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Existence for coupled Random system fractional order functional differential with infinite delay

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Abstract*In this paper we prove the existence of solutions for fractional order functional and neutral functional differential equations with infinite delay principles combined with a technique based on vector-valued metrics and convergent to zero matrices.*

1 Introduction

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. Indeed,in many cases, the mathematical models or equations used to describe phenomena in the biological, physical, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations.Therefore, it is more realistic to consider such equations as random operator equations which are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [3, 11, 12, 13, 14] among others. The problem of fixed points for random mappings was initialed by the Prague school of probabilities. The first results was studied in 1955-1956 by Špaček and Hanš in the context of Fredholm integral equations with random Kernel. In a separable metric space, random fixed point theorems for contraction mappings were proved by Hanš [6, 7], Špaček [15], Hanš and, Špaček [8] and Mukherjee [9, 10].

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many physical phenomena [21, 22, 23] . There has also been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas et al. [24], Miller and Ross [25], Podlubny [26], and for example, the paper of Kilbas and Trujillo [2].
Very recently, some basic theory for initial value problems for fractional differential equations and inclusions involving the Riemann-Liouville differential operator was discussed, see for examples, Benchohra et al. [27], B. Ahmad and J.J. Nieto [28].

This paper is concerned with the existence of solutions for initial value problems of fractional order functional differential equations with infinite delay and random effects (random parameters) of the form:

$$\left\{\begin{array}{rcl} D^{\alpha_1}x(t,w) &=& f_1(t,x_t(.,w),y_t(.,w),w),\ \textit{a.e.}\quad \textit{for each}\ \ t\in J=[0,b],\quad 0<\alpha_1<1,\\ D^{\alpha_2}y(t,w) &=& f_2(t,x_t(.,w),y_t(.,w),w),\ \textit{a.e.}\quad \textit{for each}\ \ t\in J=[0,b],\quad 0<\alpha_2<1,\\ x(t,w) &=& \phi_1(t,w),\ \ w\in\Omega,\ \ t\in(-\infty,0],\\ y(t,w) &=& \phi_2(t,w),\ \ w\in\Omega,\ \ t\in(-\infty,0], \end{array}\right. \tag{1.1}$$

where $(\Omega,\mathbb{F},\mathbb{P})$ is a complete probability space,$w\in\Omega$ and D^{α_i} is the standard Riemman-Liouville fractional derivative for each $i=1,2,$ $f_i: J\times\mathbb{B}\times\mathbb{B}\times\Omega\to\mathbb{R},$ is a given function satisfying some assumptions that will be specified later, ϕ_1,ϕ_2 are two random maps $\phi_i:\Omega\to\mathbb{B},$ $\phi_i(0,w)=0, i=1,2$ and $\mathbb{B}$ is called a phase space *that will be defined later (see Section 2). For any function x defined on $(-\infty,b]$ and any $t\in J$, we denote by $x_t(.,w)$ the element of $\mathbb{B}\times\Omega$ defined by*

$$x_t(\theta,w)=x(t+\theta,w),\ \ \theta\in(-\infty,0].$$

Here $x_t(\cdot,w)$ represents the history of the state from time $-\infty$ up to the present time, We assume that the histories $x_t(.,w)$ belong to the abstract phase $\mathbb{B}$. To our knowledge, the literature on the existence of random equations with fractional order and delay is very limited, so the present paper can be considered as a contribution to this question.
The paper is organized as follows. In Section 2 ,we introduce all the background material needed such as generalized metric spaces, examples of phase spaces , some random fixed point theorems by some new random versions of Perov's and Leray-Schauder's fixed point theorems in a vector Banach space. In Section 3, we prove some existence and compactness results for problem (1.1).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

For a stochastic process $x(\cdot,\cdot) : [0,b] \times \Omega \to X$, we will write $x(t)$ (or simply x when no confusion is possible) instead of $x(t,\omega)$.

2.1 Vector metric space

If, $x,y \in \mathbb{R}^n$ $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1,\dots,n$. Also $|x| = (|x_1|,\dots,|x_n|)$, $\max(x,y) = \max(\max(x_1,y_1),\dots,\max(x_n,y_n))$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i > 0\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1,\dots,n$.

Definition 2.1. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n_+$ with the following properties:

(i) $d(u,v) \geq 0$ for all $u,v \in X$; if $d(u,v)$ then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u,v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u,v,w \in X$.

We call the pair (X,d) a generalized metric space with $d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_n(x,y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if d_i, $i = 1,\dots,n$ are metrics on X.

For $r = (r_1,\dots,r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0,r) = \{x \in X : d(x_0,x) < r\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0,r)} = \{x \in X : d(x_0,x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 2.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.

Theorem 2.3. *[19] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) M is convergent towards zero;

(ii) $M^k \to 0$ as $k \to \infty$;

(iii) The matrix $(I - M)$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$

(iv) The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 2.4. Let (X,d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x),N(y)) \leq M d(x,y) \text{ for all } x,y \in X.$$

For $n = 1$ we recover the classical Banach's contraction fixed point result.

In this paper, we will employ an axiomatic defnition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [17] and follow the terminology used in [18]. Thus, $(\mathcal{B}, \|.\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into $\mathbb{R}$, and satisfying the following axioms :

(A1) There exist a positive constant H and functions $K(\cdot)$, $M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ with K continuous and M locally bounded, such that for any $a > 0$, if $x : (-\infty,a] \to \mathbb{R}$, $x \in {}_B$, and $x(\cdot)$ is continuous on $[0,a]$, then for every $t \in [0,a]$ the following conditions hold:

(i) x_t is in $\mathcal{B}$;

(ii) $|x(t)| \leq H\|x_t\|_{\mathcal{B}}$;

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t)\sup\{|x(s)| : \ 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}$, and H, K and M are independent of $x(\cdot)$.

(A_2) For the function $x(.)$ in (A_1), x_t is a $B-$valued continuous function on $[0,a]$.

