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Group A-cordial labeling of some spiders

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Abstract Let A denote the multiplicative group $\{1, -1, i, -i\}$. In this paper, we prove that every spider with each leg of length at least 2 is group A-cordial, and we have given an explicit group A-cordial labeling for the same. To prove this main result, we have introduced the notion of uniform spider modulo 4. We also prove that a certain type of spider having at least one leg of length 1 is not group A-cordial. In particular, we show that the spider, $S(\underbrace{2, 2, \ldots, 2}_{\alpha-\text{times}}, \underbrace{1, 1, \ldots, 1}_{\beta-\text{times}})$ is not group A-cordial if and only if $\beta \geq 2\alpha + 6$, where $\alpha, \beta \in \mathbb{N}$.

1 Introduction

Cordial labelings were introduced by Cahit [3] in 1987 as a weakened version of graceful labelings.

Definition 1.1 (Cahit[3]). Let $f : V(G) \to \{0, 1\}$ be any function. For each edge xy assign the label $|f(x) - f(y)|$. Let $v_f(0), v_f(1)$ denote the number of vertices in G with the label 0 and 1 respectively. Let $e_f(0), e_f(1)$ denote the number of edges in G with the label 0 and 1 respectively. The function f is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. The graph G is called *cordial graph* if it admits cordial labeling.

Definition 1.2. Let $G = (V, E)$ be a graph. A bijection $f : V \to \{1, 2, \ldots, |V|\}$ is called a prime labeling if for each $e = \{u, v\} \in E$, we have $\gcd(f(u), f(v)) = 1$. A graph that admits a prime labeling is called a *prime graph*.

Several variations of cordial labelings have been studied. (See [11], [12], [13], [14], [15], [16]). Motivated by these labelings, the notion of group cordial labeling of graphs was introduced by M. K. Karthik Chidambaram, S. Athisayanathan and R. Ponraj (See [1],[2],[9],[6]). For group $A = \{1, -1, i, -i\}$ the labeling for different graphs was considered by them. Further, for the group $A = S_3$, group A-cordial labeling in this case was studied by B. Chandra and R. Kala [4], [5], [6].
In this paper, we introduce the notion of a uniform spider modulo 4 and give a group A-cordial labeling for this graph as well as for any spider with legs of length at least 2, where the group considered is $A = \{1, -1, i, -i\}$.

2 Preliminaries

We begin with some necessary definitions and preliminaries in the graph labelings.

Definition 2.1 (Gallian [7]). A *spider* is a tree that has at most one vertex (called the center) of degree greater than 2. Every path from the center to a pendant, a vertex is called a *leg* of the spider.

Definition 2.2. A *uniform spider* is defined as a spider with t legs of length n each. We denote such a spider by $S(n, t)$. (See [17]).

Definition 2.3. A *modulo 4 uniform spider* is a spider with every leg of the same length modulo 4. We denote it by $S(n_1, n_2, \ldots, n_t; x)$, where $n_1 \geq n_2 \geq \ldots \geq n_t$; here $n_j, 1 \leq j \leq t$ is the number of edges on the j^{th} leg and $n_j = 4k_j + x$, where x is $0, 1, 2$ or 3. Also $V(S(n_1, n_2, \ldots, n_t; x)) = \cup_{j=1}^{t} \cup_{i=1}^{n_j} \{v_i^j\} \cup \{v\}$; here v is called the *central vertex*.

Clearly $|V(S(n_1, n_2, \ldots, n_t; x))| = 4\sum_{j=1}^{t} k_j + xt + 1$ and $|E(S(n_1, n_2, \ldots, n_t; x))| = 4\sum_{j=1}^{t} k_j + xt$.
Note that every uniform spider is a modulo 4 uniform spider.

