{\vartheta_1} \mu)(\vartheta_2, \lambda) - (\nabla_{\vartheta_2} \mu)(\vartheta_1, \lambda) = \eta(\vartheta_1) \mu(\vartheta_2, \lambda) - \eta(\vartheta_2) \mu(\vartheta_1, \lambda)$$

for some 1-form η.

Definition 2.9. ([44, 63, 85]) Let B be a $(0, 4)$ type tensor on M. If the set of 1 form ψ satisfying

$$\underset{\vartheta_1, \vartheta_2, \vartheta_3}{\mathcal{S}} \psi(\vartheta_1) \otimes B(\vartheta_2, \vartheta_3, \lambda, \vartheta) = 0,$$

configures a k-dimendional vector space with $k \geq 1$, then M is known as a B-space by Venzi.

Replacing, B by R (resp., K, C, W and P), we can get Venzi spaces for Riemann (resp., conharmonic, conformal, concircular and projective).

3 Calculation of curvature restricted geometric properties of the warped product metric

For the metric (1.2), the components are $g_{11} = g_{22} = g_{33} = (\phi)^{4/3},\ g_{44} = -1,\ g_{ij} = 0, i \neq j$ for $i, j = 1, 2, 3, 4$. and for the metric (1.1), the components are $g_{11} = g_{22} = g_{33} = (\phi)^{4/3},\ g_{44} = 1,\ g_{ij} = 0, i \neq j$ for $i, j = 1, 2, 3, 4$.

Now, we calculate the components of different type of curvature tensors of the metrics (1.1) and (1.2).

The components other than zero of the Christoffel symbols of second kind (Γ^h_{ij}) of the metric (1.2) are given by:

$$\Gamma^4_{11} = \frac{2\phi^{1/3}}{3} = \Gamma^4_{22} = \Gamma^4_{33},\ \Gamma^1_{14} = \frac{2}{3\phi} = \Gamma^2_{24} = \Gamma^3_{34};$$

and for the metric (1.1), the components are given by

$$\Gamma^4_{11} = -\frac{2\phi^{1/3}}{3} = \Gamma^4_{22} = \Gamma^4_{33},\ \Gamma^1_{14} = \frac{2}{3\phi} = \Gamma^2_{24} = \Gamma^3_{34};$$

The components other than zero of the Riemann-Christoffel curvature tensor R_{hijk} and the Ricci tensor S_{ij} of the metric (1.2) are given by:

$$R_{1212} = \frac{4\phi^{2/3}}{9} = R_{1313} = R_{2323},\ R_{1414} = \frac{2}{9\phi^{2/3}} = R_{2424} = R_{3434};$$
$$S_{11} = -\frac{2}{3\phi^{2/3}} = S_{22} = S_{33},\ S_{44} = -\frac{2}{3\phi^2};$$

and for the metric (1.1), the components are given by

$$R_{1212} = -\frac{4\phi^{2/3}}{9} = R_{1313} = R_{2323},\ R_{1414} = \frac{2}{9\phi^{2/3}} = R_{2424} = R_{3434};$$
$$S_{11} = \frac{2}{3\phi^{2/3}} = S_{22} = S_{33},\ S_{44} = -\frac{2}{3\phi^2};$$

Also the scalar curvature, for the metric (1.2), is $\kappa = -\frac{4}{3\phi^2}$, and for the metric (1.1), $\kappa = \frac{4}{3\phi^2}$. The metrics (1.1) and (1.2) are conformally flat.

From the above calculation of the components of different type of tensors of the metrics (1.1) and (1.2) we can state the following:

Proposition 3.1. *Both the metrics* (1.1) *and* (1.2) *are*

(i) quasi-Einstein as rank $(S - \alpha g) = 1$ *for* $\alpha = -\frac{2}{3\phi^2}$ *and* $\frac{2}{3\phi^2}$ *respectively,*

(ii) Einstein manifold of level 2 *i.e., fulfilled the condition* $S^2 = \beta g$ *for* $\beta = \frac{4}{9\phi^4}$*,*

(iii) Ricci tensor for both the metrics are Riemann compatible, conharmonic compatible, concircular compatible and projective compatible,

(iv) both the metrics are conformally flat.