(A_3) The space $\mathcal{B}$ is complete.

Denote by

$$K_b = \sup\{K(t) : \ t \in J\} \ \text{ and } \ M_b = \sup\{M(t) : \ t \in J\}.$$

Here after are some examples of phase spaces. For other details we refer, for instance to the book by Hino et al *[18].*

Example 2.5. The spaces BC, BUC, C^∞ and C^0. Let:

BC the space of bounded continuous functions defined from $(-\infty,0]$ to $\mathbb{R}$;

BUC the space of bounded uniformly continuous functions defined from $(-\infty,0]$ to $\mathbb{R}$;

C^{∞} $:= \{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta)$ exist in $E\}$;
C^0 $:= \{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) = 0\}$. Endowed with the uniform norm

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \in (-\infty, 0]\}.$$

We have that the spaces BUC, C^{∞} and C^0 satisfy conditions $(A_1) - (A_3)$. BC satisfies $(A_2), (A_3)$ but (A_1) is not satisfied.

Example 2.6. The spaces C_g, UC_g, C_g^{∞} and C_g^0.

Let g be a positive continuous function on $(-\infty, 0]$. We define:

C_g $:= \left\{\phi \in C((-\infty, 0], \mathbb{R}) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0]\right\}$;

C_g^0 $:= \left\{\phi \in C_g : \lim_{\theta\to-\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\}$, endowed with the uniform norm

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \in (-\infty, 0]\right\}.$$

We consider the following condition on the function g.

(g_1) For all $a > 0$, $\sup_{0\le t\le a} \sup\left\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty$.

Then we have that the spaces C_g and C_g^0 satisfy conditions (A_3). They satisfy conditions (A_1) and (A_3) if g_1 holds.

Example 2.7. The space C_{γ}.

For any real constant γ, we define the functional space C_{γ} by

$$C_{\gamma} := \left\{\phi \in C((-\infty, 0], E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } E\right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \le 0\}.$$

Then in the space C_{γ} the axioms $(A_1) - (A_3)$ are satisfied.

2.2 Random fractional derivative

Let $(\Omega, \mathcal{F})$ be a measurable space. We equip the metric space X with a $\sigma-$algebra $\mathcal{B}(X)$ of Borel subsets of X so that $(X, \mathcal{B}(X))$ becomes a measurable space. Let X and Y be two locally compact metric spaces. By $C(X, Y)$ we denote the space of continuous functions from X into Y endowed with the compact-open topology.

Lemma 2.8. *[20] Let X be a separable generalized metric space and $G : \Omega \times X \times X \to X$ be a mapping such that $G(\cdot, x, y)$ is measurable for all $x, y \in X$ and $G(w, \cdot, \cdot)$ is continuous for all $w \in \Omega$. Then the map $(w, x, y) \to G(w, x, y)$ is jointly measurable.*

Definition 2.9. function $f_i : [0, b] \times X \times X \times \Omega \to Y$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t, w) \longmapsto f_i(t, x, y, w)$ is jointly measurable for all $x, y \in X$,

(ii) The map $(x, y) \longmapsto f_i(t, x, y, w)$ is continuous for all $t \in [0, b]$ and $w \in \Omega$.

We say that $\phi(\cdot, \cdot) : [0, b] \times \Omega \to \mathbb{R}$ is sample path Lebesgue integrable on $[0, b]$ if $\phi(\cdot, w) : [0, b] \to \mathbb{R}$ is Lebesgue integrable on $[0, b]$ for a.e. $w \in \Omega$.

Let $\alpha > 0$. If $\phi : [0, b] \times \Omega \to \mathbb{R}$ is sample path Lebesgue integrable on [1, b], then we can consider the fractional integral

$$I_0^{\alpha}\phi(t, w) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi(s, w)ds$$

which will be called the sample path fractional integral of ϕ.

Remark 2.10. If $\phi(\cdot, w) : [1, b] \to \mathbb{R}$ is Lebesgue integrable on $[0, b]$ for each $w \in \Omega$, then $t \longmapsto I_0^{\alpha}\phi(t, w)$ is also Lebesgue integrable on $[0, b]$ for each $w \in \Omega$.

Recall that $\phi : [0, b] \times \Omega \to \mathbb{R}$ is a Caratheodory function if $t \longmapsto \phi(t, w)$ is continuous for a.e. $w \in \Omega$ and $w \longmapsto \phi(t, w)$ is measurable for each $t \in [0, b]$. Also, a Caratheodory function is a product measurable function.

Proposition 2.11. *If $\phi : [0, b] \times \Omega \to \mathbb{R}$ is a Caratheodory function, then the function $(t, w) \longmapsto I_0^{\alpha}\phi(t, w)$ is also a Caratheodory function*

Proof.

- It is clear that $I_0^{\alpha} : C([1, b], \mathbb{R}) \to \mathbb{R}$ is a continuous operator.
- Let $\psi : \Omega \to C([1, b], \mathbb{R})$ be defined
$$\psi(w)(\cdot) = \phi(\cdot, w).$$
By Lemma 2.8, then $\psi(\cdot)$ is measurable, so the operator $w \to (I^{\alpha} \circ \psi)(w)(\cdot)$ is measurable. Since the function $t \to I_0^{\alpha}\phi(t, w)$ is a continuous function, $(t, w) \to I_0^{\alpha}\phi(t, w)$ is a Caratheodory function

□

Definition 2.12. A function $\phi : [0,b] \times \Omega \to \mathbb{R}$ is said to have a sample path derivative at$t \in [0,b]$ if the function $t \longmapsto \phi(t,w)$ has a derivative at t for a.e. $w \in \Omega$.