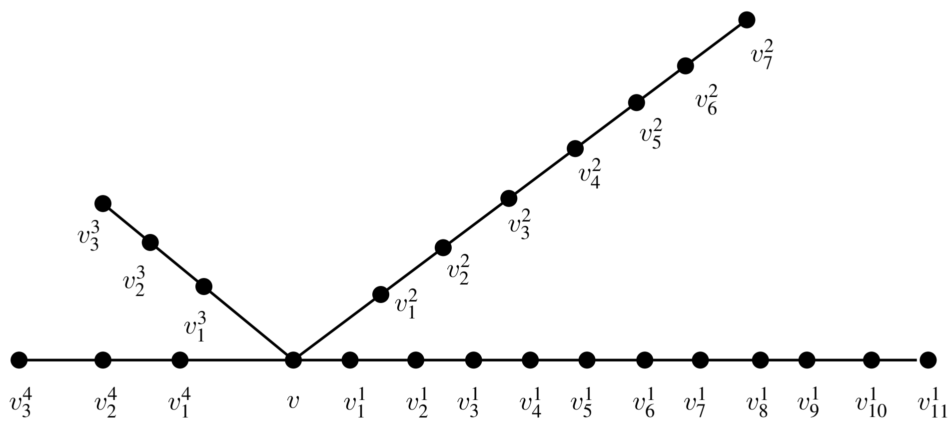

Figure 1. An example of 3 modulo 4 uniform spider $S(11, 7, 3, 3; 3)$

Definition 2.4. Let G be a graph and let A be a group. Let $o(a)$ denote the order of an element $a \in A$. Let $f : V(G) \to A$ be a function. For each edge uv, assign the label 1 if $\gcd\Big(o(f(u)), o(f(v))\Big) = 1$ and 0 otherwise. Let $v_f(a)$ denote the number of vertices in G labeled by f with the element a of the group A. Let $e_f(0), e_f(1)$ denote the number of edges with the label 0 and 1 respectively. The function f is called a *group A-cordial* labeling if $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a *group A-cordial* labeling is called group *A-cordial*.

Throughout this paper, the group under consideration will be $A = \{1, -1, i, -i\}$ with respect to multiplication of complex numbers.

3 Group A cordiality of modulo 4 uniform spider

A modulo 4 uniform spider can be of four different types depending on the length of the arms modulo 4. In what follows, we obtain a labeling to prove that the modulo 4 uniform spider is group A-cordial for each of the above types. At the end of this theorem, we have given a compendium of labelings. Whilst one labeling is enough to prove the result, we have given multiple labelings in some cases and certain other cases we have even given labelings that violate the conditions of cordiality either concerning the edges or vertices; however these will prove useful to us in the sequel when we prove the next result.

Theorem 3.1. *All modulo* 4 *uniform spiders with legs of length at least* 2 *are group A-cordial for all* $t \geq 2$.

Proof. Let $S(n_1, n_2, \ldots, n_t; x)$ be a modulo 4 uniform spider. For brevity, we denote this graph by G. Let $f : V(G) \to A$ be a labeling as defined in the sequel below:
The terms $v_f(a), e_f(0), e_f(1)$ are as defined above. We label the vertices of G in at most three stages. In stage 1, we label the central vertex v and the first $4k_j$ vertices on each leg. In stage 2, we cover the remaining vertices from the first leg onwards by grouping them into sets of 4 or 2 legs as per the requirement. In stage 3, we label the remaining vertices on the legs remaining after labeling of stage 2. Let $e_{f'}(0), e_{f'}(1)$ be the number of edges with the label 0 and 1 in the

first stage of the labeling; $e_{f''}(0), e_{f''}(1)$ be the number of edges with the label 0 and 1 in the second stage of the labeling and so on. Thus $v_f(a) = v_{f'}(a) + v_{f''}(a) + v_{f'''}(a), a \in A; e_f(0) = e_{f'}(0) + e_{f''}(0) + e_{f'''}(0)$ and $e_f(1) = e_{f'}(1) + e_{f''}(1) + e_{f'''}(1)$.
We now proceed to the labeling of vertices in stage 1:

Let $f(v) = -1$ and $f(v_r^j) = \begin{cases} 1, & \text{if } r \equiv 1 \pmod 4; \\ i, & \text{if } r \equiv 2 \pmod 4; \\ -i, & \text{if } r \equiv 3 \pmod 4; \\ -1, & \text{if } r \equiv 0 \pmod 4, \end{cases}$

that is on the j^{th} leg we label the first $4k_j$ vertices (taken in ascending order of their distance from the central vertex) as $1, i, -i, -1, \ldots, 1, i, -i, -1$. Then $v_{f'}(-1) = \sum_{j=1}^{t} k_j + 1, v_{f'}(1) = v_{f'}(i) = v_{f'}(-i) = \sum_{j=1}^{t} k_j$. Also $e_{f'}(0) = e_{f'}(1) = 2\sum_{j=1}^{t} k_j$(see Figure 2 for illustration).
In what follows, for convenience we denote $\lfloor \frac{|V(G)|}{4} \rfloor$ by d.