Let $\mathcal{V}^1 = \nabla R$ and $\mathcal{V}^2 = \nabla S$. Then the non-zero components of the covariant derivatives of the tensor R_{abcd} and S_{ab} of the metric (1.2) are given as below:

$$\mathcal{V}^1_{1212,4} = -\frac{8}{9\phi^{1/3}} = \mathcal{V}^1_{1313,4} = \mathcal{V}^1_{2323,4},\ \mathcal{V}^1_{1214,2} = -\frac{4}{9\phi^{1/3}} = \mathcal{V}^1_{1314,3} = \mathcal{V}^1_{2324,3},$$
$$\mathcal{V}^1_{1224,1} = \frac{4}{9\phi^{1/3}} = \mathcal{V}^1_{1334,1} = \mathcal{V}^1_{2334,2},\ \mathcal{V}^1_{1414,4} = -\frac{4}{9\phi^{5/3}} = \mathcal{V}^1_{2424,4} = \mathcal{V}^1_{3434,4};$$
$$\mathcal{V}^2_{11,4} = \frac{4}{3\phi^{5/3}} = \mathcal{V}^2_{22,4} = \mathcal{V}^2_{33,4},\ \mathcal{V}^2_{14,1} = \frac{8}{9\phi^{5/3}} = \mathcal{V}^2_{24,2} = \mathcal{V}^2_{34,3},\ \mathcal{V}^2_{44,4} = \frac{4}{3\phi^3};$$

and for the metric (1.1), the components are given by

$$\begin{gathered}\mathcal{V}^1_{1212,4}=\frac{8}{9\phi^{1/3}}=\mathcal{V}^1_{1313,4}=\mathcal{V}^1_{2323,4},\ \mathcal{V}^1_{1214,2}=\frac{4}{9\phi^{1/3}}=\mathcal{V}^1_{1314,3}=\mathcal{V}^1_{2324,3},\\ \mathcal{V}^1_{1224,1}=-\frac{4}{9\phi^{1/3}}=\mathcal{V}^1_{1334,1}=\mathcal{V}^1_{2334,2},\ \mathcal{V}^1_{1414,4}=-\frac{4}{9\phi^{5/3}}=\mathcal{V}^1_{2424,4}=\mathcal{V}^1_{3434,4};\\ \mathcal{V}^2_{11,4}=-\frac{4}{3\phi^{5/3}}=\mathcal{V}^2_{22,4}=\mathcal{V}^2_{33,4},\ \mathcal{V}^2_{14,1}=-\frac{8}{9\phi^{5/3}}=\mathcal{V}^2_{24,2}=\mathcal{V}^2_{34,3},\ \mathcal{V}^2_{44,4}=\frac{4}{3\phi^3};\end{gathered}$$

The components other than zero of the projective curvature tensor P_{abcd} of the metric (1.2) are given below:

$$P_{1212}=\frac{2\phi^{2/3}}{9}=P_{1313}=P_{2323},\ P_{1221}=-\frac{2\phi^{2/3}}{9}=P_{1331}=P_{2332},\ P_{1414}=\frac{4}{9\phi^{2/3}}=P_{2424}=P_{3434};$$

and for the metric (1.1), the components are given by

$$P_{1212}=-\frac{2\phi^{2/3}}{9}=P_{1313}=P_{2323},\ P_{1221}=\frac{2\phi^{2/3}}{9}=P_{1331}=P_{2332},\ P_{1414}=\frac{4}{9\phi^{2/3}}=P_{2424}=P_{3434}.$$

