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Analysis of a frictional viscoelastic contact problem with normal compliance and unilateral penetration

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Abstract In this paper, we study a contact problem between a viscoelastic body with long memory and an obstacle. The contact is modelled with a normal compliance condition with unilateral penetration. Also, we assume that the contact is frictional and we model the friction with a time-dependent Coulomb's friction law. We provide a variational formulation to the model and we prove the existence of a unique weak solution. The proof is based on results concerning quasivariational inequalities and fixed point. Finally, we consider a penalization of the variational problem in order to study the dependence of the solution with respect to the data and to prove a convergence result.

1 Introduction

Situations of contact between a deformable body and a rigid or deformable foundation appear in many systems in structure mechanics. General mathematical models in contact mechanics can be found in [1, 2, 6, 7, 8, 9, 10, 11, 15, 17, 18, 19, 21, 23, 24, 25]. In particular, processes of frictional contact are important in many industrial settings and everyday life. For this reason, considerable effort has been made in their modelling and analysis. Frictional contact problems have been considered in [3, 13, 14, 15], and more recently in [4, 19].
This paper aims to study within the variational framework a frictional contact problem for viscoelastic materials with long memory. We model the material's behavior with a constitutive law of the form

$$\boldsymbol{\sigma} = \mathcal{A}\, \boldsymbol{\varepsilon}(\mathbf{u}) + \int_0^t \mathcal{R}(t-s)\, \boldsymbol{\varepsilon}(\mathbf{u}(s))\, ds, \tag{1.1}$$

where $\mathbf{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain. Moreover, $\mathcal{A}$ is the elasticity operator, allowed to be nonlinear and $\mathcal{R}$ represents the relaxation operator, assumed to be linear. For more details on the study of variational inequalities with memory operators, see, e.g. [20, 22, 26].
The current paper has four traits of novelties which make the difference from previous papers dealing with contact processes. First, we describe the material's behavior with a viscoelastic constitutive law with long memory. Second, we model the contact with a normal compliance condition with unilateral penetration. This condition was introduced in [5] to model the contact with an elastic-rigid foundation. It represents a combination of both Signorini's condition introduced in [16] to describe the contact with a perfectly rigid foundation and normal compliance condition introduced in [12] modelling the contact with a deformable obstacle. The third novelty arises in the fact that the friction between contact surfaces is taken into account and is modelled with a time-dependent Coulomb's friction law. The latter was used in [21] modelling the friction in the study of a quasistatic contact problem with normal compliance. Finally, in contrast with a large number of references, the viscoelastic contact problem considered in this paper is formulated on a bounded interval of time $[0, T]$, $T > 0$. This implies the use of the framework

of Banach spaces of continuous functions defined on a bounded interval of time.
The rest of the paper is structured as follows. In section 2 we present the notation as well as some preliminary material. In section 3 we describe the model, list assumptions on the data and derive the variational formulation of the problem. Then, in section 4, we state and prove our main existence and uniqueness result, Theorem 4.1. The proof is based on arguments of quasi-variational inequalities and fixed point. Finally, in section 5, we state and prove a convergence result, Theorem 5.1, on the continuous dependence of the solution with respect to the data.

2 Notations and preliminaries

Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 1,2,3$ with a Lipschitz continuous boundary Γ divided into three disjoint measurable parts Γ_1, Γ_2, Γ_3 such that meas$(\Gamma_1) > 0$. We use the notation $\mathbf{x} = (x_i)$ for a generic point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal vector on Γ. Here and below the indices i, j, k, l run between 1 and d and an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$.
We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by
$\mathbf{u}.\mathbf{v} = u_i v_i \;,\quad \|\mathbf{v}\| = (\mathbf{v},\mathbf{v})^{1/2}, \quad \forall \mathbf{u},\mathbf{v} \in \mathbb{R}^d,$
$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \;,\quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau},\boldsymbol{\tau})^{1/2}, \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d.$
Moreover, for a vector $\mathbf{u}$ in $\mathbb{R}^d$, we denote by $u_{\boldsymbol{\nu}}$ and $\mathbf{u}_\tau$ its normal and tangential components on Γ respectively given by

$$u_\nu = \mathbf{u}.\boldsymbol{\nu} \;,\quad \mathbf{u}_\tau = \mathbf{u} - u_\nu.\boldsymbol{\nu}. \tag{2.1}$$

In addition, we have

$$|u_{1\nu} - u_{2\nu}| \leq \|\mathbf{u}_1 - \mathbf{u}_2\| \;,\quad |\, \|\mathbf{u}_{1\tau}\| - \|\mathbf{u}_{2\tau}\| \,| \leq \|\mathbf{u}_1 - \mathbf{u}_2\|. \tag{2.2}$$

Also, for a regular function $\boldsymbol{\sigma} : \Omega \cup \Gamma \to \mathbb{S}^d$, we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and tangential components of the vector $\boldsymbol{\sigma\nu}$ on Γ, respectively, and we recall that

$$\sigma_\nu = (\boldsymbol{\sigma\nu}).\boldsymbol{\nu} \;,\quad \boldsymbol{\sigma_\tau} = \boldsymbol{\sigma\nu} - \sigma_\nu.\boldsymbol{\nu}. \tag{2.3}$$

We use standard notations for the Lebesgue and Sobolev spaces associated to Ω and Γ and, moreover, we use the spaces

$$\begin{cases} H = \left\{\mathbf{u} = (u_i)/u_i \in L^2(\Omega)\right\}, \\ \mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \\ H_1 = \left\{\mathbf{u} = (u_i)/u_i \in H^1(\Omega)\right\}, \\ \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H}/Div\boldsymbol{\sigma} \in H\}, \\ \mathcal{H}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) : \mathcal{E}_{ijkl} = \mathcal{E}_{jiklj} = \mathcal{E}_{klij} \in L^\infty(\Omega),\ 1 \leq i,j,k,l \leq d\}. \end{cases}$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products

$$\begin{cases} (\mathbf{u},\mathbf{v})_H = \int_\Omega u_i v_i \, d\mathbf{x}, \\ (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, d\mathbf{x}, \\ (\mathbf{u},\mathbf{v})_{H_1} = (\mathbf{u},\mathbf{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (Div\,\boldsymbol{\sigma}, Div\,\boldsymbol{\tau})_H, \end{cases}$$

respectively, where $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathcal{H}$ and $Div : \mathcal{H}_1 \longrightarrow H$ are the deformation and the divergence operators given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})),\;\; \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

We denote by $\|.\|_H, \|.\|_{\mathcal{H}}, \|.\|_{H_1}$ and $\|.\|_{\mathcal{H}_1}$ the associated norms on the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$, respectively. Furthermore, we note that $\mathcal{H}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathcal{H}_\infty} = \max_{1\leq i,j,k,l\leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}. \tag{2.4}$$

In addition, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_{\mathcal{H}} \leq d\, \|\mathcal{E}\|_{\mathcal{H}_\infty} \|\boldsymbol{\tau}\|_{\mathcal{H}}\ \forall \mathcal{E} \in \mathcal{H}_\infty, \boldsymbol{\tau} \in \mathcal{H}. \tag{2.5}$$

For the displacement field, we introduce the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1/\mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}.$$

We note that for an element $\mathbf{v} \in V$ we still write $\mathbf{v}$ for the trace of $\mathbf{v}$ on the boundary Γ. Also, we note that V is a real Hilbert space endowed with the inner product

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

and the associated norm

$$\|\mathbf{v}\|_V = \|\varepsilon(\mathbf{v})\|_{\mathcal{H}}\ \ \forall \mathbf{v} \in V. \tag{2.6}$$

Compliteness of the space $(V, \|.\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality. Moreover, by the Sobolev trace theorem, there exists a positive constant $c_0 > 0$ which depends on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d}\ \leq\ c_0\, \|\mathbf{v}\|_V. \tag{2.7}$$

Also, we note that for a sufficiently regular function $\boldsymbol{\sigma}$ (say C^1), the following Green's formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\, \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma\nu}\ \mathbf{v}\ da\ \ \forall \mathbf{v} \in H_1. \tag{2.8}$$

Now, for a normed space $(X, \|.\|_X)$, we use the notation $C\,(0,T;X)$ for the space of the X-valued continuous functions defined on $[0,T]$ with values in X. Also, for a subset $K \subset X$ we still use the symbol $C\,(0,T;K)$ for the set of continuous functions defined on $[0,T]$ with values in K. The norm on the space $C\,(0,T;X)$ is given by

$$\|\mathfrak{v}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\mathfrak{v}\,(t)\|_X\,.$$

Moreover, we recall that the convergence of a sequence $(v_k)_k$ to an element v in the space $C(0,T;X)$ can be described as follows:

$$\begin{cases} v_k \to v \text{ in } C(0,T;X) \textit{ as } k \to 0 \textit{ if and only if} \\ \max_{t\in[0,T]}\ \|v_k(t) - v(t)\|_X \to 0 \textit{ as } k \to 0. \end{cases} \tag{2.9}$$

We end this section with the following results that we will use in section 4 of this paper.