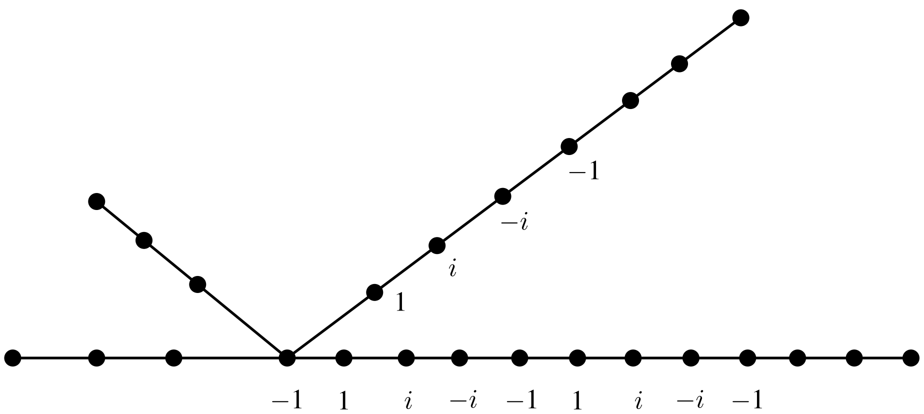

Figure 2. Stage 1 of labeling of $S(11, 7, 3, 3; 3)$

Stage 2 onwards, the labeling is dependent on x and t; hence based on the case, the resulting labeling f will be different in each case. We denote these different labelings obtained by $f_1, f_2, \ldots, f_{27}$. Depending on the values of x, we make four cases accordingly.
Case 1: $x = 0$
After the labeling of stage 1, there are no vertices remaining on any leg to be labeled in stage 2. Thus in this case, we get a labeling f for $V(G)$ such that $v_f(-1) = d + 1$. Also $v_f(1) = v_f(i) = v_f(-i) = d$ and $e_f(0) = e_f(1) = 2d$. The resulting labeling 'f' in this case will be referred to as f_1 henceforth.
Case 2: $x = 1$
We note here that since $n \geq 2$, each leg has at least 5 vertices and after the labeling of stage 1, precisely one vertex remains on each leg for labeling. Let $t = 4m + y$, where y is $0, 1, 2$ or 3. In stage 2 of the labeling, we label the pendant vertices on $4m$ legs as defined below. The remaining pendant vertices on the y legs will be labeled in stage 3. We label the pendant vertices on the first m legs by -1, next m legs by i, and the third set of m legs by $-i$. That means

$$\begin{aligned} f(v_{4k_j+1}^j) &= -1, && 1 \leq j \leq m; \\ f(v_{4k_j+1}^j) &= i, && m < j \leq 2m; \\ f(v_{4k_j+1}^j) &= -i, && 2m < j \leq 3m. \end{aligned}$$

For $3m < j \leq 4m$, we disturb the labeling of stage 1 as follows:
In stage 1, we had $f(v_{4k_j}^j) = -1$; we now define $f(v_{4k_j}^j) = 1$ and $f(v_{4k_j+1}^j) = -1$. Due to this, $e_{f'}(0)$ is reduced by m and $e_{f'}(1)$ is increased by m. Also $v_{f''}(-1) = v_{f''}(1) = v_{f''}(i) = v_{f''}(-i) = m, e_{f''}(0) = 3m, e_{f''}(1) = m$. Now $e_{f'}(0) + e_{f''}(0) = \left[2\sum_{j=1}^{t} k_j - m\right] + 3m = 2\sum_{j=1}^{t} k_j + 2m$ and $e_{f'}(1) + e_{f''}(1) = \left[2\sum_{j=1}^{t} k_j + m\right] + m = 2\sum_{j=1}^{t} k_j + 2m$. Hence the edge labels are balanced so far, that is, exactly half of the edges labeled so far have received the

label 0 and half the label 1.
Now y legs remain with one vertex on each to be labeled. We label these in stage 3 by making subcases as described below:
Subcase 2.1: $y = 0$
No legs remain in this case and hence there are no vertices left for labeling in stage 3. The labeling that we have obtained so far yields $v_f(-1) = d+1, v_f(1) = v_f(i) = v_f(-i) = d$ and $e_f(0) = e_f(1) = 2d$. This labeling will be denoted by f_2.
Subcase 2.2: $y = 1$
One leg now remains with precisely one vertex on it to be labeled. We can label this vertex in one of the following ways:

(i) Define $f(v^t_{4k_t+1}) = 1$. Thus in this case there is one vertex left for labeling in stage 3. Thus $v_{f'''}(1) = 1, v_{f'''}(-1) = v_{f'''}(i) = v_{f'''}(-i) = 0$ and $e_{f'''}(0) = 0, e_{f'''}(1) = 1$. Hence $v_f(-1) = v_f(1) = d+1, v_f(i) = v_f(-i) = d$ and $e_f(1) = e_f(0)+1$. This labeling will be denoted by f_3.

