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Multiplicative generalized skew-derivations on ideals in semiprime rings

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Abstract Our intention in this manuscript is to study the commutative structure of the semiprime rings $\mathfrak{R}$ on appropriate subsets of it. We observe several algebraic identities in the presence of multiplicative generalized skew-derivation.

1 Introduction

Suppose $\mathfrak{R}$ be a ring with associative properties. $\mathfrak{Z}$ stands for the centre of $\mathfrak{R}$. The expression $[t_1, t_2] = t_1t_2 - t_2t_1$ (resp. $t_1 \circ t_2 = t_1t_2 + t_2t_1$) is stand for commutator (resp. anti-commutator). The ring $\mathfrak{R}$ is known as prime ring if $\forall\, t_1, t_2 \in \mathfrak{R}$, we have $t_1\mathfrak{R}t_2 = (0) \implies t_1 = 0$ or $t_2 = 0$ and is called semiprime ring if for all $t_1 \in \mathfrak{R}$, $t_1\mathfrak{R}t_1 = (0)$ implies $t_1 = 0$. An additive maps d is called derivation if $d(t_1t_2) = d(t_1)t_2 + t_1d(t_2)\ \forall\, t_1, t_2 \in \mathfrak{R}$, and if d is not necessarily additive then it is called multiplicative derivation, this concept was given by Daif [6]. The map $\mathfrak{I}_a$ from $\mathfrak{R}$ to $\mathfrak{R}$ defined by $\mathfrak{I}_a(t_1) = [a, t_1]$ for $a \in \mathfrak{R}$ is known as an inner derivation of $\mathfrak{R}$. Let $\mathfrak{F} : \mathfrak{R} \to \mathfrak{R}$ be a map defined by $\mathfrak{F}(t_1t_2) = \mathfrak{F}(t_1)t_2 + t_1d(t_2)\ \forall\, t_1, t_2 \in \mathfrak{R}$ is called generalized derivation associated with d, and if $\mathfrak{F}$ is not necessarily additive then it is called multiplicative generalized derivation. For $a, b \in \mathfrak{R}$, an additive mapping $\mathfrak{G}$ from $\mathfrak{R}$ to $\mathfrak{R}$ is known as generalized inner derivation if $\mathfrak{G}(t_1) = at_1 + t_1b$. It is easy to notice that for such a mapping $\mathfrak{G}$, $\mathfrak{G}(t_1t_2) = \mathfrak{G}(t_1)t_2 + t_1[t_2, b] = \mathfrak{G}(t_1)t_2 + t_1I_b(t_2)\ \forall\, t_1, t_2 \in \mathfrak{R}$, generalized inner derivations and derivations are two examples of generalized derivations.

An additive mapping $d(t_1t_2) = d(t_1)\alpha(t_2) + t_1d(t_2)$ is term as left skew-derivation and $d(t_1t_2) = d(t_1)t_2 + \alpha(t_1)d(t_2)$ is known as right skew-derivation, where α is an associated automorphism of d. If the derivation is both left as well as right skew-derivation, then d is called as skew-derivation. Since, d is derivation if $\alpha = 1$ (identity automorphism). Similarly, we define generalized skew-derivation of $\mathfrak{R}$. An additive mapping $\mathfrak{F}$ from $\mathfrak{R}$ to $\mathfrak{R}$ is said to be generalized skew-derivation if it is both left as well as right generalized skew-derivation i.e., $\mathfrak{F}(t_1t_2) = \mathfrak{F}(t_1)\alpha(t_2) + t_1d(t_2) = \mathfrak{F}(t_1)t_2 + \alpha(t_1)d(t_2)\ \forall\, t_1, t_2 \in \mathfrak{R}$, where α and d is associated automorphism and skew-derivation of $\mathfrak{F}$. A map $\mathfrak{H}$ of $\mathfrak{R}$ of the form $\mathfrak{H}(x) = ax + \alpha(x)b$ for all $a, b \in \mathfrak{R}$ and $\alpha \in Aut(\mathfrak{R})$ is called inner generalized skew-derivation. In particular, if $a = -b$, then $\mathfrak{H}$ is called inner skew-derivation. In [4], Carini et al. studied that "when $F(u)G(u) = 0$ for all $u \in f(R)$, where F and G are generalized skew-derivations of $\mathfrak{R}$ associated to the same automorphism and then describe all possible forms of F and G". In the same year, Carini et al. [5] investigated the situation that "when generalized skew-derivations F and G of $\mathfrak{R}$ are co-commuting on $f(R)$, that is, $F(u)u - uG(u) = 0$ for all $u \in f(R)$ and then obtain all possible forms of the maps F and G".