Theorem 2.1. *Let $(X, \|.\|_X)$ be a Hilbert space and let K be a non empty closed subset of X. Let $\Lambda : C(0,T;K) \longrightarrow C(0,T;K)$ be a nonlinear operator. Assume that there exists $h \in \mathbb{N}$ with the following property: there exists $k \in [0,1)$ and $c \geq 0$ such that*

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^h \leq k\|\eta_1(t) - \eta_2(t)\|_X^h + c\int_0^t \|\eta_1(s) - \eta_2(s)\|_X^h\, ds,$$

$\forall \eta_1, \eta_2 \in C(0,T;K),\ \forall t \in [0,T]$. *Then the operator Λ has a unique fixed point $\eta^* \in C(0,T;K)$.*

Theorem 2.2. *Let X be a Hilbert space. Let K be a subset of X and consider the operator $A : K \to X$, the function $j : K \times K \to \mathbb{R}$ and the element $f \in X$ such that*

$$K \textit{ is a nonempty closed convex subset of } X. \tag{2.10}$$

$$\begin{cases} A \textit{ is strongly monotone with a constant } m_A\ , \\ A \textit{ is Lipschitz continuous with a constant } L_A. \end{cases} \tag{2.11}$$

$$\begin{cases} \textit{(a)}\ \forall\ u \in K,\ j\,(u,.) : K \to \mathbb{R}\ \textit{ is convex and lower semicontinuous.} \\ \textit{(b)}\ \exists\ \alpha > 0 \textit{ such that} \\ j\,(u_1, v_2) - j\,(u_1, v_1) + j\,(u_2, v_1) - j\,(u_2, v_2) \leq \alpha\, \|u_1 - u_2\|_X\, \|v_1 - v_2\|_X \\ \forall u_1, u_2, v_1, v_2 \in K. \end{cases} \tag{2.12}$$

$$M_A > \alpha. \tag{2.13}$$

Then, there exits a unique solution $u \in K$ of the following quasivariational inequality

$$(Au, v-u)_X + j(u,v) - j(u,u) \geq (f, v-u)_X \quad \forall v \in K. \tag{2.14}$$

The proof of Theorem 2.2 can be found in [21].

3 Problem statement and variational formulation

The mathematical formulation of the mechanical model is given in the following problem.
Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\,\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}(s))\,ds \quad \text{in } \Omega \times [0,T], \tag{3.1}$$

$$Div\,\boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = 0 \qquad \text{in } \Omega \times [0,T], \tag{3.2}$$

$$\mathbf{u}(t) = 0 \qquad \text{on } \Gamma_1 \times [0,T], \tag{3.3}$$

$$\boldsymbol{\sigma\nu}(t) = \mathbf{f}_2(t) \qquad \text{on } \Gamma_2 \times [0,T], \tag{3.4}$$

$$\begin{cases} u_\nu(t) \leq g \quad , \sigma_\nu(t) + p_\nu(u_\nu(t)) \leq 0 \\ (u_\nu(t) - g)\,\big(\sigma_\nu(t) + p_\nu(u_\nu(t))\big) = 0 \end{cases} \qquad \text{on } \Gamma_3 \times [0,T], \tag{3.5}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau(t)\| \leq p_\tau(u_\nu(t)) \\ \boldsymbol{\sigma}_\tau(t) = -p_\tau(u_\nu(t))\dfrac{\mathbf{u}_\tau(t)}{\|\mathbf{u}_\tau(t)\|} \quad \text{if } \mathbf{u}_\tau(t) \neq 0 \end{cases} \qquad \text{on } \Gamma_3 \times [\,0,T\,]. \tag{3.6}$$

Let us describe the problem (3.1) – (3.6). First, equation (3.1) represents the viscoelastic constitutive law with long memory. The equation (3.2) is the equilibrium equation which governs the mechanical process. Conditions (3.3) and (3.4) are displacement-traction boundary conditions, while (3.6) constitutes the time dependant Coulomb's friction law.
We now turn to the description of condition (3.5) which represents the contact condition with unilateral constraint. First, we assume that the penetration of the body into the foundation is bounded by $g > 0$ and, therefore, at any time $t \in [0,T]$, the normal displacement satisfies the inequality

$$u_\nu(t) \leq g \qquad \text{on } \Gamma_3 \times [0,T], \tag{3.7}$$

where g is a positive constant that represents the maximum penetration value.
Next, we assume that the normal stress has an additive decomposition of the form

$$\sigma_\nu(t) = \sigma_\nu^D(t) + \sigma_\nu^R(t) \qquad \text{on } \Gamma_3 \times [0,T], \tag{3.8}$$

where the functions $\sigma_\nu^D(t)$ and $\sigma_\nu^R(t)$ describe respectively the deformability and the rigidity of the foundation.
We assume that $\sigma_\nu^D(t)$ satisfies the normal compliance contact condition

$$-\sigma_\nu^D(t) = p_\nu(u_\nu(t)) \quad \text{on } \Gamma_3 \times [0,T], \tag{3.9}$$

where p_ν is a given nonnegative function that vanishes for negative arguments.
The part $\sigma_\nu^R(t)$ of the normal stress satisfies the Signorini condition given as a deflection function; namely,

$$\sigma_\nu^R(t) \leq 0 \quad , \quad \sigma_\nu^R(t)(u_\nu(t) - g) = 0 \qquad \text{on } \Gamma_3 \times [0,T]. \tag{3.10}$$

Details on the Signorini condition and the normal compliance function can be found in [7, 15].
We recall that the normal compliance condition describes the contact with a deformable foundation and that the Signorini condition describes the contact with a perfectly rigid foundation.
We combine (3.8) and (3.9) to see that

$$\sigma_\nu^R(t) = \sigma_\nu(t) + p_\nu(u_\nu(t)) \qquad \text{on } \Gamma_3 \times [0,T]. \tag{3.11}$$

Next, we substitute the equality (3.11) in (3.10) and we use (3.7) to obtain the contact condition (3.5).
Finally, we comment on the contact condition (3.5) which represents originality in this study. Firstly, we recall that (3.5) describes a condition with unilateral stress as the inequality (3.7) is verified at all times $t \in [0, T]$. Secondly, we assume that at some point t, there is a penetration that does not reach the boundary g; i.e.
$0 < u_\nu < g$. Then, (3.5) yields

$$-\sigma_\nu(t) = p_\nu(u_\nu(t)).$$

This equality indicates that at the time t, the foundation's reaction depends on the current penetration value (represented by the term $p_\nu(u_\nu(t))$. In conclusion, the condition (3.5) shows that when there is penetration, the contact follows a normal compliance condition but up to the limit g and then, when that limit is reached, the contact follows a unilateral condition of Signorini type with a gap g.
Moreover, condition (3.5) can be physically interpreted as follows. The foundation is assumed to be constructed of a hard material which is covered by a thin layer composed of a soft material of penetration depth g. This soft material has a rigid-elastic behavior; i.e. it is deformable and allows any penetration. Because hard material is completely rigid, it does not allow for any penetration.
To conclude, the foundation has rigid-elastic behavior; its deformation behavior is due to the layer of soft material, whereas its rigid behavior is due to the hard material.
Now, we are interested in finding a variational formulation for the suggested mathematical model. before that and in order to solve problem P, we need to assume some hypothesis on the data.
First, we assume that the elasticity operator $\mathcal{A}$ and the relaxation tensor $\mathcal{R}$ satisfy

$$\begin{cases} \text{(a)} \ \mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d. \\ \text{(b)} \ \text{There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \qquad \|\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)\| \leq L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\| \\ \ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(c)} \ \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \qquad (\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2 \\ \qquad\quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(d)} \ \text{The mapping } \mathbf{x} \longmapsto \mathcal{A}(\mathbf{x}, \varepsilon) \text{ is measurable on } \Omega, \\ \qquad \text{for all } \ \varepsilon \in \mathbb{S}^d. \\ \text{(e)} \ \text{The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{3.12}$$

$$\mathcal{R} \in C(0, T; \mathcal{H}_\infty). \tag{3.13}$$

The function $p_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}^+$ satisfies

$$\begin{cases} \text{(a)} \ \exists \, L_\nu > 0 \text{ such that} \\ \qquad |p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2)| \leq L_\nu |r_1 - r_2| \ \ \forall r_1, r_2 \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(b)} \ (p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}^d, \text{a.e .} \mathbf{x} \in \Gamma_3. \\ \text{(c)} \ \text{The mapping } \mathbf{x} \longmapsto p_\nu(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\ \ \text{for all } r \in \mathbb{R}. \\ \text{(d)} \ p_\nu(\mathbf{x}, r) = 0 \quad \forall r \leq 0, \text{a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{3.14}$$

The function p_τ satisfies

$$\begin{cases} \text{(a)} \ p_\tau : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}^+. \\ \text{(b)} \ \exists \, L_\tau > 0 \text{ such that} \\ \qquad |p_\tau(\mathbf{x}, r_1) - p_\tau(\mathbf{x}, r_2)| \leq L_\tau |r_1 - r_2| \ \ \forall r_1, r_2 \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c)} \ \text{The mapping } \mathbf{x} \longmapsto p_\tau(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\ \ \text{for all } r \in \mathbb{R}. \\ \text{(d)} \ p_\tau(\mathbf{x}, r) = 0 \quad \forall r \leq 0, \text{a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{3.15}$$

Finally, the densities of volume forces and surface tractions have the regularities

$$\mathbf{f}_0 \in C\,(0,T;H), \quad \mathbf{f}_2 \in C\,(0,T;L^2(\Gamma_2)^d). \tag{3.16}$$

We now turn to the variational formulation of Problem P. To this end, we assume that $(\mathbf{u},\boldsymbol{\sigma})$ represents a couple of regular functions that satisfy (3.1)-(3.6). We introduce the set of admissible displacements defined by

$$U = \{\mathbf{v} \in V : v_\nu \leq g \;\; \text{a.e on} \;\; \Gamma_3\}\,. \tag{3.17}$$

On U, we still use the inner product $(.,.)_V$ of V. We note that assumption $g > 0$ implies that U is a closed, convex nonempty subset of the space V.
Then we apply the Green formula (2.8) and we use (3.2) to have

$$\begin{aligned}&(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}(t)))_{\mathcal{H}} + (-\mathbf{f}_0(t),\mathbf{v}-\mathbf{u}(t))_H\\ &= \int_{\Gamma_1} \boldsymbol{\sigma\nu}(t)(\mathbf{v}-\mathbf{u}(t))\,da + \int_{\Gamma_2} \boldsymbol{\sigma\nu}(t)(\mathbf{v}-\mathbf{u}(t))\,da + \int_{\Gamma_3} \boldsymbol{\sigma\nu}(t)(\mathbf{v}-\mathbf{u}(t))\,da \;\forall \mathbf{v} \in U.\end{aligned} \tag{3.18}$$