(ii) Define $f(v^t_{4k_t+1}) = i$. Accordingly, we have $v_f(-1) = v_f(i) = d+1, v_f(1) = v_f(-i) = d$ and $e_f(0) = e_f(1)+1$. This labeling is denoted by f_4.

(iii) Define $f(v^t_{4k_t+1}) = -i$. Accordingly, we have $v_f(-1) = v_f(-i) = d+1, v_f(1) = v_f(i) = d$ and $e_f(0) = e_f(1)+1$. This labeling will be denoted by f_5.

Subcase 2.3: $y = 2$
There now remain only 2 legs with one vertex on each to be labeled. We label the last vertex on one leg with 1 and the other leg with i or $-i$; accordingly we get two labelings as below:

(i) $v_f(-1) = v_f(1) = v_f(i) = d+1, v_f(-i) = d, e_f(0) = e_f(1)$. This labeling will be denoted by f_6 (see Figure 3 for illustration).

(ii) $v_f(-1) = v_f(1) = v_f(-i) = d+1, v_f(i) = d, e_f(0) = e_f(1)$. This labeling will be denoted by f_7.

We also define two more labelings as given below for this case.
We label the last vertex on the last leg as 1 and switch the labels of the last two vertices that is we put $f(v^t_{n_t-1}) = 1$ and $f(v^t_{n_t}) = -1$; while on the earlier leg, we label the last vertex with either i or $-i$. We accordingly get two labelings as follows:

(i) $v_f(-1) = v_f(1) = v_f(i) = d+1, v_f(-i) = d$ and $e_f(1) = e_f(0)+2$. We denote this labeling by f_8.

(ii) $v_f(-1) = v_f(1) = v_f(-i) = d+1, v_f(i) = d$ and $e_f(1) = e_f(0)+2$. We denote this labeling by f_9.

Note that the labelings f_8 and f_9 violate the condition of cordiality for the edge labels hence they are not group A-cordial.
Subcase 2.4: $y = 3$
There now remain 3 legs with one vertex on each to be labeled.
We label the last vertex on one leg by 1, on one leg by i and one leg by $-i$.
Therefore $v_f(-1) = v_f(1) = v_f(i) = v_f(-i) = d$ and $e_f(0) = e_f(1)+1$. This labeling will be denoted by f_{10}.
We define three additional labelings in this case as follows:
We label the last vertex on one leg by -1, on one leg by 1 and one leg by i.
Therefore $v_f(-1) = d+1, v_f(1) = v_f(i) = d, v_f(-i) = d-1$ and $e_f(0) = e_f(1)+1$. This labeling will be denoted by f_{11}.
We label the last vertex on one leg by -1, on one leg by 1 and one leg by $-i$.
Therefore $v_f(-1) = d+1, v_f(1) = v_f(-i) = d, v_f(i) = d-1$ and $e_f(0) = e_f(1)+1$. This labeling will be denoted by f_{12}.
We label the last vertex on one leg by -1, on one leg by i and one leg by $-i$.
Therefore $v_f(-1) = d+1, v_f(1) = d-1, v_f(i) = v_f(-i) = d$ and $e_f(0) = e_f(1)+3$. This labeling will be denoted by f_{13}.
Case 3: $x = 2$
There are precisely two vertices remaining to be labeled on each leg after stage 1. Let $t = 2m+y$.