Definition 2.13. The fractional derivative of order $\alpha > 0$ of a continuous function $\phi : \Omega \times [0,b] \to \mathbb{R}$ is given by

$$\frac{d^{\alpha}\phi(t,w)}{dt^{\alpha}} = \frac{1}{\gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}\phi(s,w)ds = \frac{d}{dt}I_0^{1-\alpha}\phi(t,w).$$

Proposition 2.14. *[29] Let X be a separable Banach space, and D be a dense linear subspace of X. Let $L : \Omega \times D \to X$ be a closed linear random operator such that, for each $w \in \Omega$, $L(w)$ is one to one and onto. Then the operator $S : \Omega \times X \to X$ defined by $S(w)x = L^{-1}(w)x$ is random.*

2.3 Random fixed point theory

Definition 2.15. Recall that a mapping $F : \Omega \times X \to X$ is said to be a random operator if, for any $x \in X$, $F(\cdot, x)$ is measurable.

Definition 2.16. A random fixed point of F is a measurable function $x : \Omega \to X$ such that

$$x(w) = F(w, x(w)) \quad \text{for all} \quad w \in \Omega.$$

In this section, we give the random versions of Perov fixed point theorem in generalized metric space.

Theorem 2.17. *[20] Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $\omega \in \Omega$ the matrix, $M(\omega)$ converge to 0 and*

$$d(F(\omega,x_1), F(\omega,x_2)) \leq M(\omega)d(x_1,x_2) \text{ for each } x_1, x_2 \in X,\ \omega \in \Omega.$$

then there exists any random variable $x : \Omega \to X$ which is the unique random fixed point of F.

By above result we present the following random nonlinear alternative.

Theorem 2.18. *[20] Let X be a separable generalized Banach space and let $F : \Omega \times X \to X$ be a completely continuous random operator. Then, either*

(i) the random equation $F(\omega, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(\omega, x(\omega)) = x(\omega)$ for all $\omega \in \Omega$, or

(ii) the set $\mathcal{M} = \{x : \Omega \to X \text{ is measurable} | \lambda(\omega)F(\omega, x) = x\}$ is unbounded for some measurable $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.

2.4 Existence and Uniqueness of a Solution

Let us start by defining what we mean by a solution of problem (1.1). Let the space

$$\mathcal{E} = \{x : (-\infty, b] \to \mathbb{R} : x|_{(-\infty,0]} \in \mathcal{B} \text{ and } x|_{[0,b]} \text{ is continuous}\}.$$

Definition 2.19. A function $x, y \in \mathcal{E}$ is said to be a solution of (1.1) if x satisfies the equation $(D^{\alpha_1}x(t), D^{\alpha_2}y(t)) = (f_1(t, x_t, y_t), f_2(t, x_t, y_t))$ on J, and the condition $(x(t), y(t)) = (\phi_1(t), \phi_2(t))$ on $(-\infty, 0]$.

For the existence results on the problem (1.1) we need the following auxiliary lemma.

Lemma 2.20. *[1] Let $0 < \alpha < 1$ and let $h_i : (0,b] \to \mathbb{R}$ be continuous and $\lim_{t\to 0^+} h_i(t) = h_i(0^+) \in \mathbb{R}$ for each $i = 1, 2$. Then (x, y) is a solution of the fractional integral equation*

$$\begin{array}{rcl} x(t) & = & \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1 - 1}h_1(x(t), y(s))ds, \\ y(t) & = & \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2 - 1}h_2(x(t), y(s))ds, \end{array}$$

if and only if, (x, y) is a solution of the initial value problem for the fractional differential equation

$$\begin{array}{rcl} D^{\alpha_1}x(t) & = & h_1(x(t), y(t)),\ \ t \in (0,b] \\ D^{\alpha_2}x(t) & = & h_2(x(t), y(t)),\ \ t \in (0,b] \\ (x(0), y(0)) & = & (0,0) \end{array}$$

Our first main result is the existence and uniqueness of a random solution to the problem (1.1).

Theorem 2.21. *Let $f_i : J \times \mathcal{B} \times \mathcal{B} \times \Omega \to \mathbb{R}$ be two Carathory functions. Assume that the following conditions hold:*

(H_1) *There exist random variables $\ell_i, \bar{\ell}_i : \Omega \to \mathbb{R}^+$ such that*

$$|f_i(t, u, v, w) - f_i(t, \bar{u}, \bar{v}, w)| \leq \ell_i(w)\|u - \bar{u}\|_{\mathcal{B}} + \bar{\ell}_i(w)\|v - \bar{v}\|_{\mathcal{B}}.$$

for all $u, v, \bar{u}, \bar{v} \in \mathcal{B}$ and $w \in \Omega$.

If for every $w \in \Omega$,

$$M_{trix}(w) = \begin{pmatrix} \frac{b^{\alpha_1} K_b \ell_1(w)}{\Gamma(\alpha_1+1)} & \frac{b^{\alpha_1} K_b \bar{\ell}_1(w)}{\Gamma(\alpha_1+1)} \\ \frac{b^{\alpha_2} K_b \ell_2(w)}{\Gamma(\alpha_2+1)} & \frac{b^{\alpha_2} K_b \bar{\ell}_2(w)}{\Gamma(\alpha_2+1)} \end{pmatrix}$$

converges to 0, then problem (1.1) has a unique random solution on the interval $(-\infty, b]$.