We note that the Green formula (2.8) is satisfied for all $\mathbf{v} \in H_1$. Moreover, we know that $U \subset V \subset H_1$; so, (2.8) is satisfied for all $\mathbf{v} \in U$. By taking into account (3.3)-(3.4) in (3.18) we obtain

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}(t)))_{\mathcal{H}}+(-\mathbf{f}_0(t),\mathbf{v}-\mathbf{u}(t))_H = \int_{\Gamma_2} \mathbf{f}_2(\mathbf{v}-\mathbf{u}(t))\,da + \int_{\Gamma_3} \boldsymbol{\sigma\nu}(t)(\mathbf{v}-\mathbf{u}(t))\,da. \tag{3.19}$$

Furthermore, we know that

$$\boldsymbol{\sigma\nu}(t)(\mathbf{v}-\mathbf{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma_\tau}(t)(\mathbf{v}_\tau - \mathbf{u}_\tau(t)).$$

We substitute the last equality in (3.19) to see that

$$\begin{aligned}&(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}(t)))_{\mathcal{H}}\\ &= \int_\Omega \mathbf{f}_0(t)\,(\mathbf{v}-\mathbf{u}(t))\,d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_2(t)\,(\mathbf{v}-\mathbf{u}(t))\,da\\ &+ \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\,da + \int_{\Gamma_3} \boldsymbol{\sigma_\tau}(t)(\mathbf{v}_\tau - \mathbf{u}_\tau(t))\,da.\end{aligned} \tag{3.20}$$

Now, we write

$$\begin{aligned}&\sigma_\nu(t)(v_\nu - u_\nu(t))\\ &= [\sigma_\nu(t) + p_\nu(u_\nu(t))]\,(v_\nu - g) + [\sigma_\nu(t) + p_\nu(u_\nu(t))]\,(g - u_\nu(t)) - p_\nu(u_\nu(t))\,(v_\nu - u_\nu(t)).\end{aligned}$$

Next, we use (3.17) to see that $v_\nu - g \leq 0$; in addition, we use the contact condition (3.5) to obtain

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) \geq -p_\nu(u_\nu(t))(v_\nu - u_\nu(t)) \qquad \text{on } \Gamma_3.$$

Now, we integrate the last inequality on Γ_3 to find that

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\,da \geq -\int_{\Gamma_3} p_\nu(u_\nu(t))(v_\nu - u_\nu(t))\,da \qquad \text{on } \Gamma_3. \tag{3.21}$$

Moreover, we use the friction law (3.6) to see that for $\mathbf{u}_\tau(t) \neq \mathbf{0}$,

$$\boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \mathbf{u}_\tau(t)) = -p_\tau(u_\nu(t))\frac{\mathbf{u}_\tau(t)\,\mathbf{v}_\tau}{\|\mathbf{u}_\tau(t)\|} + p_\tau(u_\nu(t))\,\|\mathbf{u}_\tau(t)\|\,. \tag{3.22}$$

It is clear that the Cauchy-Schwartz inequality yields

$$-p_\tau(u_\nu(t))\frac{\mathbf{u}_\tau(t)\,\mathbf{v}_\tau}{\|\mathbf{u}_\tau(t)\|} + p_\tau(u_\nu(t))\,\|\mathbf{u}_\tau(t)\| \;\geq\; -p_\tau(u_\nu(t))\,\|\mathbf{v}_\tau\| + p_\tau(u_\nu(t))\,\|\mathbf{u}_\tau(t)\|\,. \tag{3.23}$$

From (3.22)-(3.23), we obtain

$$\boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \mathbf{u}_\tau(t)) \geq p_\tau(u_\nu(t))(\|\mathbf{u}_\tau(t)\| - \|\mathbf{v}_\tau\|) \text{ if } \mathbf{u}_\tau(t) \neq \mathbf{0}. \tag{3.24}$$

On the other hand, if $\mathbf{u}_\tau(t) = \mathbf{0}$, then

$$\boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \mathbf{u}_\tau(t)) = \boldsymbol{\sigma}_\tau(t)\, \mathbf{v}_\tau.$$

From the Cauchy-Schwartz inequality and (3.6), we obtain

$$\begin{aligned} \boldsymbol{\sigma}_\tau(t)\, \mathbf{v}_\tau &\geq -\|\boldsymbol{\sigma}_\tau(t)\|\, \|\mathbf{v}_\tau\| \\ &\geq -p_\tau(u_\nu(t))\, \|\mathbf{v}_\tau\|. \end{aligned}$$

Since $\mathbf{u}_\tau(t) = \mathbf{0}$, the last inequality can be written as follows

$$\boldsymbol{\sigma}_\tau(t).\mathbf{v}_\tau - \boldsymbol{\sigma}_\tau(t).\mathbf{u}_\tau(t) \geq -p_\tau(u_\nu(t))\, \|\mathbf{v}_\tau\| + p_\tau(u_\nu(t))\, \|\mathbf{u}_\tau(t)\|,$$

which yields

$$\boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \mathbf{u}_\tau(t)) \geq -p_\tau(u_\nu(t))\, (\|\mathbf{v}_\tau\| - \|\mathbf{u}_\tau(t)\|) \text{ if } \mathbf{u}_\tau(t) = \mathbf{0}. \tag{3.25}$$

We conclude from (3.24) and (3.25) that

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \mathbf{u}_\tau(t))\, da \geq \int_{\Gamma_3} -p_\tau(u_\nu(t))\, (\|\mathbf{v}_\tau\| - \|\mathbf{u}_\tau(t)\|)\, da. \tag{3.26}$$

We combine (3.20), (3.21) and (3.26) to find

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}(t)))_{\mathcal{H}} \\ &\geq \int_\Omega \mathbf{f}_0(t)(\mathbf{v} - \mathbf{u}(t))\, d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_2(t)(\mathbf{v} - \mathbf{u}(t)) da \\ &- \int_{\Gamma_3} p_\nu(u_\nu(t))(v_\nu - u_\nu(t))\, da - \int_{\Gamma_3} p_\tau(u_\nu(t))(\|\mathbf{v}_\tau\| - \|\mathbf{u}_\tau(t)\|)\, da. \end{aligned} \tag{3.27}$$

Now, we use Riesz's theorem to define the element $\mathbf{f}(t) \in V$ by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t)\, \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_2(t)\, \mathbf{v}\, da. \tag{3.28}$$

From regularities (3.16), we find

$$\mathbf{f} \in C\,(0, T; V). \tag{3.29}$$

To complete the variational formulation of our problem, we use again Riesz's theorem to define the operator $P : V \to V$ such that

$$(P\mathbf{u}, \mathbf{v})_V = \int_{\Gamma_3} p_\nu(u_\nu)\, v_\nu\, da \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{3.30}$$

It follows from (2.7) and hypothesis (3.14) that

$$(P\mathbf{u} - P\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geq 0, \; \|P\mathbf{u} - P\mathbf{v}\|_V \leq c_0^2 L_\nu \|\mathbf{u} - \mathbf{v}\|_V \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{3.31}$$

which means that $P : V \to V$ is a monotone and Lipschitz continuous operator.
Finally, we define the function $j : V \times V \to \mathbb{R}^+$ by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\tau\,(u_\nu)\, \|\mathbf{v}_\tau\|\, da. \tag{3.32}$$

Now, we substitute (3.28), (3.30) and (3.32) in (3.27) to obtain

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}(t)))_{\mathcal{H}} + (P\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_V + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \mathbf{u}(t)) \\ \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V \quad \forall \mathbf{v} \in U. \end{aligned} \tag{3.33}$$

We combine (3.1) and (3.33) to find the following variational formulation of P.
Problem PV. Find a displacement field $\mathbf{u} : [0,T] \longrightarrow U$ and a stress field $\boldsymbol{\sigma} : [0,T] \longrightarrow \mathcal{H}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\,\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{R}(t-s)\,\varepsilon(\mathbf{u}(s))\,ds, \tag{3.34}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}-\mathbf{u}(t)))_{\mathcal{H}} + (P\mathbf{u}(t), \mathbf{v}-\mathbf{u}(t))_V + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \mathbf{u}(t))\\ \geq (\boldsymbol{f}(t), \mathbf{v}-\mathbf{u}(t))_V \qquad \forall \mathbf{v} \in U.\end{aligned} \tag{3.35}$$

4 Existence and uniqueness result

The existence and uniqueness of the solution of the variational problem PV is given in the following result.

Theorem 4.1. *Assume that hypothesis (3.12)-(3.16) are satisfied. Then, there exists a constant $L_0 > 0$ such that if $L_\tau < L_0$, then problem PV has a unique solution $(\mathbf{u}, \boldsymbol{\sigma})$. Moreover, the solution satisfies*

$$\begin{aligned}\boldsymbol{u} &\in C(0,T;U),\\ \boldsymbol{\sigma} &\in C(0,T;\mathcal{H}_1).\end{aligned} \tag{4.1}$$

Now, let us move on to the proof of Theorem 4.1 which will be carried out in several steps. We assume in what follows that assumptions (3.12) -(3.16) are satisfied and, in what follows, we denote by c a generic positive constant which may change from one place to another.
***Step 1**. For all $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$, we consider the following intermediate variational problem.*
***Problem** PV_η. Find a displacement field $\mathbf{u}_\eta : [0,T] \to U$ and a stress field $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ such that*

$$\boldsymbol{\sigma}_\eta = \mathcal{A}\varepsilon(\mathbf{u}_\eta(t)) + \boldsymbol{\eta}(t), \tag{4.2}$$

$$\begin{aligned}&(\mathcal{A}\varepsilon(\mathbf{u}_\eta(t)), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_\eta(t)))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_\eta(t)))_{\mathcal{H}}\\ &+(P\mathbf{u}_\eta(t), \mathbf{v}-\mathbf{u}_\eta(t))_V + j(\mathbf{u}_\eta(t), \mathbf{v}) - j(\mathbf{u}_\eta(t), \mathbf{u}_\eta(t)) \geq (\mathbf{f}(t), \mathbf{v}-\mathbf{u}_\eta(t))_V \;\forall \mathbf{v} \in U.\end{aligned} \tag{4.3}$$

We have the following existence and uniqueness result.