The components other than zero of the concircular curvature tensor W_{abcd} of the metric (1.2) are given below:

$$W_{1212}=\frac{\phi^{2/3}}{3}=W_{1313}=W_{2323},\ W_{1414}=\frac{1}{3\phi^{2/3}}=W_{2424}=W_{3434};$$

and for the metric (1.1), the components are given by

$$W_{1212}=-\frac{\phi^{2/3}}{3}=W_{1313}=W_{2323},\ W_{1414}=\frac{1}{3\phi^{2/3}}=W_{2424}=W_{3434}.$$

The components other than zero of the conharmonic curvature tensor K_{abcd} of the metric (1.2) are given below:

$$K_{1212}=-\frac{2\phi^{2/3}}{9}=K_{1313}=K_{2323},\ K_{1414}=\frac{2}{9\phi^{2/3}}=K_{2424}=K_{3434};$$

and for the metric (1.1), the components are given by

$$K_{1212}=\frac{2\phi^{2/3}}{9}=K_{1313}=K_{2323},\ K_{1414}=\frac{2}{9\phi^{2/3}}=K_{2424}=K_{3434}.$$

From above components we get the following recurrent structures for the metrics (1.1) and (1.2):

Proposition 3.2. *For both the metrics* (1.1) *and* (1.2) *are*

(i) the Ricci 1*-forms are recurrent i.e.,* $\nabla_{\vartheta_1}S(\vartheta_2,\vartheta_3)-\nabla_{\vartheta_2}S(\vartheta_1,\vartheta_3)=\eta(\vartheta_1)\otimes S(\vartheta_2,\vartheta_3)-\eta(\vartheta_2)\otimes S(\vartheta_1,\vartheta_3)$ *for* $\eta=\left\{0,0,0,-\frac{2}{3\phi}\right\}$,

(ii) conharmonic curvature K is recurrent for the 1-form $\Pi=\left\{0,0,0,-\frac{2}{\phi}\right\}$,

(iii) projective curvature 2*-forms and concircular curvature* 2*-forms are recurrent for the same* 1*-form* $\left\{0,0,0,\frac{2}{3\phi}\right\}$.

Let $H^1=R\cdot R$, $I^1=Q(g,R)$, $I^2=Q(S,R)$. Then the components other than zero (upto symmetry) of the tensors H^1, I^1 and I^2 of the metric (1.2) are given by:

$$\begin{gathered}H^1_{1224,14}=-\frac{4}{27\phi^{4/3}}=H^1_{1334,14}=H^1_{2334,24},\ H^1_{1214,24}=\frac{4}{27\phi^{4/3}}=H^1_{1314,34}=H^1_{2324,34};\\ I^1_{1224,14}=-\frac{2\phi^{2/3}}{3}=I^1_{1334,14}=I^1_{2334,24},\ I^1_{1214,24}=\frac{2\phi^{2/3}}{3}=I^1_{1314,34}=I^1_{2324,34};\\ I^2_{1224,14}=-\frac{4}{27\phi^{4/3}}=I^2_{1334,14}=I^2_{2334,24},\ I^2_{1214,24}=\frac{4}{27\phi^{4/3}}=I^2_{1314,34}=I^2_{2324,34};\end{gathered}$$

and for the metric (1.1), the components are given by

$$\begin{gathered}H^1_{1224,14}=\frac{4}{27\phi^{4/3}}=H^1_{1334,14}=H^1_{2334,24},\ H^1_{1214,24}=-\frac{4}{27\phi^{4/3}}=H^1_{1314,34}=H^1_{2324,34};\\ I^1_{1224,14}=-\frac{2\phi^{2/3}}{3}=I^1_{1334,14}=I^1_{2334,24},\ I^1_{1214,24}=\frac{2\phi^{2/3}}{3}=I^1_{1314,34}=I^1_{2324,34};\\ I^2_{1224,14}=\frac{4}{27\phi^{4/3}}=I^2_{1334,14}=I^2_{2334,24},\ I^2_{1214,24}=-\frac{4}{27\phi^{4/3}}=I^2_{1314,34}=I^2_{2324,34};\end{gathered}$$