In 2021, Sandhu et al. [8] has studied several algebraic identities on the presence of multiplicative generalized (α,α)-derivation of semiprime rings. They have proved "Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero ideal of $\mathfrak{R}$ and α be an automorphism of $\mathfrak{R}$. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized (α,α)-derivation of $\mathfrak{R}$ associated with nonzero (α,α)-derivation d and g respectively. If $\mathfrak{G}(xy) \pm \mathfrak{F}(x)\mathfrak{F}(y) = 0 \; \forall \; x, y \in \mathfrak{I}$, then $\mathfrak{R}$ contains a nonzero central ideal and d and g maps $\mathfrak{R}$ into $\mathfrak{Z}(\mathfrak{R})$". In the present paper, we study the concept of multiplicative generalized-skew derivation. A mapping $\mathfrak{F} : \mathfrak{R} \to \mathfrak{R}$ (not necessarily additive) is called a multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with skew-derivation d from $\mathfrak{R}$ to $\mathfrak{R}$, if $\mathfrak{F}(xy) = \mathfrak{F}(x)\alpha(y) + xd(y) = \mathfrak{F}(x)y + \alpha(x)d(y) \; \forall \; x, y \in \mathfrak{R}$, where α is an automorphism of $\mathfrak{R}$. Motivated by the result of Sandhu et al., we studied those several algebraic identities with multiplicative generalized skew-derivation on semiprime rings. Precisely, we have studied the following identities: (i) $\mathfrak{G}(x_1x_2) \pm \mathfrak{F}(x_1)\mathfrak{F}(x_2) = 0$, (ii) $\mathfrak{G}(x_1x_2) \pm \mathfrak{F}(x_1)\mathfrak{F}(x_2) \pm x_1x_2 = 0$, (iii) $\mathfrak{F}(x_1x_2) \pm \mathfrak{G}(x_1)\mathfrak{G}(x_2) \pm x_1x_2 = 0$ for all $x, y \in \mathfrak{R}$ and many more.

2 PRELIMINARIES

Lemma 2.1. [1, Lemma 2.1] *If $\mathfrak{R}$ is a semiprime ring and $\mathfrak{I}$ is an ideal of $\mathfrak{R}$, then $\mathfrak{I}$ is a semiprime ring.*

Lemma 2.2. [2, Theorem 3] *Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{U}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a derivation $\mathfrak{D}$ which is nonzero on $\mathfrak{U}$ and centralizing on $\mathfrak{U}$, then $\mathfrak{R}$ contains a nonzero central ideal*

Lemma 2.3. [7, Lemma 1.1.5] *Let $\mathfrak{R}$ be a semiprime ring and ρ be a right ideal of $\mathfrak{R}$. Then $\mathfrak{Z}(\rho) \subset \mathfrak{Z}(\mathfrak{R})$.*

Lemma 2.4. [3, Lemma 1] *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero ideal of $\mathfrak{R}$ and $a \in \mathfrak{I}$ and $b \in \mathfrak{R}$. If $a\mathfrak{I}b = (0)$, then $ab = ba = 0$.*

Lemma 2.5. *Let $\mathfrak{I}$ be an ideal of a semiprime ring $\mathfrak{R}$ and d be a skew-derivation of $\mathfrak{R}$ such that $d(\mathfrak{I}) \subset \mathfrak{Z}(\mathfrak{R})$, then $d(\mathfrak{R}) \subset \mathfrak{Z}(\mathfrak{R})$.*

Proof. From the hypothesis, we get

$$[r_1, d(x_1)] = 0 \; \forall \; r_1 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.1}$$

Substituting x_1 by $x_1r_2 \; \forall \; r_2 \in \mathfrak{R}$ in (2.1), we obtain

$$[r_1, d(x_1)\alpha(r_2) + x_1d(r_2)] = 0 \; \forall \; r_1, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.2}$$

On simplifying the above relation and using (2.1) in it, then we have

$$d(x_1)[r_1, \alpha(r_2)] + x_1[r_1, d(r_2)] + [r_1, x_1]d(r_2) = 0 \tag{2.3}$$

for all $r_1, r_2 \in \mathfrak{R}$ and $x_1 \in \mathfrak{I}$. In (2.3), putting $r_1 = \alpha(r_2)$ then above relation yields that

$$x_1[\alpha(r_2), d(r_2)] + [\alpha(r_2), x_1]d(r_2) = 0 \; \forall \; r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.4}$$

Replacing x_1 by $d(r_2)x_1$ in (2.4) and using it, we find that

$$[\alpha(r_2), d(r_2)]x_1d(r_2) = 0 \; \forall \; r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.5}$$

Substituting x_1 by $x_1\alpha(r_2)$ in (2.5), we have

$$[\alpha(r_2), d(r_2)]x_1\alpha(r_2)d(r_2) = 0 \; \forall \; r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.6}$$

Multiplying (2.5) by $\alpha(r_2)$ from right, it yields that

$$[\alpha(r_2), d(r_2)]x_1d(r_2)\alpha(r_2) = 0 \; \forall \; r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$[\alpha(r_2), d(r_2)]x_1[\alpha(r_2), d(r_2)] = 0 \,\forall\, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.8}$$

Above relation implies that $[\alpha(r_2), d(r_2)]\mathfrak{I}[\alpha(r_2), d(r_2)] = (0) \,\forall\, r_2 \in \mathfrak{R}$. Using Lemma 2.1, we find that

$$[\alpha(r_2), d(r_2)] = 0 \,\forall\, r_2 \in \mathfrak{R}. \tag{2.9}$$

Using (2.9) in (2.4), we obtain

$$[\alpha(r_2), x_1]d(r_2) = 0 \,\forall\, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.10}$$

Taking $d(r_1)x_1$ in place of x_1 in (2.10) and using it, we arrive at

$$[\alpha(r_2), d(r_1)]x_1 d(r_2) = 0 \,\forall\, r_1, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.11}$$

Linearizing (2.11), taking $r_2 + r_1$ for r_2, then we have

$$[\alpha(r_2), d(r_1)]x_1 d(r_1) + [\alpha(r_1), d(r_1)]x_1 d(r_2) = 0 \tag{2.12}$$

for all $r_1, r_2 \in \mathfrak{R}$ and $x_1 \in \mathfrak{I}$. Applying (2.9) in last relation, it yields that

$$[\alpha(r_2), d(r_1)]x_1 d(r_1) = 0 \,\forall\, r_1, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.13}$$