Lemma 4.2. *There exists a unique solution $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ to problem PV_η such that $\mathbf{u}_\eta \in C(0,T;U)$ and $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H}_1)$. Moreover, if $\mathbf{u}_i = \mathbf{u}_{\eta_i}$ are two solutions of problem PV_η corresponding to $\boldsymbol{\eta}_i \in C(0,T;\mathcal{H}),\ i = 1,2$, then there exists a constant $c > 0$ such that*

$$||\mathbf{u}_1(t) - \mathbf{u}_2(t)||_V \leq c\,||\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)||_{\mathcal{H}} \quad \forall t \in [0,T]. \tag{4.4}$$

Proof. We apply Theorem 2.2 on the space $X = V$ with $K = U$. For this purpose, we recall that U given in (3.17) is a nonempty closed convex subset of V, which means that it satisfies (2.10). Next, we use Riez's theorem to define the operator $A : V \to V$ and the function $\mathbf{f}_\eta : [0,T] \to V$ by equalities

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}\varepsilon(\mathbf{u}),\ \varepsilon(\mathbf{v}))_{\mathcal{H}} + (P\mathbf{u}, \mathbf{v})_V, \tag{4.5}$$

$$(\mathbf{f}_\eta\,(t)\,, \mathbf{v})_V = (\mathbf{f}\,(t)\,, \mathbf{v})_V - (\boldsymbol{\eta}\,(t)\,, \varepsilon(\mathbf{v}))_{\mathcal{H}}, \tag{4.6}$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in [0,T]$.
First, we show that the operator A is *strongly monotone* and *Lipschitz-continous*. Let $\mathbf{u_1}, \mathbf{u_2} \in V$. We have

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V = (\mathcal{A}\varepsilon(\mathbf{u}_1) - \mathcal{A}\varepsilon(\mathbf{u}_2), \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2))_{\mathcal{H}} + (P\mathbf{u}_1 - P\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V.$$

From the monotonicity of the operator P expressed in (3.31), we obtain

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq (\mathcal{A}\varepsilon(\mathbf{u}_1) - \mathcal{A}\varepsilon(\mathbf{u}_2), \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2))_{\mathcal{H}}.$$

We use now (3.12)(c) to see that

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq m_{\mathcal{A}}||\boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2)||^2_{\mathcal{H}}.$$

Thus, (2.6) yields

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq m_{\mathcal{A}}||\mathbf{u}_1 - \mathbf{u}_2||^2_V. \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V,$$

which shows that A is strongly monotone with constant $m_{\mathcal{A}}$.
Now, for $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in V$, we have

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (P\mathbf{u}_1 - P\mathbf{u}_2, \mathbf{v})_V,$$

which implies

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{v})_V \leq ||\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2)||_{\mathcal{H}} \; ||\boldsymbol{\varepsilon}(\mathbf{v})||_{\mathcal{H}} + ||P\mathbf{u}_1 - P\mathbf{u}_2||_V \; ||\mathbf{v}||_V.$$

The last inequality combined with the Lipschitz continuity of P expressed in (3.31) as well as (3.12)(b) yields

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{v})_V \leq L_{\mathcal{A}} \; ||\boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2)||_{\mathcal{H}} \; ||\boldsymbol{\varepsilon}(\mathbf{v})||_{\mathcal{H}} + L_\nu c_0^2 \; ||\mathbf{u}_1 - \mathbf{u}_2||_V \; ||\mathbf{v}||_V.$$

We use now (2.6) to find that

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{v})_V \leq c \; ||\mathbf{u}_1 - \mathbf{u}_2||_V \; ||\mathbf{v}||_V.$$

By choosing $\mathbf{v} = A\mathbf{u}_1 - A\mathbf{u}_2$ in the previous inequality, we obtain

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|^2_V \leq c \; ||\mathbf{u}_1 - \mathbf{u}_2||_V \; ||A\mathbf{u}_1 - A\mathbf{u}_2||_V.$$

Hence,

$$||A\mathbf{u}_1 - A\mathbf{u}_2||_V \leq c \; ||\mathbf{u}_1 - \mathbf{u}_2||_V,$$

which shows that A is a Lipschitz continuous operator and then A satisfies conditions (2.11).
Next, we show conditions (2.12) on the function j. First it is easy to show that $j(\mathbf{u}, .)$ is a semi-norm on V, for all $\mathbf{u} \in V$. Moreover, we recall that $\|\mathbf{v}_\tau\| \leq \|\mathbf{v}\|$ to see that, for all $\mathbf{u} \in V$, we have

$$j(\mathbf{u}, \mathbf{v}) \quad \leq \int_{\Gamma_3} p_\tau(u_\nu) \, \|\mathbf{v}\| \, da.$$

Then, from (3.15), we deduce that

$$j(\mathbf{u}, \mathbf{v}) \leq c \, \|\mathbf{v}\|_{L^2(\Gamma_3)^d}.$$

Hence, (2.7) yields

$$j(\mathbf{u}, \mathbf{v}) \leq c \, \|\mathbf{v}\|_V.$$

We can then deduce that $j(\mathbf{u}, .)$ is a continuous semi-norm on V; therefore it is convex and lower semi-continuous.
Now, for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$, we have

$$\begin{aligned} & j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ & \qquad = \int_{\Gamma_3} p_\tau(u_{1\nu}) \; \|\mathbf{v}_{2\tau}\| \, da - \int_{\Gamma_3} p_\tau(u_{1\nu}) \; \|\mathbf{v}_{1\tau}\| \, da \\ & \qquad + \int_{\Gamma_3} p_\tau(u_{2\nu}) \; \|\mathbf{v}_{1\tau}\| \, da - \int_{\Gamma_3} p_\tau(u_{2\nu}) \; \|\mathbf{v}_{2\tau}\| \, da. \end{aligned}$$

Therefore,

$$\begin{aligned} & j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ & \qquad \leq \int_{\Gamma_3} |p_\tau(u_{1\nu}) - p_\tau(u_{2\nu})| \; | \, \|\mathbf{v}_{2\tau}\| - \|\mathbf{v}_{1\nu}\| \, | \, da \, . \end{aligned}$$

Condition (3.15) (b) yields

$$j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \leq L_\tau \int_{\Gamma_3} |u_{1\nu} - u_{2\nu}| \; | \, \|\mathbf{v}_{1\tau}\| - \|\mathbf{v}_{2\tau}\| \, | \; da \, ,$$

Next, using inequalities (2.2), we have

$$j(\mathbf{u}_1,\mathbf{v}_2)-j(\mathbf{u}_1,\mathbf{v}_1)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)\leq L_\tau\int_{\Gamma_3}\|\mathbf{u}_1-\mathbf{u}_2\|\ \|\mathbf{v}_1-\mathbf{v}_2\|\ da\,.$$

Therefore,

$$j(\mathbf{u}_1,\mathbf{v}_2)-j(\mathbf{u}_1,\mathbf{v}_1)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)\leq L_\tau\ \|\mathbf{u}_1-\mathbf{u}_2\|_{L^2(\Gamma_3)^d}\,\|\mathbf{v}_1-\mathbf{v}_2\|_{L^2(\Gamma_3)^d}\,.$$

Thus, inequality (2.7) yields

$$\begin{aligned}&j(\mathbf{u}_1,\mathbf{v}_2)-j(\mathbf{u}_1,\mathbf{v}_1)+j(\mathbf{u}_2,\mathbf{v}_1)-j(\mathbf{u}_2,\mathbf{v}_2)\\&\leq\alpha\,\|\mathbf{u}_1-\mathbf{u}_2\|_V\,\|\mathbf{v}_1-\mathbf{v}_2\|_V\ ,\end{aligned}\tag{4.7}$$

for all $\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2\in V$, with $\alpha=c_0^2\,L_\tau$. Note that (4.7) shows that condition (2.12) (b) is satisfied for $\alpha=c_0^2\,L_\tau$.
Finally, for the condition (2.13) to be satisfied, we must have

$$m_{\mathcal{A}}>c_0^2\,L_\tau.$$

Therefore,

$$L_\tau<\frac{m_{\mathcal{A}}}{c_0^2}.$$

Thus, we take $L_0=\dfrac{m_{\mathcal{A}}}{c_0^2}$. Hence, if $L_\tau<L_0$, this implies $m_{\mathcal{A}}>c_0^2\,L_\tau$, which means that condition (2.13) of Theorem 2.2 is actually satisfied.
Consequently, it follows from Theorem 2.2 that there exists a unique solution $\mathbf{u}_\eta(t)\in U$ of the variational inequality

$$(A\mathbf{u}_\eta(t),\mathbf{v}-\mathbf{u}_\eta(t))_V+j(\mathbf{u}_\eta(t),\mathbf{v})-j(\mathbf{u}_\eta(t),\mathbf{u}_\eta(t))\geq(\mathbf{f}_\eta(t),\mathbf{v}-\mathbf{u}_\eta(t))_V\quad\forall\mathbf{v}\in U.\tag{4.8}$$

In the end, we combine (4.5), (4.6) and (4.8) to see that $\mathbf{u}_\eta(t)\in U$ is the unique solution of Problem PV_η.
Let us now prove the regularity of the solution $\mathbf{u}_\eta$. To this end, let $t_1,t_2\in[0,T]$ and let use the notations $\mathbf{u}_\eta(t_i)=\mathbf{u}_i,\ \boldsymbol{\eta}(t_i)=\boldsymbol{\eta}_i,\ \mathbf{f}(t_i)=\mathbf{f}_i$, for i=1,2. We write (4.3) by replacing t with t_1 and by taking $\mathbf{v}=\mathbf{u}_2$ and then by replacing t with t_2 and by taking $\mathbf{v}=\mathbf{u}_1$; after an elementary calculation we obtain