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator $N : \mathcal{E} \times \mathcal{E} \times \Omega \longrightarrow \mathcal{E} \times \mathcal{E}$ defined by

$$(x, y) \longmapsto (N_1(x, y, w), N_2(x, y, w)),$$

where

$$N_1(x(t, w), y(t, w), w) = \begin{cases} \phi_1(t, w), & t \in (-\infty, 0], \\ \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} f(s, x_s, y_s, w) ds, & t \in [0, b], \end{cases}$$

and

$$N_2(x(t, w), y(t, w), w) = \begin{cases} \phi_2(t, w), & t \in (-\infty, 0], \\ \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2 - 1} f(s, x_s, y_s, w) ds, & t \in [0, b]. \end{cases}$$

First we show that N is a random operator on $\mathcal{E} \times \mathcal{E}$. Since f_1 and f_2 are Caratheodory functions, $w \longmapsto f_1(t, x, y, w)$ and $w \longmapsto f_2(t, x, y, w)$ are measurable maps in view of Lemma 2.8 and Proposition2.11. Further, the integral is a limit of a finite sum of measurable functions; therefore, the maps

$$w \longmapsto N_1(x(t, w), y(t, w), w), \qquad w \longmapsto N_2(x(t, w), y(t, w), w)$$

are measurable. As a result, N is a random operator on $\mathcal{E} \times \mathcal{E} \times \Omega$ into $\mathcal{E} \times \mathcal{E}$. We next show that N satisfies all the conditions of Theorem 2.17 on $\mathcal{E} \times \mathcal{E}$. Let $(x(\cdot, w), y(\cdot, w)) \in \mathcal{E} \times \mathcal{E}$; then

Let $x_i(\cdot, \cdot) : (-\infty, b] \times \Omega \to \mathbb{R}$ be the function defined by

$$x_i(t, w) = \begin{cases} \phi(0, w), & \text{if } t \in [0, b], \\ \phi_i(t, w), & \text{if } t \in (-\infty, 0]. \end{cases}$$

Then $(x_1(0, w), x_2(0, w)) = (\phi_1(w), \phi_2(w))$. For each $z^i \in C([0, b] \times \Omega, \mathbb{R})$ with $z^i(0, w) = 0$, we denote by $\bar{z}^i$ the function defined by

$$\bar{z}^i(t, w) = \begin{cases} z^i(t, w), & \text{if } t \in [0, b], \\ 0, & \text{if } t \in (-\infty, 0]. \end{cases}$$

If $(x(\cdot, w), y(\cdot, w))$ satisfies the integral equation,

$$x(t, w) = \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} f_1(s, x_s, y_s, w) ds,$$

and

$$y(t, w) = \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2 - 1} f_2(s, x_s, y_s, w) ds,$$

we can decompose $(x(\cdot, w), y(\cdot, w))$ as $(x(t, w), y(t, w)) = (\bar{z}^1(t, w) + x^1(t, w), \bar{z}^2(t, w) + x^2(t, w)), 0 \le t \le b$, which implies $(x_t(., w), y_t(., w)) = (\bar{z}^1_t(., w) + x^1_t(., w), \bar{z}^2_t(., w) + x^2_t(., w))$ for every $0 \le t \le b$, and the function $z^i(\cdot)$ satisfies

$$z^1(t, w) = \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} f_1(s, \bar{z}^1_s + x^1_s, \bar{z}^2_s + x^2_s, w) ds.$$

and

$$z^2(t, w) = \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2 - 1} f_2(s, \bar{z}^1_s + x^1_s, \bar{z}^2_s + x^2_s, w) ds.$$

Set

$$C_0 = \{z^i \in C([0, b], \mathbb{R}) : z(0, w) = z_0(w) = 0, \ w \in \Omega\},$$

and let $\| \cdot \|_{\mathcal{D}}$ be the seminorm in C_0 defined by

$$\|z^i\|_{\mathcal{D}} = \|z^i_0\|_{\mathcal{B}} + \sup\{|z^i(t)| : 0 \le t \le b\} = \sup\{|z^i(t)| : 0 \le t \le b\}, \ z^i \in C_0.$$

C_0 is a Banach space with norm $\| \cdot \|_{\mathcal{D}}$.
Consider the operator $P : C_0 \times C_0 \times \Omega \to C_0 \times C_0$ defined by

$$(z^1, z^2) \longmapsto (P_1(z^1, z^2, w), P_2(z^1, z^2, w)),$$

$$P_1(z^1(t, w), z^2(t, w), w) = \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} f_1(s, \bar{z}^1_s + x^1_s, \bar{z}^2_s + x^2_s, w) ds, \quad t \in [0, b]. \tag{2.1}$$

and

$$P_2(z^1(t, w), z^2(t, w), w) = \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2 - 1} f_2(s, \bar{z}^1_s + x^1_s, \bar{z}^2_s + x^2_s, w) ds, \quad t \in [0, b]. \tag{2.2}$$

That the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point. We shall show that $P: C_0\times C_0\times\Omega \to C_0\times C_0$ is a contraction map. Indeed, consider $(z^1(.,w), z^2(.,w)), (z^{1,*}(.,w), z^{2,*}(.,w)) \in C_0 \times C_0$. Then we have for each $t \in [0, b]$

$$\begin{aligned}
&|P_1(z^1(t), z^2(t), w) - P_1(z^{1,*}(t), z^{2,*}(t), w)|\\
&\le \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}|f_1(s, \bar{z}_s^1 + x_s^1, \bar{z}_s^2 + x_s^2, w) - f_1(s, \bar{z}_s^{1,*} + x_s^1, \bar{z}_s^{2,*} + x_s^2, w)|ds\\
&\le \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\Big(\ell_1(w)\|\bar{z}_s^1 - \bar{z}_s^{1,*}\|_{\mathcal{B}} + \bar{\ell}_1(w)\|\bar{z}_s^2 - \bar{z}_s^{2,*}\|_{\mathcal{B}}\Big)ds\\
&\le \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}K_b \sup_{s\in[0,t]}\Big(\ell_1(w)\|z(s)^1 - z^{1,*}(s)\| + \bar{\ell}_1(w)\|z(s)^2 - z^{2,*}(s)\|\Big)ds\\
&\le \frac{K_b}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\ell_1 ds\|z^1 - z^{1,*}\|_{\mathcal{D}} + \frac{K_b}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\bar{\ell}_1 ds\|z^2 - z^{2,*}\|_{\mathcal{D}}.
\end{aligned}$$