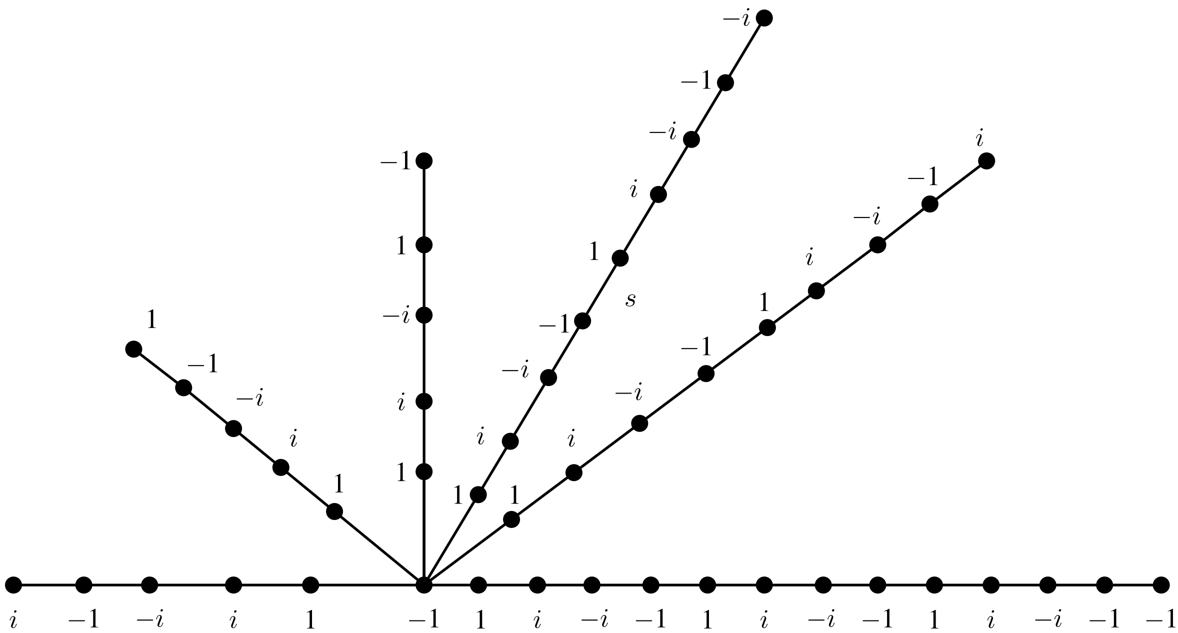

Figure 3. $S(13,9,9,5,5,5;1); t=6, m=1, y=2$, illustration of labeling f_6

In stage 2 of the labeling, we label the pendant vertices on $2m$ legs as defined below. We label the last two vertices on m of the legs as $1, i$ and on m of the legs as $-1, -i$ that is

$$f(v^j_{4k_j+1}) = 1, f(v^j_{4k_j+2}) = i; 1 \le j \le m,$$
$$f(v^j_{4k_j+1}) = -1, f(v^j_{4k_j+2}) = -i; m < j \le 2m.$$

Thus $v_{f''}(-1) = v_{f''}(1) = v_{f''}(i) = v_{f''}(-i) = m$ and $e_{f''}(0) = e_{f''}(1) = 2m$.
We are now left with y legs on which the last two vertices require labeling. We make two subcases accordingly:
Subcase 3.1: $y = 0$
There are no vertices remaining in this case and we have $v_f(1) = v_f(i) = v_f(-i) = d, v_f(-1) = d+1$ and $e_f(0) = e_f(1)$. This labeling will be denoted by f_{14}.
Subcase 3.2: $y = 1$
We label the last two vertices on the last leg as $i, 1$ or $-i, 1$. Accordingly, we get 2 labelings:

(i) $v_f(-1) = v_f(1) = v_f(i) = d+1, v_f(-i) = d$ and $e_f(0) = e_f(1)$. This labeling will be denoted by f_{15}.

(ii) $v_f(-1) = v_f(1) = v_f(-i) = d+1, v_f(i) = d$ and $e_f(0) = e_f(1)$. This labeling will be denoted by f_{16}.

We define an additional labeling in this case as below.
We label the last two vertices on the last leg as $1, i$. This gives $v_f(-1) = v_f(1) = v_f(i) = d+1$, $v_f(-i) = d$ and $e_f(0) + 2 = e_f(1)$. This labeling will be denoted by f_{17}.
Case 4: $x = 3$
There now remain 3 vertices on each leg yet to be labeled. Let $t = 4m + y; y = 0, 1, 2, 3$. On m of these legs, label the last 3 vertices as $1, i, -i$ in that order, that is, $f(v^j_{4k_j+1}) = 1, f(v^j_{4k_j+2}) = i, f(v^j_{4k_j+3}) = -i$; for $1 \le j \le m$.
On m of these legs label the last 3 vertices as $1, i, -1$;
On m of these legs label the last 3 vertices as $1, -i, -1$; and
On m of these legs label the last 3 vertices as $i, -i, -1$; then $v_{f''}(-1) = v_{f''}(1) = v_{f''}(i) = v_{f''}(-i) = 3m$ and $e_{f''}(0) = e_{f''}(1) = 6m$.
We are now left with y legs on which the last three vertices need to be labeled. We make subcases accordingly:

Subcase 4.1: $y = 0$
There are no vertices remaining in this case and we have $v_f(-1) = d + 1, v_f(1) = v_f(i) = v_f(-i) = d$ and $e_f(0) = e_f(1)$. This labeling will be denoted by f_{18}.
Subcase 4.2: $y = 1$
In stage 3, we label the 3 remaining vertices on the last leg as $1, i, -i$ or $i, -i, 1$. Accordingly we get 2 labelings as follows:

(i) $v_f(-1) = v_f(1) = v_f(i) = v_f(-i) = d$ and $e_f(1) = e_f(0) + 1$. This labeling will be denoted by f_{19}.