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1)-\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2),\boldsymbol{\varepsilon}(\mathbf{u}_1-\mathbf{u}_2))_{\mathcal{H}}+(P\mathbf{u}_1-P\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)_V\leq(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2,\boldsymbol{\varepsilon}(\mathbf{u}_2-\mathbf{u}_1))_{\mathcal{H}}$$

$$+j(\mathbf{u}_1,\mathbf{u}_2)-j(\mathbf{u}_1,\mathbf{u}_1)+j(\mathbf{u}_2,\mathbf{u}_1)-j(\mathbf{u}_2,\mathbf{u}_2)+(\mathbf{f}_1-\mathbf{f}_2,\mathbf{u}_1-\mathbf{u}_2)_V,$$

By tacking into account (3.12)(c) and the monotonicity of P expressed in (3.31), we obtain

$$\begin{aligned}m_{\mathcal{A}}||\boldsymbol{\varepsilon}(\mathbf{u}_1-\mathbf{u}_2)||^2_{\mathcal{H}}&\leq(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2,\boldsymbol{\varepsilon}(\mathbf{u}_2-\mathbf{u}_1))_{\mathcal{H}}+(\mathbf{f}_1-\mathbf{f}_2,\mathbf{u}_1-\mathbf{u}_2)_V\\&+j(\mathbf{u}_1,\mathbf{u}_2)-j(\mathbf{u}_1,\mathbf{u}_1)+j(\mathbf{u}_2,\mathbf{u}_1)-j(\mathbf{u}_2,\mathbf{u}_2)\\&\leq||\boldsymbol{\eta}_1-\boldsymbol{\eta}_2||_{\mathcal{H}}\ ||\boldsymbol{\varepsilon}(\mathbf{u}_2-\mathbf{u}_1)||_{\mathcal{H}}+||\mathbf{f}_1-\mathbf{f}_2||_V||\mathbf{u}_1-\mathbf{u}_2||_V\\&+j(\mathbf{u}_1,\mathbf{u}_2)-j(\mathbf{u}_1,\mathbf{u}_1)+j(\mathbf{u}_2,\mathbf{u}_1)-j(\mathbf{u}_2,\mathbf{u}_2).\end{aligned}$$

We recall (2.6) and (4.7) to find

$$\begin{aligned}m_{\mathcal{A}}||\mathbf{u}_1-\mathbf{u}_2||^2_V&\leq||\boldsymbol{\eta}_1-\boldsymbol{\eta}_2||_{\mathcal{H}}\ ||\mathbf{u}_2-\mathbf{u}_1||_V+||\mathbf{f}_1-\mathbf{f}_2||_V\ ||\mathbf{u}_1-\mathbf{u}_2||_V\\&+c_0^2\,L_\tau\,\|\mathbf{u}_1-\mathbf{u}_2\|^2_V.\end{aligned}$$

Since $m_{\mathcal{A}}>c_0^2\,L_\tau$, we deduce that

$$||\mathbf{u}_1-\mathbf{u}_2||_V\leq c\,(||\boldsymbol{\eta}_1-\boldsymbol{\eta}_2||_{\mathcal{H}}+||\mathbf{f}_1-\mathbf{f}_2||_V).$$

We combine the last inequality with (3.29) and the regularity $\boldsymbol{\eta}\in C(0,T;\mathcal{H})$ to obtain $\mathbf{u}_\eta\in C(0,T;U)$.

Now, let $\boldsymbol{\sigma}_\eta$ be the function defined by (4.2), then, $\mathbf{u}_\eta(t) \in U \subset V$ et $\boldsymbol{\eta}(t) \in \mathcal{H}$ imply $\boldsymbol{\sigma}_\eta(t) \in \mathcal{H}$. Also, for all $t_1, t_2 \in [0, T]$,

$$||\boldsymbol{\sigma}_\eta(t_1) - \boldsymbol{\sigma}_\eta(t_2)||_{\mathcal{H}} \leq ||\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t_1)) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t_2))||_{\mathcal{H}} + ||\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2)||_{\mathcal{H}},$$

From (3.12)(b) and (2.6), we obtain

$$||\boldsymbol{\sigma}_\eta(t_1) - \boldsymbol{\sigma}_\eta(t_2)||_{\mathcal{H}} \leq L_{\mathcal{A}}\, ||\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)||_V + ||\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2)||_{\mathcal{H}},$$

and by taking into account the regularities $\mathbf{u}_\eta \in C(0, T; U)$ and $\boldsymbol{\eta} \in C(0, T; \mathcal{H})$, we deduce that $\boldsymbol{\sigma}_\eta \in C(0, T; \mathcal{H})$.
In order to have the regularity $\boldsymbol{\sigma}_\eta \in C(0, T; \mathcal{H}_1)$ of the stress field, we test (4.3) with $\mathbf{v} = \mathbf{u}_\eta(t) + \boldsymbol{\varphi}$ then with $\mathbf{v} = \mathbf{u}_\eta(t) - \boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \in C_0^\infty(\boldsymbol{\Omega})^d$, we obtain

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}} + (P\mathbf{u}_\eta(t), \boldsymbol{\varphi})_V$$
$$+ j(\mathbf{u}_\eta(t), \boldsymbol{\varphi}) - j(\mathbf{u}_\eta(t), \mathbf{u}_\eta(t)) \geq (\mathbf{f}(t), \boldsymbol{\varphi})_V,$$

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(-\boldsymbol{\varphi}))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(-\boldsymbol{\varphi}))_{\mathcal{H}} + (P\mathbf{u}_\eta(t), -\boldsymbol{\varphi})_V$$
$$+ j(\mathbf{u}_\eta(t), -\boldsymbol{\varphi}) - j(\mathbf{u}_\eta(t), \mathbf{u}_\eta(t)) \geq (\mathbf{f}(t), -\boldsymbol{\varphi})_V,$$

which is equivalent to

$$(\boldsymbol{\sigma}_{\boldsymbol{\eta}}(t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}} + (P\mathbf{u}_\eta(t), \boldsymbol{\varphi})_V + j(\mathbf{u}_\eta(t), \boldsymbol{\varphi}) - j(\mathbf{u}_\eta(t), \mathbf{u}_\eta(t)) \geq (\mathbf{f}(t), \boldsymbol{\varphi})_V,$$
$$(\boldsymbol{\sigma}_{\boldsymbol{\eta}}(t), \boldsymbol{\varepsilon}(-\boldsymbol{\varphi}))_{\mathcal{H}} + (P\mathbf{u}_\eta(t), -\boldsymbol{\varphi})_V + j(\mathbf{u}_\eta(t), -\boldsymbol{\varphi}) - j(\mathbf{u}_\eta(t), \mathbf{u}_\eta(t)) \geq (\mathbf{f}(t), -\boldsymbol{\varphi})_V.$$

We recall that $j(\mathbf{u}_\eta(t), \mathbf{u}_\eta(t)) \geq 0$, so we obtain

$$(\boldsymbol{\sigma}_{\boldsymbol{\eta}}(t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}} + (P\mathbf{u}_\eta(t), \boldsymbol{\varphi})_V + j(\mathbf{u}_\eta(t), \boldsymbol{\varphi}) \geq (\mathbf{f}(t), \boldsymbol{\varphi})_V,$$
$$(\boldsymbol{\sigma}_{\boldsymbol{\eta}}(t), \boldsymbol{\varepsilon}(-\boldsymbol{\varphi}))_{\mathcal{H}} + (P\mathbf{u}_\eta(t), -\boldsymbol{\varphi})_V + j(\mathbf{u}_\eta(t), -\boldsymbol{\varphi}) \geq (\mathbf{f}(t), -\boldsymbol{\varphi})_V.$$

Since $\varphi \in C_0^\infty(\Omega)^d$, it follows that

$$(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}} \geq (\mathbf{f}(t), \boldsymbol{\varphi})_V,$$

$$(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(-\boldsymbol{\varphi}))_{\mathcal{H}} \geq (\mathbf{f}(t), -\boldsymbol{\varphi})_V.$$

which yields

$$(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}} = (\mathbf{f}(t), \boldsymbol{\varphi})_V \quad \forall t \in [0, T].$$

Therefore, by using (3.28) and the definition of the weak divergence, we obtain

$$(Div\ \boldsymbol{\sigma}_\eta(t), \boldsymbol{\varphi})_{L^2(\Omega)^d} = (-\mathbf{f}_0(t), \boldsymbol{\varphi})_{L^2(\Omega)^d} \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\boldsymbol{\Omega})^d.$$

Since the space $C_0^\infty(\Omega)^d$ is dense in $L^2(\Omega)^d$, we conclude that

$$Div\ \boldsymbol{\sigma}_\eta(t) = -\mathbf{f}_0(t) \quad \forall t \in [0, T]. \tag{4.9}$$

The last equality combined with the hypothesis (3.16) on $\mathbf{f}_0$ implies $Div\ \boldsymbol{\sigma}_\eta(t) \in H$ and, consequently, $\boldsymbol{\sigma}_\eta \in \mathcal{H}_1$. Finally, we note that the induced norm of the scalar product on $\mathcal{H}_1$ which has been defined in section 2, allows us to write

$$\|\boldsymbol{\sigma}_\eta(t_1) - \boldsymbol{\sigma}_\eta(t_2)\|^2_{\mathcal{H}_1} \quad = \|\boldsymbol{\sigma}_\eta(t_1) - \boldsymbol{\sigma}_\eta(t_2)\|^2_{\mathcal{H}} + \|Div\ (\boldsymbol{\sigma}_\eta(t_1)) - Div\ (\boldsymbol{\sigma}_\eta(t_2))\|^2_H\ .$$