Therefore

$$\|P_1(z^1, z^2, w) - P_1(\bar{z}^{1,*}, \bar{z}^{2,*}, w)\|_{\mathcal{D}} \le \frac{b^{\alpha_1}K_b\ell_1(w)}{\Gamma(\alpha_1+1)}\|z^1 - z^{1,*}\|_{\mathcal{D}} + \frac{b^{\alpha_1}K_b\bar{\ell}_1(w)}{\Gamma(\alpha_1+1)}\|z^2 - z^{2,*}\|_{\mathcal{D}}.$$

Similarly, we obtain

$$\|P_2(z^1, z^2, w) - P_2(\bar{z}^{1,*}, \bar{z}^{2,*}, w)\|_{\mathcal{D}} \le \frac{b^{\alpha_2}K_b\ell_2(w)}{\Gamma(\alpha_2+1)}\|z^1 - z^{1,*}\|_{\mathcal{D}} + \frac{b^{\alpha_2}K_b\bar{\ell}_2(w)}{\Gamma(\alpha_2+1)}\|z^2 - z^{2,*}\|_{\mathcal{D}},$$

Hence

$$d(P(z^1, z^2, w), P(\bar{z}^{1,*}, \bar{z}^{2,*}, w)) \le M_{trix}(w)d((z^1, z^2), (\bar{z}^{1,*}, \bar{z}^{2,*})),$$

where

$$d(z^1, z^2) = \begin{pmatrix} \|z^1 - z^2\|_{\mathcal{D}} \\ \|z^1 - z^2\|_{\mathcal{D}} \end{pmatrix}$$

and

$$M_{trix}(w) = \begin{pmatrix} \frac{b^{\alpha_1}K_b\ell_1(w)}{\Gamma(\alpha_1+1)} & \frac{b^{\alpha_1}K_b\bar{\ell}_1(w)}{\Gamma(\alpha_1+1)} \\ \frac{b^{\alpha_2}K_b\ell_2(w)}{\Gamma(\alpha_2+1)} & \frac{b^{\alpha_2}K_b\bar{\ell}_2(w)}{\Gamma(\alpha_2+1)} \end{pmatrix}$$

It is clear that the radius spectral $\rho(M_{trix}(w)) < 1$. By Lemma 2.3 $M_{trix}(w)$ converges to zero. From Theorem 2.17 there exists a unique random solution of problem (1.1)

□

We recall Gronwalls lemma for singular kernels, whose proof can be found in ([30], Lemma 7.1.1), which will be essential for the main result of this section.

Lemma 2.22. *Let $v : [0, b] \to [0, \infty)$ be a real function and $W(\cdot)$ is a nonnegative, locally integrable function on $[0, b]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le W(t) + a\int_0^t \frac{v(s)}{(t-s)^\alpha}ds,$$

then,there exists a constant $K = K(\alpha)$ such that

$$v(t) \le W(t) + Ka\int_0^t \frac{W(s)}{(t-s)^\alpha}ds,$$

for every $t \in [0, b]$.

Next, we present an existence result that does not assume Lipschitz conditions. We need the following conditions:

(H_2) *For every $w \in \Omega$, the functions $f_i(\cdot,\cdot,w)$ are continuous for each $i = 1, 2$, and $w \to f_i(\cdot,\cdot,w))$ are measurable*

(H_3) *There exist measurable and bounded functions $p_i, q_i, \gamma_i : J \times \Omega \to \mathbb{R}_+$ such that*

$$|f_i(t, x, y, w)| \le p_i(t, w) + q_i(t, w)\|x\|_{\mathcal{B}} + \gamma_i(t, w)\|y\|_{\mathcal{B}}$$

for $t \in J$ and each $x, y \in \mathcal{B}$, and $\|I^\alpha p_i\|_\infty < +\infty$, $i = 1, 2$

We prove an existence result for problem (1.1) by using a Leray-Schauder type random fixed point theorem in generalized Banach spaces.

Theorem 2.23. *Assume that (H_2) and (H_3) hold. Then the problem (1.1) has a random solution defined on $(-\infty, b]$.*

Proof. Let $P : C_0 \times C_0 \times \Omega \to C_0 \times C_0$ be the random operator defined in 2.21.
Clearly, the random fixed points of P are solutions to (1.1). In order to apply theorem 2.18. we first show that N is completely continuous. The proof will be given in several steps.
Step 1: *$P(\cdot,\cdot,w) = (P_1(\cdot,\cdot,w), P_2(\cdot,\cdot,w))$ is continuous.*

Let $\{(z_n^1, z_n^2)\}$ be a sequence such that $(z_n^1, z_n^2) \to (z^1, z^2) \in C_0 \times C_0$ as $n \to \infty$. Then

$$\begin{aligned}&|(P_1(z_n^1(t,w), z_n^2(t,w), w)) - P_1(z^1(t,w), z^2(t,w), w))|\\ &\leq \frac{1}{\Gamma(\alpha_1)} \int_0^b (t-s)^{\alpha_1 - 1} |f_1(s, \bar{z}_{n_s}^1 + x_s^1, \bar{z}_{n_s}^2 + x_s^2, w) - f_1(s, \bar{z}_s^1 + x_s^1, \bar{z}_s^2 + x_s^2, w)| ds.\end{aligned}$$

Since f_i is a continuous function, we have

$$\begin{aligned}&\|P_1(z_n^1(\cdot,w), z_n^2(\cdot,w), w)) - P_1(z^1(\cdot,w), z^2(\cdot,w), w))\|_{\mathcal{D}}\\ &\leq \frac{b^{\alpha_1}}{\Gamma(\alpha_1+1)} \|f_1(\cdot, \bar{z}_{n.}^1 + x_.^1, \bar{z}_{n.}^2 + x_.^2, w) - f_1(\cdot, \bar{z}_.^1 + x_.^1, \bar{z}_.^2 + x_.^2, w)\|_\infty \to 0 \text{ as } n \to \infty.\end{aligned}$$