Replacing x_1 by $x_1\alpha(r_2)$ in (2.13), we see that

$$[\alpha(r_2), d(r_1)]x_1\alpha(r_2)d(r_1) = 0 \,\forall\, r_1, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.14}$$

Multiplying (2.13) from right by $\alpha(r_2)$, we get

$$[\alpha(r_2), d(r_1)]x_1 d(r_1)\alpha(r_2) = 0 \,\forall\, r_1, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.15}$$

On combining (2.14) and (2.15), we find that

$$[\alpha(r_2), d(r_1)]x_1[\alpha(r_2), d(r_1)] = 0 \,\forall\, r_1, r_2 \in \mathfrak{R} \text{ and } x_1 \in \mathfrak{I}. \tag{2.16}$$

Using Lemma 2.1, this implies that $[\alpha(r_2), d(r_1)] = 0 \,\forall\, r_1, r_2 \in \mathfrak{R}$. Since, α is an automorphism that means $d(\mathfrak{R}) \subset \mathfrak{Z}(\mathfrak{R})$, we get the result. □

3 MAIN RESULTS

Theorem 3.1. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\mathfrak{G}(x_1x_2) \pm \mathfrak{F}(x_1)\mathfrak{F}(x_2) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{R}$ contains a nonzero central ideal and f and g maps $\mathfrak{R}$ into $\mathfrak{Z}(\mathfrak{R})$.*

Proof. From the hypothesis, first we consider that

$$\mathfrak{G}(x_1x_2) + \mathfrak{F}(x_1)\mathfrak{F}(x_2) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}. \tag{3.1}$$

Replacing x_2 by x_2r_1 in (3.1) for all $r_1 \in \mathfrak{R}$, we get

$$\mathfrak{G}(x_1x_2)\alpha(r_1) + x_1x_2g(r_1) + \mathfrak{F}(x_1)\mathfrak{F}(x_2)\alpha(r_1) + \mathfrak{F}(x_1)x_2f(r_1) = 0 \tag{3.2}$$

for all $x_1, x_2 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Using the hypothesis in the last relation, we obtain

$$x_1x_2g(r_1) + \mathfrak{F}(x_1)x_2f(r_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.3}$$

Substituting x_1 by x_1r_1 in (3.3), we have

$$x_1r_1x_2g(r_1) + \mathfrak{F}(x_1)r_1x_2f(r_1) + \alpha(x_1)f(r_1)x_2f(r_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.4}$$

Again, substituting x_2 by r_1x_2 in (3.3), we get

$$x_1r_1x_2g(r_1) + \mathfrak{F}(x_1)r_1x_2f(r_1) = 0\ \forall\ x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.5}$$

On combining (3.4) and (3.5), we find that

$$\alpha(x_1)f(r_1)x_2f(r_1) = 0\ \forall\ x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.6}$$

Replacing x_2 by $x_2\alpha(x_1)$ in (3.6), we see that

$$\alpha(x_1)f(r_1)x_2\alpha(x_1)f(r_1) = 0\ \forall\ x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.7}$$

Since, $\mathfrak{I}$ is semiprime ring due to Lemma 2.1, we have

$$\alpha(x_1)f(r_1) = 0\ \forall\ x_1 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.8}$$

Putting x_1x_2 for x_1 in (3.8), we obtain

$$\alpha(x_1)\alpha(x_2)f(r_1) = 0\ \forall\ x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.9}$$

Since, $\mathfrak{I}$ is an ideal of $\mathfrak{R}$. So is $\alpha(\mathfrak{I})$. In view of Lemma 2.4, we get $\alpha(x_1)f(r_1) = f(r_1)\alpha(x_1) = 0\ \forall\ x_1 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. This implies that

$$[\alpha(x_1), f(r_1)] = 0\ \forall\ x_1 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.10}$$

Particularly, (3.10) implies that $[x_1, \phi(x_1)] = 0\ \forall\ x_1 \in \mathfrak{I}$, where $\phi = \alpha^{-1}f$ is an ordinary derivation of $\mathfrak{R}$. Due to Lemma 2.2, $\mathfrak{R}$ have a nonzero central ideal of $\mathfrak{R}$. Moreover, from (3.10), we get $f(r_1) \in \mathfrak{Z}(\mathfrak{I})\ \forall\ r_1 \in \mathfrak{R}$. Using Lemma 2.3, f maps $\mathfrak{R}$ into $\mathfrak{Z}(\mathfrak{R})$.

Now, replacing x_2 by $x_2\alpha(x_3)\ \forall\ x_3 \in \mathfrak{I}$ in (3.3), we have

$$x_1x_2\alpha(x_3)g(r_1) + \mathfrak{F}(x_1)x_2\alpha(x_3)f(r_1) = 0 \tag{3.11}$$

for all $x_1, x_2, x_3 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Using $\alpha(x_1)f(r_1) = 0$ in (3.11), we obtain

$$x_1x_2\alpha(x_3)g(r_1) = 0\ \forall\ x_1, x_2, x_3 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.12}$$

Substituting x_1 by $x_2\alpha(x_3)g(r_1)x_1$ in (3.12), it yields that

$$x_2\alpha(x_3)g(r_1)x_1x_2\alpha(x_3)g(r_1) = 0\ \forall\ x_1, x_2, x_3 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.13}$$

Since, for $x_1 \in \mathfrak{I}$ and by Lemma 2.1, $\mathfrak{I}$ is semiprime ring. From the last relation we have $x_2\alpha(x_3)g(r_1) = 0\ \forall\ x_2, x_3 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Replacing x_2 by $\alpha(x_3)g(r_1)x_2$ and by using the previous argument we get $\alpha(x_3)g(r_1) = 0\ \forall\ x_3 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. This relation is same as (3.9) but in place of f there is g. So, using similar argument we conclude the result.