(ii) $v_f(-1) = v_f(1) = v_f(i) = v_f(-i)$ and $e_f(0) = e_f(1) + 1$. This labeling will be denoted by f_{20}.

Subcase 4.3: $y = 2$
There now remain two legs with the last three vertices on each to be labeled. On these 2 legs, we label the vertices respectively as $1, i, -i$ on one leg and on the other $-1, i, 1$ (or $i, -i, 1$ or $-1, -i, 1$). Accordingly, we get the following:

(i) $v_f(-1) = v_f(1) = v_f(i) = d + 1, v_f(-i) = d$ and $e_f(0) = e_f(1)$. This labeling will be denoted by f_{21}.

(ii) $v_f(-1) = d, v_f(1) = v_f(i) = v_f(-i) = d + 1$ and $e_f(0) = e_f(1)$. This labeling will be denoted by f_{22}.

(iii) $v_f(-1) = v_f(1) = v_f(-i) = d + 1, v_f(i) = d$ and $e_f(0) = e_f(1)$. This labeling will be denoted by f_{23}.

We give additional labeling in this case as follows:
Label the last three vertices on one leg as $1, i, -i$ and $i, -i, -1$ on the other. This gives $v_f(-1) = v_f(i) = v_f(-i) = d+1, v_f(1) = d$ and $e_f(0) = e_f(1)+2$. This labeling will be denoted by f_{24}.
Subcase 4.4: $y = 3$
There now remain three legs with three vertices on each to be labeled. On two of the legs, we label the last three vertices as $1, i, -i$, and on the third leg we label it as $-1, -1, 1$ or $-1, -1, i$ or $-1, -1, -i$. Accordingly, we get four labelings as given below:

(i) $v_f(-1) = v_f(1) = d + 1, v_f(i) = v_f(-i) = d$ and $e_f(0) + 1 = e_f(1)$. This labeling will be denoted by f_{25}.

(ii) $v_f(-1) = v_f(i) = d + 1, v_f(1) = v_f(-i) = d$ and $e_f(0) = e_f(1) + 1$. This labeling will be denoted by f_{26}.

(iii) $v_f(-1) = v_f(-i) = d + 1, v_f(1) = v_f(i) = d$ and $e_f(0) = e_f(1) + 1$. This labeling will be denoted by f_{27}.

Thus in every case we have given a labeling which proves that modulo 4 uniform spider is group A-cordial. □

In some cases we have also provided alternate labelings and this complete information is listed in a tabular form as given below in Table 1. The relation between the corresponding vertex labels and edge labels for each label has been shown alongside it. Also, we state the conditions of cordiality satisfied by the vertex labels and edge labels for each of the functions in terms of parity. The stars shown alongside some of the functions in column 1, indicate that the condition of cordiality is violated for that function either for the vertices or the edges.

Table 1. List of functions

f_k	x	t	$v_{f_k}(-1)$	$v_{f_k}(1)$	$v_{f_k}(i)$	$v_{f_k}(-i)$	$e_f(0)$	$e_f(1)$
f_1	0	For all t	1	0	0	0	0	0
f_2	1	$t \equiv 0 \pmod 4$	1	0	0	0	0	0
f_3		$t \equiv 1 \pmod 4$	1	1	0	0	0	1
f_4			1	0	1	0	1	0
f_5			1	0	0	1	1	0
f_6		$t \equiv 2 \pmod 4$	1	1	1	0	0	0
f_7			1	1	0	1	0	0
f_8*			1	1	1	0	0	2
f_9*			1	1	0	1	0	2
f_{10}		$t \equiv 3 \pmod 4$	0	0	0	0	1	0
$f_{11}*$			1	0	0	-1	1	0
$f_{12}*$			1	0	-1	0	1	0
$f_{13}*$			1	-1	0	0	3	0
f_{14}	2	$t \equiv 0 \pmod 4$	1	0	0	0	0	0
f_{15}		$t \equiv 1 \pmod 4$	1	1	1	0	0	0
f_{16}			1	1	0	1	0	0
$f_{17}*$			1	1	1	0	0	2
f_{18}	3	$t \equiv 0 \pmod 4$	1	0	0	0	0	0
f_{19}		$t \equiv 1 \pmod 4$	0	0	0	0	0	1
f_{20}			0	0	0	0	1	0
f_{21}		$t \equiv 2 \pmod 4$	1	1	1	0	0	0
f_{22}			0	1	1	1	0	0
f_{23}			1	1	0	1	0	0
$f_{24}*$			1	0	1	1	2	0
f_{25}		$t \equiv 3 \pmod 4$	1	1	0	0	0	1
f_{26}			1	0	1	0	1	0
f_{27}			1	0	0	1	1	0