Similarly

$$\begin{aligned}&\|P_2(z_n^1(\cdot,w), z_n^2(\cdot,w), w)) - P_2(z^1(\cdot,w), z^2(\cdot,w), w))\|_{\mathcal{D}}\\ &\leq \frac{b^{\alpha_1}}{\Gamma(\alpha_1+1)} \|f_2(\cdot, \bar{z}_{n.}^1 + x_.^1, \bar{z}_{n.}^2 + x_.^2, w) - f_2(\cdot, \bar{z}_.^1 + x_.^1, \bar{z}_.^2 + x_.^2, w)\|_\infty \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus P is continous.
Step 2: *P maps bounded sets into bounded sets in C_0.*

Indeed, it is enough to show that for any $\eta^1, \eta^2 > 0$, there exists a positive constant $\ell = (\ell_1, \ell_2)$ such that for each $z^1, z^2 \in \mathcal{B}_\eta = \{z^1, z^2 \in C_0 : \|z^1\|_{\mathcal{D}} \leq \eta^1, |z^2\|_{\mathcal{D}} \leq \eta^2\}$ one has $\|P(z^1, z^2)\|_\infty \leq (\ell_1, \ell_2)$. Let $z^i \in \mathcal{B}_{\eta^i}$. Since f_i is a continuous function,Then for each $t \in [0, b]$, we get

$$\begin{aligned}&|P_1(z^1(t,w), z^2(t,w), w))|\\ &\leq \frac{1}{\Gamma(\alpha_1)} \int_0^b (t-s)^{\alpha_1-1} |f_1(s, \bar{z}_s^1 + x_s^1, \bar{z}_s^2 + x_s^2, w)| ds\\ &\leq \frac{1}{\Gamma(\alpha_1)} \int_0^b (t-s)^{\alpha_1-1} \Big(p_1(s,w) + q_1(s,w)\|\bar{z}_s + x_s\|_{\mathcal{B}} + \gamma_1(s,w)\|\bar{z}_s + x_s\|_{\mathcal{B}}\Big) ds\\ &\leq \frac{b^{\alpha_1}\int_0^b p_1(s,w)ds}{\Gamma(\alpha_1+1)} + \frac{b^{\alpha_1}\int_0^b q_1(s,w)ds}{\Gamma(\alpha_1+1)}\eta_1^* + \frac{b^{\alpha_1}\int_0^b \gamma_1(s,w)ds}{\Gamma(\alpha_1+1)}\eta_2^* =: \ell_1,\end{aligned}$$

where

$$\|\bar{z}_s^i + x_s^i\|_{\mathcal{B}} \leq \|\bar{z}_s^i\|_{\mathcal{B}} + \|x_s^i\|_{\mathcal{B}} \leq K_b\eta^i + K_b\|\phi_i(0)\| + M_b\|\phi_i\|_{\mathcal{B}} := \eta_i^*,$$

and

$$M_b = \sup\{|M(t)| : t \in [0, b]\}.$$

Similarly, we have

$$\begin{aligned}&|P_2(z^1(t,w), z^2(t,w), w))|\\ &\leq \frac{b^{\alpha_2}\int_0^b p_2(s,w)ds}{\Gamma(\alpha_2+1)} + \frac{b^{\alpha_2}\int_0^b q_2(s,w)ds}{\Gamma(\alpha_2+1)}\eta_1^* + \frac{b^{\alpha_2}\int_0^b \gamma_2(s,w)ds}{\Gamma(\alpha_2+1)}\eta_2^* =: \ell_2.\end{aligned}$$

Step 3: *P maps bounded sets into equicontinuous sets of C_0.*

Let $t_1, t_2 \in [0, b]$, $t_1 < t_2$ and let $\mathcal{B}_\eta$ be a bounded set of C_0 as in Step 2. Let $z \in \mathcal{B}_\eta$. Then for each $t \in [0, b]$, we have

$$\begin{aligned}&|P_1(z^1(t_2,w), z^2(t_2,w), w)) - P_1(z^1(t_1,w), z^2(t_1,w), w))|\\ &= \tfrac{1}{\Gamma(\alpha_1)}\Big|\int_0^{t_1}((t_2-s)^{\alpha_1-1} - (t_1-s)^{\alpha_1-1}) f_1(s, \bar{z}_s^1 + x_s^1, \bar{z}_s^2 + x_s^2, w) ds\\ &+ \tfrac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1} f_1(s, \bar{z}_s^1 + x_s^1, \bar{z}_s^2 + x_s^2, w) ds\Big|\\ &\leq \tfrac{\|p_1(w)\|_\infty + \|q_1(w)\|_\infty\eta_1^* + \|\gamma_1(w)\|_\infty\eta_2^*}{\Gamma(\alpha_1)} \int_0^{t_1}[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha_1-1}] ds\\ &+ \tfrac{\|p_1(w)\|_\infty + \|q_1(w)\|_\infty\eta_1^* + \|\gamma_1(w)\|_\infty\eta_2^*}{\Gamma(\alpha_1)} \int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1} ds\\ &\leq \tfrac{\|p_1(w)\|_\infty + \|q_1(w)\|_\infty\eta_1^* + \|\gamma_1(w)\|_\infty\eta_2^*}{\Gamma(\alpha_1+1)} \Big((t_2-t_1)^{\alpha_1} + t_1^{\alpha_1} - t_2^{\alpha_1}\Big)\\ &+ \tfrac{\|p_1(w)\|_\infty + \|q_1(w)\|_\infty\eta_1^* + \|\gamma_1(w)\|_\infty\eta_2^*}{\Gamma(\alpha_1+1)} (t_2-t_1)^{\alpha_1}.\end{aligned}$$

Hence,

$$\begin{aligned}&|P_1(z^1(t_2,w), z^2(t_2,w), w)) - P_1(z^1(t_1,w), z^2(t_1,w), w))|\\ &\leq \tfrac{\|p_1(w)\|_\infty + \|q_1(w)\|_\infty\eta_1^* + \|\gamma_1(w)\|_\infty\eta_2^*}{\Gamma(\alpha_1+1)} (t_2-t_1)^{\alpha_1},\end{aligned}$$

and

$$\begin{aligned}&|P_2(z^1(t_2,w), z^2(t_2,w), w)) - P_2(z^1(t_1,w), z^2(t_1,w), w))|\\ &\leq \tfrac{\|p_2(w)\|_\infty + \|q_2(w)\|_\infty\eta_1^* + \|\gamma_2(w)\|_\infty\eta_2^*}{\Gamma(\alpha_2+1)} (t_2-t_1)^{\alpha_2}.\end{aligned}$$

The right-hand term tends to zero as $|t_2 - t_1| \to 0$.
As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $P : C_0 \times C_0 \longrightarrow C_0 \times C_0$ is continuous and completely continuous.