Proposition 3.3. *The metric* (1.2) *realizes the curvature conditions*

$$R \cdot R = \frac{2}{9\phi^2} Q(g, R) \text{ and } R \cdot R = Q(S, R)$$

and the metric (1.1) *fulfills* $R \cdot R = -\frac{2}{9\phi^2} Q(g, R)$ *and* $R \cdot R = Q(S, R)$ *i.e., both the metrics are pseudosymmetric and also Ricci generalized peudosymmetric.*

Let $H^2 = P \cdot R$, $I^3 = P \cdot K$ and $I^4 = Q(S, K)$. Then the components other than zero (upto symmetry) of the tensors H^2, I^3 and I^4 of the metric (1.2) are given by:

$$\begin{gathered}
H^2_{1224,14} = -\tfrac{8}{81\phi^{4/3}} = H^2_{1334,14} = H^2_{2334,24} = H^2_{1214,42} = H^2_{1314,43} = H^2_{2324,43},\\
H^2_{1214,24} = \tfrac{8}{81\phi^{4/3}} = H^2_{1314,34} = H^2_{2324,34} = H^2_{1224,41} = H^2_{1334,41} = H^2_{2334,42};\\
I^3_{1224,14} = -\tfrac{8}{81\phi^{4/3}} = I^3_{1334,14} = I^3_{2334,24} = I^3_{1214,42} = I^3_{1314,43} = I^3_{2324,43},\\
I^3_{1214,24} = \tfrac{8}{81\phi^{4/3}} = I^3_{1314,34} = I^3_{2324,34} = I^3_{1224,41} = I^3_{1334,41} = I^3_{2334,42};\\
I^4_{1224,14} = \tfrac{8}{27\phi^{4/3}} = I^4_{1334,14} = I^4_{2334,24},\ I^4_{1214,24} = -\tfrac{8}{27\phi^{4/3}} = I^4_{1314,34} = I^4_{2324,34};
\end{gathered}$$

and for the metric (1.1), the components are given by

$$\begin{gathered}
H^2_{1224,14} = \tfrac{8}{81\phi^{4/3}} = H^2_{1334,14} = H^2_{2334,24} = H^2_{1214,42} = H^2_{1314,43} = H^2_{2324,43},\\
H^2_{1214,24} = -\tfrac{8}{81\phi^{4/3}} = H^2_{1314,34} = H^2_{2324,34} = H^2_{1224,41} = H^2_{1334,41} = H^2_{2334,42};\\
I^3_{1224,14} = \tfrac{8}{81\phi^{4/3}} = I^3_{1334,14} = I^3_{2334,24} = I^3_{1214,42} = I^3_{1314,43} = I^3_{2324,43},\\
I^3_{1214,24} = -\tfrac{8}{81\phi^{4/3}} = I^3_{1314,34} = I^3_{2324,34} = I^3_{1224,41} = I^3_{1334,41} = I^3_{2334,42};\\
I^4_{1224,14} = -\tfrac{8}{27\phi^{4/3}} = I^4_{1334,14} = I^4_{2334,24},\ I^4_{1214,24} = \tfrac{8}{27\phi^{4/3}} = I^4_{1314,34} = I^4_{2324,34}.
\end{gathered}$$

Proposition 3.4. *The metric* (1.2) *fulfills the curvature relations*

$$P \cdot R = \frac{4}{27\phi^2} Q(g, R) \text{ and } P \cdot R = Q(S, R)$$

and the metric (1.1) *satisfies the curvature conditions* $P \cdot R = \frac{4}{27\phi^2} Q(g, R)$ *and* $P \cdot R = Q(S, R)$ *i.e., both the metrics are pseudosymmetric due to projective curvature tensor and Ricci generalized projective pseudosymmetric manifold.*