Similarly, we will prove the case $\mathfrak{G}(x_1x_2) - \mathfrak{F}(x_1)\mathfrak{F}(x_2) = 0\ \forall\ x_1, x_2 \in \mathfrak{I}$. □

Theorem 3.2. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\mathfrak{G}(x_1x_2) \pm \mathfrak{F}(x_1)\mathfrak{F}(x_2) \pm x_1x_2 = 0\ \forall\ x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{R}$ contains a nonzero central ideal and f maps $\mathfrak{R}$ into $\mathfrak{Z}(\mathfrak{R})$.*

Proof. We consider that

$$\mathfrak{G}(x_1x_2) + \mathfrak{F}(x_1)\mathfrak{F}(x_2) + x_1x_2 = 0\ \forall\ x_1, x_2 \in \mathfrak{I}. \tag{3.14}$$

Replacing x_2 by x_2r_1 in (3.14), we obtain

$$\begin{aligned}\mathfrak{G}(x_1x_2)\alpha(r_1) + x_1x_2g(r_1) + \mathfrak{F}(x_1)\mathfrak{F}(x_2)\alpha(r_1)\\ + \mathfrak{F}(x_1)x_2f(r_1) + x_1x_2r_1 = 0\end{aligned} \tag{3.15}$$

for all $x_1, x_2 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Using (3.14) in (3.15), we have

$$x_1x_2g(r_1) + \mathfrak{F}(x_1)x_2f(r_1) + x_1x_2r_1 - x_1x_2\alpha(r_1) = 0 \tag{3.16}$$

for all $x_1, x_2 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Substituting x_1 by x_1r_1 in (3.16), we get

$$\begin{aligned} x_1r_1x_2g(r_1) + \mathfrak{F}(x_1)r_1x_2f(r_1) + \alpha(x_1)f(r_1)x_2f(r_1) \\ + x_1r_1x_2r_1 - x_1r_1x_2\alpha(r_1) = 0 \end{aligned} \tag{3.17}$$

for all $x_1, x_2 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Putting r_1x_2 for x_2 in (3.16), we find that

$$x_1r_1x_2g(r_1) + \mathfrak{F}(x_1)r_1x_2f(r_1) + x_1r_1x_2r_1 - x_1r_1x_2\alpha(r_1) = 0 \tag{3.18}$$

for all $x_1, x_2 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Combining (3.17) and (3.18), its yields that

$$\alpha(x_1)f(r_1)x_2f(r_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.19}$$

Replacing x_2 by $x_2\alpha(x_1)$ in (3.19), we have $\alpha(x_1)f(r_1)x_2\alpha(x_1)f(r_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. By Lemma 2.1, $\mathfrak{I}$ is semiprime ring, so from previous equation we get $\alpha(x_1)f(r_1) = 0 \,\forall\, x_1 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Previous equation is same as (3.8), by using same technique we get the result.

Using similar argument, we arrives at $\mathfrak{G}(x_1x_2) - \mathfrak{F}(x_1)\mathfrak{F}(x_2) - x_1x_2 = 0$, $\mathfrak{G}(x_1x_2) + \mathfrak{F}(x_1)\mathfrak{F}(x_2) - x_1x_2 = 0$ and $\mathfrak{G}(x_1x_2) - \mathfrak{F}(x_1)\mathfrak{F}(x_2) + x_1x_2 = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}$. □

Theorem 3.3. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\mathfrak{F}(x_1x_2) \pm \mathfrak{G}(x_1)\mathfrak{G}(x_2) \pm x_1x_2 = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{R}$ contains a nonzero central ideal and g maps $\mathfrak{R}$ into $\mathfrak{Z}(\mathfrak{R})$.*

Proof. By using similar argument as we have done in previous theorem, we get our conclusion. □

Theorem 3.4. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\mathfrak{F}(x_1)x_2 \pm x_2\mathfrak{G}(x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{F}$ and $\mathfrak{G}$ maps $\mathfrak{I}$ into $\mathfrak{Z}(\mathfrak{R})$. Moreover, $f = -g$.*

Proof. From the hypothesis, we have

$$\mathfrak{F}(x_1)x_2 + x_2\mathfrak{G}(x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}. \tag{3.20}$$

Replacing x_1 by x_1x_3 in (3.20), we obtain

$$\begin{aligned} \mathfrak{F}(x_1)\alpha(x_3)x_2 + x_1f(x_3)x_2 + x_2\mathfrak{G}(x_1)\alpha(x_3) \\ + x_2x_1g(x_3) = 0 \,\forall\, x_1, x_2, x_3 \in \mathfrak{I}. \end{aligned} \tag{3.21}$$

Substituting x_2 by $\alpha(x_3)x_2$ in (3.20), we have

$$\mathfrak{F}(x_1)\alpha(x_3)x_2 + \alpha(x_3)x_2\mathfrak{G}(x_1) = 0 \,\forall\, x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.22}$$