Step 4: (A priori bounds).

$$\mathcal{A}(w) = \{(z^1, z^2) \in C_0 \times C_0 \ : (z^1, z^2) = \lambda(w)P_1(z^1, z^2), \ \lambda(w) \in (0, 1)\},$$

is bounded.

Let $(z^1, z^2) \in C_0 \times C_0$. Then $z^1 = \lambda P_1(z^1, z^2)$ and $z^2 = \lambda P_2(z^1, z^2)$ for some $0 < \lambda(w) < 1$. Then for each $t \in [0, b]$ we have

$$z^1(t,w) = \lambda(w)\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1} f_1(s, \bar{z}_s^1 + x_s^1, \bar{z}_s^2 + x_s^2, w)ds\Big).$$

Hence

$$\begin{aligned}|z^1(t,w)| &\leq \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\Big(q_1(s,w)\|\bar{z}_s^1 + x_s^1\|_{\mathcal{B}} + \gamma_1(s,w)\|\bar{z}_s^2 + x_s^2\|_{\mathcal{B}}\Big)ds \\ &+ \frac{b^{\alpha}\|p_1(w)\|_\infty}{\Gamma(\alpha_1+1)}.\ t \in [0,b].\end{aligned}$$

But

$$\begin{aligned}\textstyle\sum_{i=1}^2 \|\bar{z}_s^i + x_s^i\|_{\mathcal{B}} &\leq \textstyle\sum_{i=1}^2 \Big(\|\bar{z}_s^i\|_{\mathcal{B}} + \|x_s^i\|_{\mathcal{B}}\Big) \\ &\leq \textstyle\sum_{i=1}^2 \Big(K(t)\sup\{|z^i(s)| : 0 \leq s \leq t\} + M(t)\|z_0^i\|_{\mathcal{B}} \\ &\quad + K(t)\sup\{|x^i(s)| : 0 \leq s \leq t\} + M(t)\|x_0^i\|_{\mathcal{B}}\Big) \\ &\leq \textstyle\sum_{i=1}^2 \Big(K_b\sup\{|z^i(s)| : 0 \leq s \leq t\} + M_b\|\phi_i\|_{\mathcal{B}} + K_b|\phi_i(0)|\Big) \\ &\leq \Big(K_b\sup\{|z^1(s,w)| + |z^2(s,w)| : 0 \leq s \leq t\} \\ &\quad + M_b(\|\phi_1(w)\|_{\mathcal{B}} + \|\phi_2(w)\|_{\mathcal{B}})\Big).\end{aligned} \tag{2.3}$$

If we name $W_*(t,w)$ the right hand side of (2.3), then we have

$$\sum_{i=1}^2 \|\bar{z}_s^i + x_s^i\|_{\mathcal{B}} \leq \sum_{i=1}^2 W_i(t,w) = W_*(t,w).$$

Therefore

$$\begin{aligned}|z^1(t,w)| + |z^2(t,w)| &\leq \frac{1}{\Gamma(\alpha_*)}\int_0^t (t-s)^{\alpha_*-1}(q_1(s,w) + q_2(s,w))W_1(s,w)ds \\ &+ \frac{1}{\Gamma(\alpha_*)}\int_0^t (t-s)^{\alpha_*-1}(\gamma_1(s,w) + \gamma_2(s,w))W_2(s,w)ds \\ &+ \frac{b^{\alpha_*}(\|p_1(w)\|_\infty + \|p_2(w)\|_\infty)}{\Gamma(\alpha_*+1)} \\ &\leq \frac{2}{\Gamma(\alpha_*)}\int_0^t (t-s)^{\alpha_*-1} q_*(s,w)W_*(s,w)ds + \frac{b^{\alpha_*}\|p_*(w)\|_\infty}{\Gamma(\alpha_*+1)},\quad t \in [0,b],\end{aligned}$$

where

$$\|p_*(w)\|_\infty = \max\{\|p_1(w)\|_\infty, \|p_2(w)\|_\infty\}$$

and

$$\|q_*(w)\|_\infty = \max\{\|q_1(w)\|_\infty + \|q_2(w)\|_\infty\ ,\ \|\gamma_1(w)\|_\infty + \|\gamma_2(w)\|_\infty\},$$
$$\alpha_* = \max\{\alpha_1, \alpha_2\},\quad \|\gamma_*(w)\|_\infty = \max\{\|\gamma_1(w)\|_\infty, \|\gamma_2(w)\|_\infty\}.$$

Using the above inequality and the definition of W_* we have that

$$W_*(t,w) \quad \leq \quad \Lambda + \frac{2K_b\|q_*(w)\|_\infty}{\Gamma(\alpha_*)}\int_0^t (t-s)^{\alpha_*-1}W_*(s,w)ds,\quad t \in [0,b],$$

where

$$\Lambda = M_b(\|\phi_1\|_{\mathcal{B}} + \|\phi_2\|_{\mathcal{B}}) + \frac{K_b b^{\alpha_*}\|p_*(w)\|_\infty}{\Gamma(\alpha_*+1)}.$$

Then from Lemma 2.22, there exists $K = K(\alpha_*)$ such that we have

$$|W_*(t,w)| \quad \leq \quad \Lambda + K(\alpha_*)\frac{2K_b\|q_*(w)\|_\infty}{\Gamma(\alpha_*)}\int_0^t (t-s)^{\alpha_*-1}\Lambda ds.$$