Therefore, (4.9), (3.16) and the regularity $\boldsymbol{\sigma}_\eta \in C(0, T; \mathcal{H})$ show that $\boldsymbol{\sigma}_\eta \in C(0, T; \mathcal{H}_1)$. Thus, we complete the demonstration of the existence of the solution $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ to Problem PV_η. The uniqueness of the latter comes from the unique solvability of the equality (4.3).
Let us move on to the demonstration of the estimate (4.4). For that purpose, let $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ be of $C(0, T; \mathcal{H})$. Let us use the notations $\mathbf{u}_i = \mathbf{u}_{\eta_i}$, for $i = 1, 2$. Next, we write (4.3) for $\boldsymbol{\eta} = \boldsymbol{\eta}_1$ by taking $\mathbf{v} = \mathbf{u}_2$ and then for $\boldsymbol{\eta} = \boldsymbol{\eta}_2$ by taking $\mathbf{v} = \mathbf{u}_1$, after an elementary calculation we obtain

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2), \boldsymbol{\varepsilon}(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} + (P\mathbf{u}_1 - P\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V$$
$$\leq (\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\varepsilon}(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} + j(\mathbf{u}_1, \mathbf{u}_2) - j(\mathbf{u}_1, \mathbf{u}_1) + j(\mathbf{u}_2, \mathbf{u}_1) - j(\mathbf{u}_2, \mathbf{u}_2).$$

From the monotonicity of P expressed in (3.31), while using (3.12)(c) and (4.7), we obtain

$$m_{\mathcal{A}} \, ||\boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2)||^2_{\mathcal{H}} \le ||\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1||_{\mathcal{H}} \, ||\boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2)||_{\mathcal{H}} + c_0^2 \, L_\tau \, \|\mathbf{u}_1 - \mathbf{u}_2\|^2_V.$$

We know that $m_{\mathcal{A}} > c_0^2 \, L_\tau$ and we use (2.6) to find

$$||\mathbf{u_1} - \mathbf{u}_2||_V \le c \, ||\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2||_{\mathcal{H}},$$

which completes the proof of the estimate (4.4). □

Step 2*. We use the solution $\mathbf{u}_\eta \in C(0,T;U)$ of problem PV_η and we consider the operator Λ which associates to any element $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$, the element*

$$\Lambda\boldsymbol{\eta}(t) = \int_0^t \mathcal{R}(t-s) \, \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)) \, ds, \tag{4.10}$$

for all $t \in [0,T]$. We get the following result.

Lemma 4.3. *The operator Λ takes its values in the set $C(0,T;\mathcal{H})$. Moreover, it has only one fixed point $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H})$.*

Proof. For $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$, we use (4.10), (2.5), (2.6), (3.13) and the regularity $\mathbf{u}_\eta \in C(0,T;V)$ to conclude that $\Lambda\boldsymbol{\eta}(t) \in \mathcal{H}$. Also, it is easy to see that $t \mapsto \Lambda\boldsymbol{\eta}(t)$ is continuous from $[0,T]$ to the space $\mathcal{H}$.
Let us move on to the proof of the second part of Lemma 4.3. For this purpose, we consider $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C(0,T;\mathcal{H})$ and, for simplicity, we use the notations $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $i = 1,2$. We use (2.5) to deduce that

$$||\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)||_{\mathcal{H}} \le \int_0^t ||\mathcal{R}(t-s) \, (\boldsymbol{\varepsilon}(\mathbf{u}_1(s)) - \boldsymbol{\varepsilon}(\mathbf{u}_2(s)))||_{\mathcal{H}} \, ds$$

$$\le d \max_{s\in[0,T]} ||\mathcal{R}(s) \, ||_{\mathcal{H}_\infty} \int_0^t ||\boldsymbol{\varepsilon}(\mathbf{u}_1(s)) - \boldsymbol{\varepsilon}(\mathbf{u}_2(s)))||_{\mathcal{H}} \, ds.$$

Thus, (2.6), (3.13) and the last inequality provide

$$||\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)||_{\mathcal{H}} \le c \int_0^t ||\mathbf{u}_1(s) - \mathbf{u}_2(s)||_V \, ds. \tag{4.11}$$

We now combine (4.11) and (4.4) to deduce that

$$||\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)||_{\mathcal{H}} \le c \int_0^t ||\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)||_{\mathcal{H}} \, ds.$$

Finally, the application of Theorem 2.1 allows us to conclude the proof. □

We now have all necessary to proof the Theorem 4.1.

Proof. Let $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H})$ be the fixed point of the operator Λ and let $\mathbf{u}^*$ and $\boldsymbol{\sigma}^*$ be the functions defined by

$$\mathbf{u}^*(t) = \mathbf{u}_{\boldsymbol{\eta}^*}(t), \tag{4.12}$$

$$\boldsymbol{\sigma}^*(t) = \mathcal{A} \, \boldsymbol{\varepsilon}(\mathbf{u}^*(t)) + \int_0^t \mathcal{R}(t-s) \, \boldsymbol{\varepsilon}(\mathbf{u}^*(s)) \, ds, \tag{4.13}$$

For all $t \in [0,T]$. We recall that $\boldsymbol{\eta}^* = \Lambda\boldsymbol{\eta}^*$ and we use (4.10) and (4.12) to obtain

$$\boldsymbol{\eta}^*(t) = \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}^*(s))ds. \tag{4.14}$$

Now, we show that the couple $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ satisfies the system (3.34)-(3.35). Firstly, we note that (3.34) is an immediate consequence of (4.13). Next, we write (4.3) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and we use (4.12)-(4.14) to see that (3.35) is verified. This means that the couple $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ is a solution of the problem PV. The regularity of the latter expressed in (4.1) is a direct consequence of the Lemma 4.2.

Uniqueness. The uniqueness of the solution arises from the uniqueness of the fixed point of the operator Λ defined by (4.10) combined with the unique solvability of the Problem PV_η. In fact, let $(\mathbf{u}, \boldsymbol{\sigma})$ be a solution of Problem PV satisfying (4.1) and let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$ be given by

$$\boldsymbol{\eta}(t) = \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}(s))\,ds \;\; \forall t \in [0,T]. \tag{4.15}$$

We substitute the equality (3.34) in (3.35) and we use (4.15) to deduce that $\mathbf{u}$ satisfies the inequality (4.3), for all $t \in [0,T]$. On the other hand, it follows from Lemma 4.2 that the Problem PV_η has a unique solution denoted $\mathbf{u}_\eta$ having the regularity $\mathbf{u}_\eta \in C(0,T;U)$. So we conclude that

$$\mathbf{u} = \mathbf{u}_\eta. \tag{4.16}$$

We use (4.16) to see that

$$\int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}_\eta(s))\,ds = \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}(s))\,ds \;\; \forall t \in [0,T].$$

Consequently, (4.10) and (4.15) show that $\Lambda\boldsymbol{\eta} = \boldsymbol{\eta}$ and, using the uniqueness of the fixed point of the operator Λ, it comes that

$$\boldsymbol{\eta} = \boldsymbol{\eta}^*. \tag{4.17}$$

Now, we use (4.16), (4.17) and (4.12) to see that

$$\mathbf{u} = \mathbf{u}_\eta = \mathbf{u}_{\eta^*} = \mathbf{u}^*. \tag{4.18}$$

Next, we use (3.34), (4.18) and (4.13) to deduce that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\,\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}(s))\,ds, \\ &= \mathcal{A}\,\boldsymbol{\varepsilon}(\mathbf{u}^*(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}^*(s))\,ds, \\ &= \boldsymbol{\sigma}^*(t), \end{aligned} \tag{4.19}$$

for all $t \in [0,T]$. The uniqueness of the solution $(\mathbf{u}, \boldsymbol{\sigma})$ is a consequence of equalities (4.18) and (4.19). □

5 A convergence result

In this section, we study the dependence of the solution of the Problem PV with respect to data perturbations. To this end, we assume in what follows that (3.12)-(3.16) are satisfied. In addition, we assume that

$$L_\tau < L_0 \quad \text{where} \quad L_0 = \frac{m_{\mathcal{A}}}{c_0^2}, \tag{5.1}$$

and we note by $\mathbf{u}$ the solution of the Problem PV obtained in Theorem 4.1.

for all $\rho > 0$, $\mathbf{f}_{0\rho}$ and $\mathbf{f}_{2\rho}$ represent perturbations of $\mathbf{f}_0$ and $\mathbf{f}_2$ and $p_{\nu\rho}$, $p_{\tau\rho}$ are the perturbations of

p_ν et p_τ, respectively. Let us assume that $\mathbf{f}_{0\rho}$ and $\mathbf{f}_{2\rho}$ satisfy (3.16) and that $p_{\nu\rho}$, $p_{\tau\rho}$ satisfy (3.14) and (3.15) with Lipschitz's constants $L_{\nu\rho}$ and $L_{\tau\rho}$ respectively. In addition, we assume that

$$L_{\tau\rho} < L_0 \qquad \forall \rho > 0, \tag{5.2}$$

where $L_0 = \dfrac{m_{\mathcal{A}}}{c_0^2}$. By using these new data, we define the operator $P_\rho : V \longrightarrow V$ and the functions $\mathbf{f}_\rho : [0,T] \longrightarrow V$ and $j_\rho : V \times V \longrightarrow \mathbb{R}^+$ by the equalities

$$(P_\rho\, \mathbf{u}, \mathbf{v})_V = \int_{\Gamma_3} p_{\nu\rho}(u_\nu)\, v_\nu\, da \qquad \forall \mathbf{u}, \mathbf{v} \in V, \tag{5.3}$$

$$(\mathbf{f}_\rho(t), \mathbf{v})_V = \int_{\Omega} \mathbf{f}_{0\rho}(t)\, \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_{2\rho}(t)\, \mathbf{v}\, da \qquad \forall \mathbf{u}, \mathbf{v} \in V, t \in [0,T], \tag{5.4}$$

$$j_\rho(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_{\tau\rho}(u_\nu)\, \| \mathbf{v}_\tau \|\, da \qquad \forall \mathbf{u}, \mathbf{v} \in V. \tag{5.5}$$