On combining (3.21) and (3.22), we get

$$x_1f(x_3)x_2 + x_2\mathfrak{G}(x_1)\alpha(x_3) + x_2x_1g(x_3) - \alpha(x_3)x_2\mathfrak{G}(x_1) = 0 \tag{3.23}$$

for all $x_1, x_2, x_3 \in \mathfrak{I}$. Using (3.20) in (3.23), we find that

$$x_1f(x_3)x_2 - \mathfrak{F}(x_1)x_2\alpha(x_3) + x_2x_1g(x_3) - \alpha(x_3)x_2\mathfrak{G}(x_1) = 0 \tag{3.24}$$

for all $x_1, x_2, x_3 \in \mathfrak{I}$. Subtracting (3.24) from (3.21), yields that

$$\mathfrak{F}(x_1)\alpha(x_3)x_2 + (\mathfrak{F}(x_1)x_2 + x_2\mathfrak{G}(x_1))\alpha(x_3) + \alpha(x_3)x_2\mathfrak{G}(x_1) \tag{3.25}$$

for all $x_1, x_2, x_3 \in \mathfrak{I}$. Due to hypothesis, last relation yields that

$$\mathfrak{F}(x_1)\alpha(x_3)x_2 - \alpha(x_3)\mathfrak{F}(x_1)x_2 = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.26}$$

This implies that

$$[\mathfrak{F}(x_1), \alpha(x_3)]x_2 = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.27}$$

Since, $\mathfrak{I}$ is semiprime ring by Lemma 2.1. So, from 3.27, we have $[\mathfrak{F}(x_1), \alpha(x_3)] = 0$ for all $x_1, x_3 \in \mathfrak{I}$. That is $\mathfrak{F}(\mathfrak{I}) \subset \mathfrak{Z}(\mathfrak{I})$, from Lemma 2.3 we have $\mathfrak{F}(\mathfrak{I}) \subset \mathfrak{Z}(\mathfrak{R})$.

Now, multiplying (3.20) by $\alpha(x_3)$ from right, we have

$$\mathfrak{F}(x_1)x_2\alpha(x_3) + x_2\mathfrak{G}(x_1)\alpha(x_3) = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.28}$$

Substituting x_2 by $x_2\alpha(x_3)$ in (3.20), we obtain

$$\mathfrak{F}(x_1)x_2\alpha(x_3) + x_2\alpha(x_3)\mathfrak{G}(x_1) = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.29}$$

Subtracting (3.29) from (3.28), we get

$$x_2(\mathfrak{G}(x_1)\alpha(x_3) - \alpha(x_3)\mathfrak{G}(x_1)) = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.30}$$

That is

$$x_2[\mathfrak{G}(x_1), \alpha(x_3)] = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.31}$$

Using the similar argument after (3.27), we get $\mathfrak{G}(\mathfrak{I}) \subset \mathfrak{Z}(\mathfrak{R})$.

Since, $\mathfrak{F}(\mathfrak{I}) \subset \mathfrak{Z}(\mathfrak{R})$, our hypothesis becomes

$$x_2\mathfrak{F}(x_1) + x_2\mathfrak{G}(x_1) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}. \tag{3.32}$$

This implies that

$$x_2(\mathfrak{F}(x_1) + \mathfrak{G}(x_1)) = 0 \tag{3.33}$$

$$or\; x_2((\mathfrak{F} + \mathfrak{G})(x_1)) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}.$$

Replacing x_1 by $x_1r_1 \;\forall\; r_1 \in \mathfrak{R}$ in (3.33) and using the hypothesis, we have

$$x_2x_1((f+g)(r_1)) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.34}$$

Substituting x_2 by $x_1((f+g)(r_1))x_2$ in (3.34), we see that

$$x_1((f+g)(r_1))x_2x_1((f+g)(r_1)) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.35}$$

Since, $\mathfrak{I}$ is semiprime ring by Lemma 2.1, from last relation we obtain $x_1((f+g)(r_1)) = 0$. Again, replacing x_1 by $((f+g)(r_1))x_1$ in previous equation and using similar argument we obtain $(f+g)(r_1) = 0$. That is, for all $r_1 \in \mathfrak{R}$ we get $f = -g$ on $\mathfrak{R}$.

In a similar way we can prove the case $\mathfrak{F}(x_1)x_2 - x_2\mathfrak{G}(x_1) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}$. □

Theorem 3.5. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\mathfrak{F}(x_1x_2) \pm \mathfrak{G}(x_1x_2) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}$, then $f \pm g$ maps $\mathfrak{R}$ into $\mathfrak{Z}(\mathfrak{R})$ and $\mathfrak{R}$ contains a nonzero central ideal.*

Proof. From the hypothesis $\mathfrak{F}(x_1x_2) \pm \mathfrak{G}(x_1x_2) = 0$ can be written as $(\mathfrak{F} \pm \mathfrak{G})(x_1x_2) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}$. Since sum(difference) of two multiplicative generalized skew-derivation of $\mathfrak{R}$ is again a multiplicative generalized skew-derivation of $\mathfrak{R}$. We assume $\mathfrak{F} \pm \mathfrak{G} = \mathfrak{H}$, then our hypothesis becomes $\mathfrak{H}(x_1x_2) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}$ which is a special case of Theorem 3.1 where we consider $\mathfrak{F} = 0$. Hence, we arrives at the result. □

Theorem 3.6. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\mathfrak{F}(x_1x_2) \pm \mathfrak{G}(x_2x_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I}$, then f and g is commuting on $\mathfrak{I}$.*