Therefore

$$\|W_*(w)\|_\infty \leq \Lambda + \frac{2\Lambda K(\alpha_*)b^{\alpha_*}K_b}{\Gamma(\alpha_*+1)} := \widetilde{M}.$$

Consequently

$$\|z^1\|_\infty \leq \widetilde{M}\|I^{\alpha_*}q_*(w)\|_\infty + \frac{b^{\alpha_*}\|p_*(w)\|_\infty}{\Gamma(\alpha_*)+1} := M^* \quad \text{and} \quad \|z^2\|_\infty \leq M^*.$$

This shows that $\mathcal{A}$ is bounded.
As a consequence of Theorem 2.18 we conclude that N has a random fixed point $w \to ((x(\cdot,w), y(\cdot,w))$ that is a solution to (1.1).

□

References

[1] *D. Delboso, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation,* J. Math. Anal. Appl. **204** *(1996) 609–625.*

[2] *A.A. Kilbas and J.J. Trujillo, Differential equations of fractional order: Methods, results and problems,* II. Appl. Anal. **81** *(2002), 435-493.*

[3] *A. T. Bharuch-Reid,* Random Integral Equations*, New York, Academic Press, 1972.*

[4] *S. Itoh, Random fixed point theorems with an application to random differential equations in Banach spaces,* J. Math. Anal. Appl. **67** *(1979), 261-273.*

[5] *S. Itoh, Nonlinear random equatios with monotone operators in Banach spaces,* Math. Ann.*;* **236** *(1978), 133-146*

[6] *O. Hanš, Random operator equations,* Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Univ. California Press, Berkeley, Calif.*, (1961),* **II** *185–202.*

[7] *O. Hanš, Random fixed point theorems. 1957 Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956 pp. 105–125 Publishing House of the Czechoslovak Academy of Sciences, Prague.*

[8] *O. Hanš and A. Špacek, Random fixed point approximation by differentiable trajectories. 1960 Trans. 2nd Prague Conf. Information Theory pp. 203–213 Publ. House Czechoslovak Acad. Sci., Prague, Academic Press, New York.*

[9] *A. Mukherjea, Transformations aléatoires separables. Théorème du point fixe aloire,* C. R. Acad. Sei. Paris Sér. *A-B* **263** *(1966), 393-395.*

[10] *A. Mukherjea,* Random Transformations of Banach Spaces*; Ph. D. Dissertation, Wayne State Univ. Detriot, Michigan, 1968.*

[11] *W. Padgett and C. Tsokos,* Random Integral Equations with Applications to Life Science and Engineering, *Academic Press, New York, 1976.*

[12] *D. O'Regan, N. Shahzad and R. P. Agarwal, Random fixed point theory in spaces with two metrics, J. Appl. Math. Stoch. Anal.* **16** *(2003), 171-176.*

[13] *N. Shahzad, Random fixed point theorems for various classes of 1-set-ontractive maps in Banach spaces,* J. Math. Anal. Appl. **203** *(1996), 712718.*

[14] *N. Shahzad and S. Latif, Random fixed points for several classes of 1-ball-contractive and 1-set-contractive random maps,* J. Math. Anal. Appl. **237** *(1999), 83–92.*

[15] *A. Špaček, Zulfallige Gleichungen,* Czechoslovak, Math. J. , **5** *(1995), 462-466. mappings,* Proc. Amer. Math. Soc. **95** *(1985), 91–94*

[16] *J. M. A. Toledano, T. D. Benavides, and G. L. Azedo,* Measures of Noncompactness in Metric Fixed Point Theory, *Birkhauser, Basel, 1997.*

[17] *JK.Hale, J.Kato Phase space for retarded equations with infinite delay.* Funkcialaj Ekvacioj *1978,* **21***, 11–41.*

[18] *Y. Hino,S. Murakami,T. Naito* .Functional Differential Equations with Infinite Delay*, Lectures Notes in Mathematics, vol. 1473. Springer-Verlag: Berlin,1991.*

[19] *I. A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances,* Fixed Point Theory **9** *(2008), 541–559.*

[20] *ML. Sinacer, JJ.Nieto, A. Ouahab. Random fixed point theorem in generalized Banach space and applications.* Random Oper. Stoch. Equ. **24***, 93–112 (2016)*

[21] *K. Diethelm and A. D. Freed, On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity. In:* Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties *(Eds. F. Keil, W. Mackens, H. Voss, and J. Werther), 217-224, Springer-Verlag, Heidelberg, 1999.*

[22] *L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators.* Mech. Systems Signal Processing **5** *(1991), 81-88.*

[23] *A.B. Malinowska and D.F.M. Torres, Towards a combined fractional mechanics and quantization.* Fract. Calc. Appl. Anal. **15***, No 3 (2012), 407-417*

[24] *A.A. Kilbas, H.M. Srivastava and J.J.* Trujillo, Theory and Applications of Fractional Differential Equations. *North-Holland Mathematics Studies* **204***, Elsevier Science B. V. Amsterdam, 2006.*

[25] *K.S. Miller and B. Ross,* An Introduction to the Fractional Calculus and Differential Equations. *John Wiley, New York, 1993.*

[26] *I. Podlubny,* Fractional Differential Equations. *Academic Press, San Diego, 1999.*

[27] *M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay.* J. Math. Anal. Appl. **338** *(2008), 13401350.*

[28] *B. Ahmad and J.J. Nieto, Anti-periodic fractional boundary value problems with nonlinear term depending on lower order derivative.*Fract. Calc. Appl. Anal. **15***, No 3 (2012), 451-462*

[29] *D. Krawaritis,N.Stavrakakis, Perturbations of maximal monotone random operators.* Linear Algebra Appl. **84***, 301–310 (1986)*

[30] *D. Henry,* Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer-Verlag, Berlin, 1989*

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