Theorem 3.2. *All spiders with legs of length at least* 2 *are group A-cordial.*

Proof. Let G be a spider with every leg of length at least 2. Observe that G is a one-point union of four graphs G_0, G_1, G_2, G_3; where G_i is a modular 4 uniform spider in which each leg is of length i congruent modulo 4. Let t_i be the number of legs in the graph G_i, $i = 0, 1, 2, 3$. Depending on the values of t_0, t_1, t_2, t_3 we get 32 cases in all, and in each one of these cases, we choose an appropriate function for G_i from the library of functions available in Table 1. We then define a labeling $f : V(G) \to A$ which will agree on G_i with the chosen function for G_i.
We note in passing here, that for each of the functions given in the above library, the label for the central vertex is always -1. The choice of the function made in each case for G_i is given in Table 2.

For G_0, we choose the labeling f_1 in all cases and hence we have not included the column corresponding to G_0 in the table. Also, it can be easily verified that if a particular component G_i is missing in the spider, then the case is covered under $t_i \equiv 0 \pmod 2$, or $t_i \equiv 0 \pmod 4$ as the the case may be. □

Note that we have tackled all spiders with legs of length at least 2. There are only two types of spiders left as mentioned below:

(a) Spiders with all legs of length 1.

(b) Spiders with atleast 2 legs, with atleast 1 leg of length 1 and atleast one leg of length more than 1.

Table 2. List of functions

No.	G_1	i	G_2	i	G_3	i	$v_f(-1)$	$v_f(1)$	$v_f(i)$	$v_f(-i)$	$e_f(0)$	$e_f(1)$
1	0	2	0	14	0	18	1	0	0	0	0	0
2	0	2	0	14	1	19	1	0	0	0	1	0
3	0	2	0	14	2	21	1	1	1	0	0	0
4	0	2	0	14	3	26	1	0	1	0	1	0
5	0	2	1	15	0	18	1	1	1	0	0	0
6	0	2	1	15	1	19	1	1	1	0	1	0
7	0	2	1	15	2	23	0	1	0	0	0	0
8	0	2	1	15	3	27	0	0	0	0	1	0
9	1	3	0	14	0	18	1	1	0	0	0	1
10	1	3	0	14	1	20	0	1	0	0	0	0
11	1	4	0	14	2	23	0	0	0	0	1	0
12	1	3	0	14	3	27	1	1	0	1	0	0
13	1	4	1	16	0	18	0	0	0	0	1	0
14	1	5	1	17	1	20	0	1	1	1	0	0
15	1	5	1	15	2	21	0	1	1	0	1	0
16	1	5	1	15	3	25	0	1	0	0	0	0
17	2	6	0	14	0	18	1	1	1	0	0	0
18	2	6	0	14	1	19	1	1	1	0	1	0
19	2	6	0	14	2	22	0	1	0	0	0	0
20	2	6	0	14	3	27	0	0	0	0	1	0
21	2	6	1	16	0	18	0	1	0	0	0	0
22	2	6	1	16	1	19	0	1	0	0	1	0
23	2	8	1	16	2	24	0	1	1	1	0	0
24	2	6	1	15	3	27	0	1	1	0	1	0
25	3	10	0	14	0	18	1	1	1	0	1	0
26	3	10	0	14	1	19	0	1	1	1	0	0
27	3	10	0	14	2	21	0	1	1	0	1	0
28	3	10	0	14	3	25	0	1	0	0	0	0
29	3	10	1	15	0	18	0	1	1	0	1	0
30	3	11	1	16	1	19	0	1	0	0	0	0
31	3	13	1	17	2	23	0	0	0	0	1	0
32	3	12	1	17	3	26	1	1	1	0	0	0

A spider of type (a) is nothing but the star graph. The star graph $K_{1,n}$ is group A-cordial if and only if $n \leq 5$. See [10]. For spiders of type (b), we now prove:

Theorem 3.3. *$S(\underbrace{2,2,\dots,2}_{\alpha-\text{times}},\underbrace{1,1,\dots,1}_{\beta-\text{times}})$ is not group A-cordial if and only if $\beta \geq 2\alpha + 6$, where $\alpha, \beta \in \mathbb{N}$.*

Proof. For ease of reference, we denote the graph $S(\underbrace{2,2,\dots,2}_{\alpha-\text{times}},\underbrace{1,1,\dots,1}_{\beta-\text{times}})$ by $S(2^\alpha, 1^\beta)$.