Proof. Assuming that

$$\mathfrak{F}(x_1x_2) + \mathfrak{G}(x_2x_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I}. \tag{3.36}$$

Replacing x_1 by x_1r_1 in (3.36) $\forall \; r_1 \in \mathfrak{R}$, we get

$$\mathfrak{F}(x_1r_1x_2) + \mathfrak{G}(x_2x_1r_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.37}$$

Substituting x_2 by r_1x_2 in (3.36), we have

$$\mathfrak{F}(x_1r_1x_2) + \mathfrak{G}(r_1x_2x_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.38}$$

On comparing (3.37) and (3.38), we find that

$$\mathfrak{G}(x_2x_1r_1) = \mathfrak{G}(r_1x_2x_1) \; \forall \; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.39}$$

Putting x_2x_3 for x_2 in (3.36) and on simplifying, we see that

$$\begin{aligned} &\mathfrak{F}(x_1x_2)\alpha(x_3) + x_1x_2f(x_3) + \mathfrak{G}(x_2x_3)\alpha(x_1) \\ &\quad + x_2x_3g(x_1) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I}. \end{aligned} \tag{3.40}$$

Using the hypothesis in (3.40), yields that

$$\begin{aligned} &-\mathfrak{G}(x_2x_1)\alpha(x_3) + x_1x_2f(x_3) + \mathfrak{G}(x_2x_3)\alpha(x_1) \\ &\quad + x_2x_3g(x_1) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I}. \end{aligned} \tag{3.41}$$

Replacing x_1 by x_1r_2 for all $r_2 \in \mathfrak{R}$ in (3.41) and on solving, we arrives at

$$\begin{aligned} &-\mathfrak{G}(x_2x_1r_2)\alpha(x_3) + x_1r_2x_2f(x_3) + \mathfrak{G}(x_2x_3)\alpha(x_1r_2) \\ &\quad + x_2x_3g(x_1r_2) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I} \text{ and } r_2 \in \mathfrak{R}. \end{aligned} \tag{3.42}$$

Substituting x_2 by $r_2x_2 \; \forall \; r_2 \in \mathfrak{R}$ in (3.41) and on solving, we get

$$\begin{aligned} &-\mathfrak{G}(r_2x_2x_1)\alpha(x_3) + x_1r_2x_2f(x_3) + \mathfrak{G}(r_2x_2x_3)\alpha(x_1) \\ &\quad + r_2x_2x_3g(x_1) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I} \text{ and } r_2 \in \mathfrak{R}. \end{aligned} \tag{3.43}$$

On comparing (3.42) and (3.43) and by using (3.39), we obtain

$$\begin{aligned} &\mathfrak{G}(x_2x_3)\alpha(x_1r_2) + x_2x_3g(x_1r_2) - \mathfrak{G}(x_2x_3r_2)\alpha(x_1) \\ &\quad - r_2x_2x_3g(x_1) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I} \text{ and } r_2 \in \mathfrak{R}. \end{aligned} \tag{3.44}$$

This implies that

$$\begin{aligned} &\mathfrak{G}(x_2x_3)\alpha(x_1)\alpha(r_2) + x_2x_3g(x_1)\alpha(r_2) + x_2x_3x_1g(r_2) \\ &-\mathfrak{G}(x_2x_3)\alpha(r_2)\alpha(x_1) - x_2x_3g(r_2)\alpha(x_1) - r_2x_2x_3g(x_1) = 0 \end{aligned} \tag{3.45}$$

for all $x_1, x_2, x_3 \in \mathfrak{I}$ and $r_2 \in \mathfrak{R}$. In particular, for $r_2 = x_1$ in (3.45), we get

$$x_2x_3x_1g(x_1) - x_1x_2x_3g(x_1) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.46}$$

That is

$$[x_2x_3, x_1]g(x_1) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.47}$$

Substituting x_2 by $r_1x_2 \; \forall \; r_1 \in \mathfrak{R}$ in (3.47) and using it, we have

$$[r_1, x_1]x_2x_3g(x_1) = 0 \; \forall \; x_1, x_2, x_3 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R} \tag{3.48}$$

for all $x_3 \in \mathfrak{I}$ and using Lemma 2.4, we obtain

$$[r_1, x_1]x_2 g(x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.49}$$

Taking $r_1 = g(x_1)$ and $x_2 = x_2x_1$ in (3.49), we find that

$$[g(x_1), x_1]x_2x_1g(x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}. \tag{3.50}$$

In (3.49), we replace r_1 by $g(x_1)$ and post multiply by x_1, we see that

$$[g(x_1), x_1]x_2g(x_1)x_1 = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}. \tag{3.51}$$

From (3.50) and (3.51), we arrives at $[g(x_1), x_1]x_2[g(x_1), x_1] = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}$. By Lemma 2.1, we conclude that $[g(x_1), x_1] = 0 \,\forall\, x_1 \in \mathfrak{I}$. This implies, g is commuting on $\mathfrak{I}$.

Replacing x_2 by x_2r_1 for all $r_1 \in \mathfrak{R}$ in (3.36), we get

$$\mathfrak{F}(x_1x_2r_1) + \mathfrak{G}(x_2r_1x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.52}$$

Substituting x_1 by r_1x_1 in (3.36), we obtain

$$\mathfrak{F}(r_1x_1x_2) + \mathfrak{G}(x_2r_1x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.53}$$

On combining (3.52) and (3.53), we have

$$\mathfrak{F}(r_1x_1x_2) = \mathfrak{F}(x_1x_2r_1) \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.54}$$

This relation has already existed above in this prove for $\mathfrak{G}$ after interchanging the role of x_1 and x_2 in (3.39). So, by following same step we arrives at conclusion. That is, we get f is commuting on $\mathfrak{I}$.