Let $H^3 = W \cdot R$ and $H^4 = K \cdot R$. Then the components other than zero (upto symmetry) of the tensors H^3 and H^4 of the metric (1.2) are given below:

$$\begin{gathered}
H^3_{1224,14} = -\tfrac{2}{9\phi^{4/3}} = H^3_{1334,14} = H^3_{2334,24},\ H^3_{1214,24} = \tfrac{2}{9\phi^{4/3}} = H^3_{1314,34} = H^3_{2324,34};\\
H^4_{1224,14} = -\tfrac{4}{27\phi^{4/3}} = H^4_{1334,14} = H^4_{2334,24},\ H^4_{1214,24} = \tfrac{4}{27\phi^{4/3}} = H^4_{1314,34} = H^4_{2324,34};
\end{gathered}$$

and for the metric (1.1), the components are given below :

$$\begin{gathered}
H^3_{1224,14} = \tfrac{2}{9\phi^{4/3}} = H^3_{1334,14} = H^3_{2334,24},\ H^3_{1214,24} = -\tfrac{2}{9\phi^{4/3}} = H^3_{1314,34} = H^3_{2324,34};\\
H^4_{1224,14} = \tfrac{4}{27\phi^{4/3}} = H^4_{1334,14} = H^4_{2334,24},\ H^4_{1214,24} = -\tfrac{4}{27\phi^{4/3}} = H^4_{1314,34} = H^4_{2324,34};
\end{gathered}$$

Proposition 3.5. *The metric* (1.2) *yields the following pseudosymmetric type curvature conditions*

$$W \cdot R = \frac{1}{3\phi^2} Q(g, R),\ K \cdot R = \frac{2}{9\phi^2} Q(g, R),\ W \cdot R = Q(S, R) \text{ and } K \cdot R = Q(S, R)$$

and the metric (1.1) *fulfills the conditions* $W \cdot R = -\frac{1}{3\phi^2} Q(g, R)$, $K \cdot R = -\frac{2}{9\phi^2} Q(g, R)$, $W \cdot R = Q(S, R)$ *and* $K \cdot R = Q(S, R)$ *i.e., both the metrics are pseudosymmetric due to concircular curvature, pseudosymmetric due to conharmonic curvature and Ricci generalized concircular pseudosymmetric as well as Ricci generalized conharmonic pseudosymmetric.*

Thus we can conclude that the curvature properties of the metrics (1.1) and (1.2) can be stated as follows:

Theorem 3.6. *The metrics* (1.1) *and* (1.2) *admit the following curvature properties:*

(i) both are pseudosymmetric and consequently pseudosymmetric for Ricci curvature, projective curvature, concircular curvature and conharmonic curvature,

(ii) both are pseudosymmetric due to concircular, conharmonic and projective curvatures,

(iii) both are special Ricci generalized pseudosymmetric and Ricci generalized pseudosymmetric for projective, concircular and conharmonic curvatures as well,

(iv) both are quasi-Einstein manifolds and Einstein manifolds of level 2*,*

(v) conharmonically semisymmetric $R \cdot K = 0$ *and semisymmetric type conditions* $W \cdot K = 0$ *and* $K \cdot K = 0$ *are realized by both the metrics,*

(vi) conharmonic curvature is recurrent for both the metrics,

(vii) projective curvature 2*-forms, concircular curvature* 2*-forms are recurrent for both the metrics,*

(viii) both are Ricci 1*-forms are recurrent for both the metrics,*

(ix) Ricci tensor is compatible for Riemann, projective, conharmonic and concircular curvatures for both the metrics.

Remark 3.7. The metrics (1.1) and (1.2) do not fulfill the following geometric structures:

(i) B-Venzi space for B=R, P, W, K,

(ii) Codazzi type Ricci tensor or cyclic parallel Ricci tensor,

(iii) Super generalized recurrence, hyper generalized recurrence and weakly generalized recurrence,

(iv) Chaki pseudosymmetry.