It is clear that the assumptions (3.14) on $p_{\nu\rho}$ show that the operator P_ρ satisfies (3.31); i.e. that it is a monotone and Lipschitz-continuous operator. Also, the assumptions (3.15) on $p_{\tau\rho}$ show that the function j_ρ satisfies (4.7) for $\alpha = c_0^2\, L_{\tau\rho}$. We consider the following variational problem.
Problem PV_ρ. Find a displacement field $\mathbf{u}_\rho : [0,T] \to U$ such that, for all $t \in [0,T]$,

$$\begin{aligned}&(\mathcal{A}\varepsilon(\mathbf{u}_\rho(t)), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_\rho(t)))_{\mathcal{H}} + (\int_0^t \mathcal{R}(t-s)\, \varepsilon(\mathbf{u}_\rho(s))\, ds, \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_\rho(t)))_{\mathcal{H}}\\ &+(P_\rho \mathbf{u}_\rho(t), \mathbf{v} - \mathbf{u}_\rho(t))_V + j_\rho(\mathbf{u}_\rho, \mathbf{v}) - j_\rho(\mathbf{u}_\rho(t), \mathbf{u}_\rho(t)) \geq (\mathbf{f}_\rho(t), \mathbf{v} - \mathbf{u}_\rho(t))_V\ \forall \mathbf{v} \in U.\end{aligned} \tag{5.6}$$

Since the problem's data satisfy the assumptions (3.12)-(3.16) and (5.2), then it follows from Theorem 4.1 that for all $\rho > 0$, the Problem PV_ρ has a unique solution $\mathbf{u}_\rho$ such that

$$\mathbf{u}_\rho \in C(0,T;U). \tag{5.7}$$

We now introduce some supplementary assumptions on the functions $p_{\nu\rho}$, $p_{\tau\rho}$, $\mathbf{f}_{0\rho}$ and $\mathbf{f}_{2\rho}$ which are given by

$$\begin{cases}\text{For } e = \nu, \tau :\\ (a)\ \text{There exists } G_e : \mathbb{R}^+ \longrightarrow \mathbb{R} \text{ and } q \in \mathbb{R}^+ \text{ such that}\\ |p_{e\rho}(\mathbf{x}, r) - p_e(\mathbf{x}, r)| \leq G_e(\rho)\ (|r| + q)\\ \forall r \in \mathbb{R}^+,\ \text{a.e. } \mathbf{x} \in \Gamma_3,\ \text{for all } \rho > 0.\\ (b)\ G_e(\rho) \longrightarrow 0 \text{ when } \rho \longrightarrow 0.\end{cases} \tag{5.8}$$

$$\mathbf{f}_{0\rho} \longrightarrow \mathbf{f}_0 \text{ in } C(0,T;H) \text{ when } \rho \longrightarrow 0. \tag{5.9}$$

$$\mathbf{f}_{2\rho} \longrightarrow \mathbf{f}_2 \text{ in } C(0,T;L^2(\Gamma_2)^d) \text{ when } \rho \longrightarrow 0. \tag{5.10}$$

We have the following convergence result.

Theorem 5.1. *Under the assumptions (5.8) – (5.10), the solution $\mathbf{u}_\rho$ of Problem PV_ρ converges to the solution $\mathbf{u}$ of Problem PV; that is to say*

$$\mathbf{u}_\rho \longrightarrow \mathbf{u} \quad \textit{in} \quad C(0,T;V) \quad \textit{when} \quad \rho \longrightarrow 0. \tag{5.11}$$

Proof. Let $\rho > 0$ and $t \in [0,T]$. We take $\mathbf{v} = \mathbf{u}(t)$ in (5.6) and $\mathbf{v} = \mathbf{u}_\rho(t)$ in (3.35); after an elementary calculation we obtain

$$\begin{aligned}&(\mathcal{A}\varepsilon(\mathbf{u}_\rho(t)) - \mathcal{A}\varepsilon(\mathbf{u}(t)), \varepsilon(\mathbf{u}_\rho(t)) - \varepsilon(\mathbf{u}(t)))_{\mathcal{H}} \leq\\ &(\int_0^t \mathcal{R}(t-s)\ (\varepsilon(\mathbf{u}_\rho(s)) - \varepsilon(\mathbf{u}(s))\ ds, \varepsilon(\mathbf{u}(t)) - \varepsilon(\mathbf{u}_\rho(t)))_{\mathcal{H}}\\ &+(P_\rho \mathbf{u}_\rho(t) - P\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_\rho(t))_V + j_\rho(\mathbf{u}_\rho(t), \mathbf{u}(t)) - j_\rho(\mathbf{u}_\rho(t), \mathbf{u}_\rho(t))\\ &+j(\mathbf{u}(t), \mathbf{u}_\rho(t)) - j(\mathbf{u}(t), \mathbf{u}(t)) + (\mathbf{f}_\rho(t) - \mathbf{f}(t), \mathbf{u}_\rho(t) - \mathbf{u}(t))_V,\end{aligned} \tag{5.12}$$

First, from assumptions (3.12) (c) and (2.6), it follows that

$$(\mathcal{A}\varepsilon(\mathbf{u}_\rho(t)) - \mathcal{A}\varepsilon(\mathbf{u}(t)), \varepsilon(\mathbf{u}_\rho(t)) - \varepsilon(\mathbf{u}(t)))_{\mathcal{H}} \geq m_{\mathcal{A}} \parallel \mathbf{u}_\rho(t) - \mathbf{u}(t) \parallel_V^2 . \tag{5.13}$$

Also,

$$\begin{aligned}&(\int_0^t \mathcal{R}(t-s)\ (\varepsilon(\mathbf{u}_\rho(s)) - \varepsilon(\mathbf{u}(s)))\ ds, \varepsilon(\mathbf{u}(t)) - \varepsilon(\mathbf{u}_\rho(t)))_{\mathcal{H}} \\ &\leq (\int_0^t \parallel \mathcal{R}(t-s)\ (\varepsilon(\mathbf{u}_\rho(s)) - \varepsilon(\mathbf{u}(s)))\ \parallel_{\mathcal{H}} ds) \parallel \varepsilon(\mathbf{u}(t)) - \varepsilon(\mathbf{u}_\rho(t)) \parallel_{\mathcal{H}} .\end{aligned}$$

Therefore, (2.5) and (2.6) yield

$$\begin{aligned}&(\int_0^t \mathcal{R}(t-s)\ (\varepsilon(\mathbf{u}_\rho(s)) - \varepsilon(\mathbf{u}(s)))\ ds, \varepsilon(\mathbf{u}(t)) - \varepsilon(\mathbf{u}_\rho(t)))_{\mathcal{H}} \\ &\leq d\ (\int_0^t \parallel \mathcal{R}(t-s) \parallel_{\mathcal{H}_\infty} \ \parallel \mathbf{u}_\rho(s) - \mathbf{u}(s) \parallel_V ds) \parallel \mathbf{u}(t) - \mathbf{u}_\rho(t) \parallel_V .\end{aligned}$$

Hence,

$$\begin{aligned}&(\int_0^t \mathcal{R}(t-s)\ (\varepsilon(\mathbf{u}_\rho(s)) - \varepsilon(\mathbf{u}(s)))\ ds, \varepsilon(\mathbf{u}(t)) - \varepsilon(\mathbf{u}_\rho(t)))_{\mathcal{H}} \\ &\leq d\ \max_{z\in[0,T]} \parallel \mathcal{R}(z) \parallel_{\mathcal{H}_\infty} (\int_0^t \ \parallel \mathbf{u}_\rho(s) - \mathbf{u}(s) \parallel_V ds) \parallel \mathbf{u}(t) - \mathbf{u}_\rho(t) \parallel_V .\end{aligned} \tag{5.14}$$

Next, we use (5.3) and (3.30) to see that

$$\begin{aligned}&(P_\rho \mathbf{u}_\rho(t) - P\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_\rho(t))_V = \int_{\Gamma_3} (p_{\nu\rho}(u_{\rho\nu}(t)) - p_\nu(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t))\ da \\ &= \int_{\Gamma_3} (p_{\nu\rho}(u_{\rho\nu}(t)) - p_{\nu\rho}(u_\nu(t)) + p_{\nu\rho}(u_\nu(t)) - p_\nu(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t))\ da.\end{aligned} \tag{5.15}$$

We write,

$$\begin{aligned}&(p_{\nu\rho}(u_{\rho\nu}(t)) - p_{\nu\rho}(u_\nu(t)) + p_{\nu\rho}(u_\nu(t)) - p_\nu(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t)) \\ &= (p_{\nu\rho}(u_{\rho\nu}(t)) - p_{\nu\rho}(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t)) + (p_{\nu\rho}(u_\nu(t)) - p_\nu(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t)).\end{aligned}$$

From (3.14)(b), the first term of this sum is non positive; hence

$$\begin{aligned}&(p_{\nu\rho}(u_{\rho\nu}(t)) - p_{\nu\rho}(u_\nu(t)) + p_{\nu\rho}(u_\nu(t)) - p_\nu(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t)) \\ &\leq (p_{\nu\rho}(u_\nu(t)) - p_\nu(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t)).\end{aligned} \tag{5.16}$$

From (5.15)-(5.16), we deduce that

$$\begin{aligned}(P_\rho \mathbf{u}_\rho(t) - P\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_\rho(t))_V \quad &\leq \int_{\Gamma_3} (p_{\nu\rho}(u_\nu(t)) - p_\nu(u_\nu(t)))\ (u_\nu(t) - u_{\rho\nu}(t))\ da \\ &\leq \int_{\Gamma_3} |p_{\nu\rho}(u_\nu(t)) - p_\nu(u_\nu(t))|\ |u_\nu(t) - u_{\rho\nu}(t)|\ da.\end{aligned} \tag{5.17}$$