In the similar way we can prove the case $\mathfrak{F}(x_1x_2) + \mathfrak{G}(x_2x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}$. □

Theorem 3.7. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\alpha(x_1) \circ \mathfrak{F}(x_2) \pm \mathfrak{G}(x_2x_1) = 0$, then f maps $\mathfrak{I}$ into $\mathfrak{Z}(\mathfrak{R})$ and $\mathfrak{R}$ contains a nonzero central ideal.*

Proof. Let us assume that

$$\alpha(x_1)\mathfrak{F}(x_2) + \mathfrak{F}(x_2)\alpha(x_1) + \mathfrak{G}(x_2x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}. \tag{3.55}$$

Replacing x_2 by x_2x_1 in (3.55), we get

$$\begin{aligned} &\alpha(x_1)\mathfrak{F}(x_2)\alpha(x_1) + \alpha(x_1)x_2f(x_1) + \mathfrak{F}(x_2)\alpha(x_1)\alpha(x_1) \\ &+ x_2f(x_1)\alpha(x_1) + \mathfrak{G}(x_2x_1)\alpha(x_1) + x_2x_1g(x_1) = 0 \end{aligned} \tag{3.56}$$

for all $x_1, x_2 \in \mathfrak{I}$. Using (3.55) in (3.56), we have

$$\alpha(x_1)x_2f(x_1) + x_2f(x_1)\alpha(x_1) + x_2x_1g(x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}. \tag{3.57}$$

Substituting x_2 by $r_1x_2 \,\forall\, r_1 \in \mathfrak{R}$ in (3.57), we obtain

$$\alpha(x_1)r_1x_2f(x_1) + r_1x_2f(x_1)\alpha(x_1) + r_1x_2x_1g(x_1) = 0 \tag{3.58}$$

for all $x_1, x_2 \in \mathfrak{I}$ and $r_1 \in \mathfrak{R}$. Pre-multiplying (3.57) by r_1, we find that

$$r_1\alpha(x_1)x_2f(x_1) + r_1x_2f(x_1)\alpha(x_1) + r_1x_2x_1g(x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I}. \tag{3.59}$$

Subtracting (3.59) from (3.58), we see that

$$[\alpha(x_1), r_1]x_2f(x_1) = 0 \,\forall\, x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.60}$$

Replacing x_2 by $r_2x_2 \;\forall\; r_2 \in \mathfrak{R}$ in (3.60), we find that

$$[\alpha(x_1), r_1]r_2x_2f(x_1) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I} \text{ and } r_1, r_2 \in \mathfrak{R}. \tag{3.61}$$

This implies that

$$[\alpha(x_1), r_1]\mathfrak{R}\mathfrak{I}f(x_1) = (0) \;\forall\; x_1 \in \mathfrak{I}. \tag{3.62}$$

Since $\mathfrak{R}$ contains a family of prime ideals say $\mathfrak{S}$ such that $\cap\mathfrak{P}_\lambda = (0)$. Let $\mathfrak{P}$ be a member of this family and for all $x_1 \in \mathfrak{I}$, from (3.62), we have

$$[\alpha(x_1), r_1] \subset \mathfrak{P} \text{ or } \mathfrak{I}f(x_1) \subset \mathfrak{P}. \tag{3.63}$$

Let $\mathfrak{A} = \{x_1 \in \mathfrak{I} : [\alpha(x_1), r_1] \subset \mathfrak{P}\}$ and $\mathfrak{B} = \{x_1 \in \mathfrak{I} : \mathfrak{I}f(x_1) \subset \mathfrak{P}\}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are additive subgroups of $\mathfrak{R}$ and also $\mathfrak{A} \cup \mathfrak{B} = \mathfrak{I}$. Due to Brauer's trick we arrives at

$$[\alpha(\mathfrak{I}), \mathfrak{R}] \subset \mathfrak{P} \text{ or } \mathfrak{I}f(\mathfrak{I}) \subset \mathfrak{P}. \tag{3.64}$$

Considering these cases together, we get $[\alpha(\mathfrak{I}), \mathfrak{R}]\mathfrak{I}f(\mathfrak{I}) \subset \cap\mathfrak{P}_\lambda$. That is

$$[\alpha(x_1), r_1]x_2f(x_3) = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.65}$$

Taking $r_1 = f(x_3)$ and $x_2 = x_2\alpha(x_1)$ in (3.65), we find that

$$[\alpha(x_1), f(x_3)]x_2\alpha(x_1)f(x_3) = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.66}$$

In (3.65), we replace r_1 by $f(x_3)$ and post-multiply by $\alpha(x_1)$, we see that

$$[\alpha(x_1), f(x_3)]x_2f(x_3)\alpha(x_1) = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}. \tag{3.67}$$

Subtracting (3.67) and (3.66), we arrives at $[\alpha(x_1), f(x_3)]x_2[\alpha(x_1), f(x_3)] = 0 \;\forall\; x_1, x_2, x_3 \in \mathfrak{I}$. By Lemma 2.1, we conclude that $[\alpha(x_1), f(x_3)] = 0 \;\forall\; x_1, x_3 \in \mathfrak{I}$. That is $f(\mathfrak{I}) \subset \mathfrak{Z}(\mathfrak{I})$, by Lemma 2.3 we conclude that $f(\mathfrak{I}) \subset \mathfrak{Z}(\mathfrak{R})$ i.e., f map $\mathfrak{I}$ into $\mathfrak{Z}(\mathfrak{R})$.