Firstly, we will give constructions of the existence of group A labeling for the case when $\beta < 2\alpha + 6$. Let $\beta + \gamma = 2\alpha + 5$, where γ is a nonnegative integer.
Clearly $|V(G)| = 2\alpha + \beta + 1 = 4\alpha + 6 - \gamma$ and $|E(G)| = 2\alpha + \beta = 4\alpha + 5 - \gamma$.
Let $\gamma = 4k + r$, where $k \geq 0$ and $0 \leq r < 4$.
We shall prove that for each value of r, there is a group A-cordial labeling of $S(2^\alpha, 1^\beta)$. We label

the central vertex by -1 in all cases.
Case 1: $r = 0$
Now $|V(G)| = 4\alpha + 6 - 4k$ and $|E(G)| = 4\alpha + 5 - 4k$.
We begin by noting that since $\gamma \leq 2\alpha + 5$, hence $\alpha - k + 1$ is positive. Let $\alpha - k + 1$ vertices of degree 2 on arms of length 2 be labeled by 1 and all remaining vertices can be almost equitably distributed with the labels -1, i, and $-i$. Thus we have $v_f(1) = \alpha - k + 1 = \lfloor \frac{|V(G)|}{4} \rfloor + 1$ and $e_f(1) = 2\alpha - 2k + 2$, $e_f(0) = 2\alpha - 2k + 3$, as required.
Case 2: $r = 1, 2$
These two cases can be dealt with similarly.
Case 3: $r = 3$
Now $|V(G)| = 4\alpha+3-4k$ and $|E(G)| = 4\alpha+2-4k$. We label $\alpha-k$ vertices of degree 2 on arms of length 2 and 1 pendant vertex on an arm of length 1 with the label 1; all the remaining vertices almost equitably labeled with -1, i and $-i$. Thus we have $v_f(1) = \alpha - k + 1 = \lfloor \frac{|V(G)|}{4} \rfloor + 1$ and $e_f(1) = 2\alpha - 2k + 1 = e_f(0)$, as required.

Secondly, let $\beta \geq 2\alpha + 6$ that is $\beta = 2\alpha + 6 + \gamma$, where γ is a non negative integer.
Clearly $|V(G)| = 2\alpha + \beta + 1 = 4\alpha + 7 + \gamma$ and $|E(G)| = 2\alpha + \beta = 4\alpha + 6 + \gamma$.
Let $\gamma = 4k + r$, where $k \geq 0$ and $0 \leq r < 4$.
We shall prove that for each value of r, there is no A-cordial labeling of $S(2^\alpha, 1^\beta)$.
Case 1: $r = 0$
Suppose f is an A-cordial labeling of $S(2^\alpha, 1^\beta)$, then $v_f(1) = \alpha + k + 1$ or $\alpha + k + 2$ and $e_f(0) = 2\alpha + 2k + 3 = e_f(1)$.
If we label the central vertex as 1, then $e_f(1) \geq \alpha + \beta = 3\alpha + 4k + 6$, which is too large a value, hence the central vertex should not be labeled as 1.
Further, we observe that the only way to get the label 1 for an edge e is to have at least one end vertex of e labeled as 1. We can thus maximize the number of edges with the label 1 by labeling every vertex of degree 2 on each arm of length 2 with the label 1. This will result in 2α edges on arms of length 2 with the label 1.
We still need some more edges with the label 1. Thus we will have to label some of the pendant vertices on the arms of length 1 by 1. However, α vertices have already received the label 1; hence we can label at most $k + 1$ of the pendant vertices on arms of length 1 by 1. Thus maximum value of $e_f(1)$ is $2\alpha + k + 2$ which is strictly less than $2\alpha + k + 3$.
Case 2: $r = 1$
Arguing as before, we require $v_f(1) = \alpha + k + 2$ and $e_f(1)$ should be either $2\alpha + k + 3$ or $2\alpha + k + 4$. Once again, the maximum value of $e_f(1)$ that can be obtained is $2\alpha + k + 2$ which is less than the required value.
Case 3: $r = 2, 3$
These two cases can be dealt with similarly. □

4 Conclusion remarks

Thus we have proved that every spider with each leg of length at least 2 is group A-cordial by giving explicit labelings in each case. Also the spider, $S(\underbrace{2, 2, \ldots, 2}_{\alpha-\text{times}}, \underbrace{1, 1, \ldots, 1}_{\beta-\text{times}})$ is not group A-cordial if and only if $\beta \geq 2\alpha + 6$, where $\alpha, \beta \in \mathbb{N}$.

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