4 Some geometric properties of energy momentum tensor of the metric (1.2)

In the well known theory of general relativity, energy momentum tensor describes the physics of a spacetime and Einstein field equation made a bridge between the physical quantity energy momentum tensor and geometrical quantity 'curvature' of a spacetime via

$$S - \frac{k}{2}g = \frac{8\pi G}{c^4}T$$

G being Newton's gravitational constant, T being the energy momentum tensor, c being the speed of light in vacuum. We assume $\frac{8\pi G}{c^4} = 1$ and compute the stress energy momentum tensor in terms of its components for the metric (1.2) by the above equation.

The only non-vanishing component of the energy momentum tensor is

$$T_{44} = -\tfrac{1}{6\pi\phi^2}.$$

For the metric (1.2), the components other than zero of $R \cdot T$, $W \cdot T$ and $K \cdot T$ are

$$(R \cdot T)_{1414} = -\tfrac{1}{27\pi\phi^{8/3}} = (R \cdot T)_{2424} = (R \cdot T)_{3434},$$

$$(W \cdot T)_{1414} = -\tfrac{4}{9\pi\phi^{8/3}} = (W \cdot T)_{2424} = (W \cdot T)_{3434},$$

$$(K \cdot T)_{1414} = -\tfrac{8}{27\pi\phi^{8/3}} = (K \cdot T)_{2424} = (K \cdot T)_{3434}.$$

Also, let $I^5 = Q(g,T)$ and $I^6 = Q(T,R)$. For the metric (1.2), the components other than zero of I^5 and I^6 are given by:

$$I^5_{1414} = -\frac{1}{6\pi\phi^{2/3}} = I^5_{2424} = I^5_{3434},$$

$$I^6_{122414} = -\frac{16}{27\phi^{4/3}} = I^6_{133414} = -I^6_{121424}$$
$$= I^6_{233424} = -I^6_{131434} = -I^6_{232434}.$$

From the above components we get the following:

Theorem 4.1. *The following pseudosymmetric type curvature conditions are represented by the energy momentum tensor T of the metric (1.2):*

(i) $R \cdot T = -\frac{2}{9\phi^2}Q(g,T)$ and $R \cdot R = \frac{1}{4}Q(T,R)$ i.e., the nature of the energy momentum tensor is pseudosymmetric and fulfills Ricci generalized pseudosymmety as well,
(ii) $W \cdot T = \frac{2}{9\phi^2}Q(g,T)$ and $W \cdot R = \frac{1}{2}Q(T,R)$,
(iii) $K \cdot T = \frac{2}{9\phi^2}Q(g,T)$ and $K \cdot R = \frac{1}{4}Q(T,R)$,
(iv) $P \cdot R = \frac{1}{6}Q(T,R)$ and also
(v) the energy momentum tensor T is compatible for Riemann, projective, conharmonic and concircular curvatures.

5 Conclusions

In this article the curvature restricted geometric properties of a warped product metric with 1-dimensional base and 3-dimensional fibre are studied. In differential geometry one of the worthy notion of symmetry is pseudosymmetry and we find that this is admitted by both the metrics (1.1) and (1.2). These metrics also admit special Ricci generalized pseudosymmetry. Several kinds of pseudosymmetries such as pseudosymmetry due to concircular curvature, conharmonic curvature and projective curvature are also fulfilled by both the metrics. These metrics are also quasi-Einstein, $Ein(2)$ but conformally flat manifolds. The nature of the conharmonic curvature is recurrent and semisymmetric type. The novelty of the work is that the energy momentum tensor of the warped product metric (1.2) is pseudosymmetric and realized several types of pseudosymmetries. We can consider the metric (1.2) as model of a pseudosymmetric, special Ricci generalized pseudosymmetric spacetime which has pseudosymmetric energy momentum tensor.

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