In particular, $[\alpha(x_1), f(x_3)] = 0 \;\forall\; x_1, x_3 \in \mathfrak{I}$ implies that $[x_1, \phi(x_1)] = 0 \;\forall\; x_1 \in \mathfrak{I}$, where $\phi = \alpha^{-1}f$ is an ordinary derivation of $\mathfrak{R}$. Due to Lemma 2.2, $\mathfrak{R}$ have a nonzero central ideal of $\mathfrak{R}$.

In the similar way we can prove the case $\alpha(x_1)\mathfrak{F}(x_2) + \mathfrak{F}(x_2)\alpha(x_1) - \mathfrak{G}(x_2x_1) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}$.

□

Theorem 3.8. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $[\alpha(x_1), \mathfrak{F}(x_2)] \pm \mathfrak{G}(x_2x_1) = 0$, then f maps $\mathfrak{I}$ into $\mathfrak{Z}(\mathfrak{R})$ and $\mathfrak{R}$ contains a nonzero central ideal.*

Proof. Implications of similar steps as in above theorem with necessary changes, we get the result. □

Theorem 3.9. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $\alpha(x_1) \circ \mathfrak{F}(x_2) \pm \alpha([x_1, x_2]) = 0$, then f maps $\mathfrak{I}$ into $\mathfrak{Z}(\mathfrak{R})$ and $\mathfrak{R}$ contains a nonzero central ideal.*

Proof. First we assume that

$$\alpha(x_1)\mathfrak{F}(x_2) + \mathfrak{F}(x_2)\alpha(x_1) + \alpha([x_1, x_2]) = 0 \;\forall\; x_1, x_2 \in \mathfrak{I}. \tag{3.68}$$

Replacing x_2 by x_2x_1 in (3.68), we get

$$\begin{aligned} &\alpha(x_1)\mathfrak{F}(x_2)\alpha(x_1) + \alpha(x_1)x_2f(x_1) + \mathfrak{F}(x_2)\alpha(x_1)\alpha(x_1) \\ &\quad + x_2f(x_1)\alpha(x_1) + \alpha([x_1, x_2])\alpha(x_1) = 0 \end{aligned} \tag{3.69}$$

for all $x_1, x_2 \in \mathfrak{I}$. Using (3.68) in (3.69), we have

$$\alpha(x_1)x_2 f(x_1) + x_2 f(x_1)\alpha(x_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I}. \tag{3.70}$$

Replacing x_2 by $r_1 x_2 \; \forall \; r_1 \in \mathfrak{R}$ in (3.70), we find that

$$\alpha(x_1)r_1 x_2 f(x_1) + r_1 x_2 f(x_1)\alpha(x_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.71}$$

Pre-multiply (3.70) by r_1, we see that

$$r_1\alpha(x_1)x_2 f(x_1) + r_1 x_2 f(x_1)\alpha(x_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.72}$$

From (3.71) and (3.72), we obtain

$$[\alpha(x_1), r_1]x_2 f(x_1) = 0 \; \forall \; x_1, x_2 \in \mathfrak{I} \text{ and } r_1 \in \mathfrak{R}. \tag{3.73}$$

Above expression is same as (3.60), we get the conclusion by similar manner. □

Theorem 3.10. *$\mathfrak{I}$ and α be an ideal and automorphism of a semiprime ring $\mathfrak{R}$, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be multiplicative generalized skew-derivation of $\mathfrak{R}$ associated with nonzero skew-derivation f and g satisfying $[\alpha(x_1), \mathfrak{F}(x_2)] \pm \alpha(x_1 \circ x_2) = 0$, then f maps $\mathfrak{I}$ into $\mathfrak{Z}(\mathfrak{R})$ and $\mathfrak{R}$ contains a nonzero central ideal.*

Proof. We get the result by using similar argument as in Theorem 3.9 with necessary changes. □

Conflict of interest

There is no conflict of interest among the authors.

References

[1] S. Ali, B. Dhara, N. A. Dar and A. N. Khan: *On Lie ideals with multiplicative (generalized) derivations in prime and semiprime rings,* Beitr. Algebra. Geom., **56**, 325-337, (2015).

[2] H. E. Bell and W. S. Martindale: *Centralizing mappings of semiprime rings,* Canad. Math. Bull., **30(1)**, 92–101, (1987).

[3] M. Brešar and J. Vukman: *Orthogonal derivations and extension of a theorem of Posner,* Rad. Mat., **5**, 237–246, (1989).

[4] L. Carini, V. De Filippis and G. Scudo: *Identities with product of generalized skew derivations on multilinear polynomials,* Comm. Algebra, **44**, 3122–3138, (2016).

[5] L. Carini, V. De Filippis and F. Wei: *Generalized skew derivations co-centralizing multilinear polynomials,* Mediterr. J. Math., **13**, 2397–2424, (2016).

[6] M. N. Daif: *When is a multiplicative derivation additive,* Internat. J. Math. and Math. Sci., **14(3)**, 615-618, (1991).

[7] I. N. Herstein: *Rings with Involution,* University of Chicago Press, 1976.

[8] G. S. Sandhu, A. Ayran and N. Aydin: *Identities with multiplicative generalized (α, α)-derivation of semiprime rings,* Kragujevac J. Math., **48(3)**, 365-382, (2021).

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