Now we use (5.8) (a) and (5.17) to obtain

$$(P_\rho \mathbf{u}_\rho(t) - P\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_\rho(t))_V \leq \int_{\Gamma_3} G_\nu(\rho)(|u_\nu(t)| + q)\ |u_\nu(t) - u_{\rho\nu}(t)|\ da,$$

and by using (2.7), after an elementary computation, we find that

$$\begin{aligned}&(P_\rho \mathbf{u}_\rho(t) - P\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_\rho(t))_V \\ &\leq G_\nu(\rho)\ c_0^2\ \parallel \mathbf{u}(t) \parallel_V \parallel \mathbf{u}_\rho(t) - \mathbf{u}(t) \parallel_V + G_\nu(\rho)\ q\ mes(\Gamma_3)^{1/2}\ c_0 \parallel \mathbf{u}_\rho(t) - \mathbf{u}(t) \parallel_V ,\end{aligned}$$

which yields

$$\begin{aligned}&(P_\rho \mathbf{u}_\rho(t) - P\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_\rho(t))_V \\ &\leq G_\nu(\rho)(c_0^2 \parallel \mathbf{u}(t) \parallel_V + c_0\, q\ mes(\Gamma_3)^{1/2}) \parallel \mathbf{u}_\rho(t) - \mathbf{u}(t) \parallel_V .\end{aligned} \tag{5.18}$$

On the other hand, definitions (5.5) and (3.32) imply

$$\begin{aligned}&j_\rho(\mathbf{u}_\rho(t),\mathbf{u}(t)) - j_\rho(\mathbf{u}_\rho(t),\mathbf{u}_\rho(t)) + j(\mathbf{u}(t),\mathbf{u}_\rho(t)) - j(\mathbf{u}(t),\mathbf{u}(t))\\ &= \int_{\Gamma_3} (p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_\nu(t)))(\| \mathbf{u}_\tau(t) \| - \| \mathbf{u}_{\rho\tau}(t) \|)\, da\\ &\le \int_{\Gamma_3} |p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_\nu(t))| \; |\, \| \mathbf{u}_\tau(t) \| - \| \mathbf{u}_{\rho\tau}(t) \| \,|\, da,\end{aligned}$$

which yields by using (2.2),

$$\begin{aligned}&j_\rho(\mathbf{u}_\rho(t),\mathbf{u}(t)) - j_\rho(\mathbf{u}_\rho(t),\mathbf{u}_\rho(t)) + j(\mathbf{u}(t),\mathbf{u}_\rho(t)) - j(\mathbf{u}(t),\mathbf{u}(t))\\ &\le \int_{\Gamma_3} |p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_\nu(t))| \; \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \| \; da.\end{aligned} \tag{5.19}$$

We now use (5.8)(a) and (3.15)(b) to see that

$$\begin{aligned}&|p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_\nu(t))| = |p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_{\rho\nu}(t)) + p_\tau(u_{\rho\nu}(t)) - p_\tau(u_\nu(t))|\\ &\le |p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_{\rho\nu}(t))| + |p_\tau(u_{\rho\nu}(t)) - p_\tau(u_\nu(t))|\\ &\le G_\tau(\rho)(|u_{\rho\nu}(t)| + q) + L_\tau\, |u_{\rho\nu}(t) - u_\nu(t)|.\end{aligned} \tag{5.20}$$

Thus, (5.20) and (2.2) yield

$$\begin{aligned}&\int_{\Gamma_3} |p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_\nu(t))| \; \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \| \; da\\ &\le \int_{\Gamma_3} G_\tau(\rho)(\| \mathbf{u}_{\rho\nu}(t) \| + q) \; \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \| \; da + \int_{\Gamma_3} L_\tau \; \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|^2 \; da.\end{aligned}$$

By an elementary calculation, we find from the last inequality,

$$\begin{aligned}&\int_{\Gamma_3} |p_{\tau\rho}(u_{\rho\nu}(t)) - p_\tau(u_\nu(t))| \; \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \| \; da\\ &\le G_\tau(\rho)\, c_0^2 \, \| \mathbf{u}_\rho(t) \|_V \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \|_V \; + G_\tau(\rho)\, c_0\, q\, mes(\Gamma_3)^{1/2} \; \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \|_V\\ &+ L_\tau\, c_0^2 \, \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V^2 \; .\end{aligned} \tag{5.21}$$

We deduce from (5.19) and (5.21) that

$$\begin{aligned}&j_\rho(\mathbf{u}_\rho(t),\mathbf{u}(t)) - j_\rho(\mathbf{u}_\rho(t),\mathbf{u}_\rho(t)) + j(\mathbf{u}(t),\mathbf{u}_\rho(t)) - j(\mathbf{u}(t),\mathbf{u}(t))\\ &\le G_\tau(\rho)\, (c_0^2 \, \| \mathbf{u}_\rho(t) \|_V + c_0\, q\, mes(\Gamma_3)^{1/2}) \, \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \|_V + L_\tau\, c_0^2 \, \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V^2 \; .\end{aligned} \tag{5.22}$$

Finally, we note that

$$\begin{aligned}&(\mathbf{f}_\rho(t) - \mathbf{f}(t), \mathbf{u}_\rho(t) - \mathbf{u}(t))_V \le \| \mathbf{f}_\rho(t) - \mathbf{f}(t) \|_V \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V\\ &\le \delta_\rho \, \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V,\end{aligned}$$

where

$$\delta_\rho = \max_{z\in[0,T]} \| \mathbf{f}_\rho(z) - \mathbf{f}(z) \|_V \tag{5.23}$$

We combine (5.12), (5.13), (5.14), (5.18), (5.22) and (5.23) to deduce that

$$\begin{aligned}&m_{\mathcal{A}} \, \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V^2\\ &\le d\, \max_{z\in[0,T]} \| \mathcal{R}(z) \|_{\mathcal{H}_\infty} \, \Big(\int_0^t \| \mathbf{u}_\rho(s) - \mathbf{u}(s) \|_V \, ds\Big) \, \| (\mathbf{u}(t) - \mathbf{u}_\rho(t)) \|_V\\ &+ G_\nu(\rho)\, (c_0^2 \, \| \mathbf{u}(t) \|_V + c_0\, q\, mes(\Gamma_3)^{1/2}) \, \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V\\ &+ G_\tau(\rho)\, (c_0^2 \, \| \mathbf{u}_\rho(t) \|_V + c_0\, q\, mes(\Gamma_3)^{1/2}) \, \| \mathbf{u}(t) - \mathbf{u}_\rho(t) \|_V + L_\tau\, c_0^2 \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V^2\\ &+ \delta_\rho \, \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V \; .\end{aligned}$$

Since $m_{\mathcal{A}} > L_\tau\, c_0^2$, we obtain

$$\begin{aligned} &\| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V \\ &\leq \frac{d\, \max_{z\in[0,T]} \| \mathcal{R}(z) \|_{\mathcal{H}_\infty}}{m_{\mathcal{A}} - L_\tau\, c_0^2} \int_0^t \| \mathbf{u}_\rho(s) - \mathbf{u}(s) \|_V\, ds \\ &+ \frac{G_\nu(\rho)}{m_{\mathcal{A}} - L_\tau\, c_0^2} (c_0^2 \| \mathbf{u}(t) \|_V + c_0\, q\, mes(\Gamma_3)^{1/2}) \\ &+ \frac{G_\tau(\rho)}{m_{\mathcal{A}} - L_\tau\, c_0^2} (c_0^2 \| \mathbf{u}_\rho(t) \|_V + c_0\, q\, mes(\Gamma_3)^{1/2}) \\ &+ \frac{\delta_\rho}{m_{\mathcal{A}} - L_\tau\, c_0^2}. \end{aligned} \tag{5.24}$$

Let $\xi = \max(\xi_u, \xi_{u_\rho}, \dfrac{1}{m_{\mathcal{A}} - L_\tau\, c_0^2})$ where

$$\xi_u = \frac{1}{m_{\mathcal{A}} - L_\tau\, c_0^2} (c_0^2 \max_{z\in[0,T]} \| \mathbf{u}(z) \|_V + c_0\, q\, mes(\Gamma_3)^{1/2}),$$

$$\xi_{u_\rho} = \frac{1}{m_{\mathcal{A}} - L_\tau\, c_0^2} (c_0^2 \max_{z\in[0,T]} \| \mathbf{u}_\rho(z) \|_V + c_0\, q\, mes(\Gamma_3)^{1/2}).$$

Then, (5.24) implies

$$\| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V \leq (G_\nu(\rho) + G_\tau(\rho) + \delta_\rho)\xi + \frac{d\, \max_{z\in[0,T]} \| \mathcal{R}(z) \|_{\mathcal{H}_\infty}}{m_{\mathcal{A}} - L_\tau\, c_0^2} \int_0^t \| \mathbf{u}_\rho(s) - \mathbf{u}(s) \|_V\, ds.$$

Using Gronwell's inequality, we obtain

$$\| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V \leq (G_\nu(\rho) + G_\tau(\rho) + \delta_\rho)\, \xi\, e^{\dfrac{d\, \| \mathcal{R} \|_{C(0,T;\mathcal{H}_\infty)}}{m_{\mathcal{A}} - L_\tau\, c_0^2} t},$$

and, by using the inequality $e^{ct} \leq e^{cT}$ where $t \in [0, T]$, we infer that

$$\max_{t\in[0,T]} \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V \leq (G_\nu(\rho) + G_\tau(\rho) + \delta_\rho)\, \xi\, e^{\dfrac{d\, \| \mathcal{R} \|_{C(0,T;\mathcal{H}_\infty)}}{m_{\mathcal{A}} - L_\tau\, c_0^2} T}. \tag{5.25}$$

From assumptions (5.8)(b), (5.9), (5.10) and the definition (5.23), we have

$$G_\nu(\rho) \longrightarrow 0\,,\ G_\tau(\rho) \longrightarrow 0\,,\ \delta_\rho \longrightarrow 0 \text{ when } \rho \longrightarrow 0. \tag{5.26}$$

By combining the convergence (5.26) with the inequality (5.25), we obtain

$$\max_{t\in[0,T]} \| \mathbf{u}_\rho(t) - \mathbf{u}(t) \|_V \longrightarrow 0 \text{ when } \rho \longrightarrow 0. \tag{5.27}$$

Finally, from (5.27) and the definition (2.9), we conclude that the convergence (5.11) is satisfied. □

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