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weakly 1-absorbing δ-primary ideal

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Abstract *Throughout this study, we present a new class of δ-primary ideals, called weakly 1-absorbing δ-primary ideal. Let R be a commutative ring with a non-zero identity. Let $\mathcal{I}(R)$ be the set of all ideals of R and let $\delta : \mathcal{I}(R) \to \mathcal{I}(R)$ be a function. δ is called an expansion function of ideals of R if $I \subseteq \delta(I)$ and if $L \subseteq J$, then $\delta(L) \subseteq \delta(J)$, for each I, L and J are ideals of R. A proper ideal I of R is said to be a weakly 1-absorbing δ-primary ideal if $0 \neq abc \in I$, then $ab \in I$ or $c \in \delta(I)$ for each a, b, c non-unit elements of R. We investigate some basic properties of this class of ideals and we study the weakly 1-absorbing δ-primary ideals of the localization of rings, the direct product of rings, and the trivial ring extensions.*

1 Introduction

Let R be a commutative ring with a non-zero identity. We called I a proper ideal of R if $I \neq R$. Suppose that I is an ideal of R. We mean by $\sqrt{I}$ the radical of I defined by $\sqrt{I} = \{a \in R : a^n \in I$ for some $n \in \mathbb{N}\}$. In particular, $\sqrt{0}$ is the set of all nilpotents in R; i.e, $\{a \in R : a^n = 0$ for some $n \in \mathbb{N}\}$. Let S be a nonempty subset of R. Then the ideal $\{a \in R : aS \subseteq I\}$, which contains I, will be designated by $(I : S)$.

The prime ideal, which is an important subject of ideal theory, has been widely studied by various authors. Among the many generalizations of the notion of prime ideals in the literature, we find the following, due to Anderson and Smith [1]. A proper ideal I of R is called a weakly prime ideal of R if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in I$. Then Atani and Farzalipour introduced the concept of weakly primary ideals which is a generalization of primary ideals in [3]. A proper ideal I of R is called a weakly primary ideal of R if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in \sqrt{I}$. In recent studies [4] and [5] A. Badawi and Y. Celikel introduced the concept of 1-absorbing primary ideal and weakly 1-absorbing primary ideal. A proper ideal I of R is called a 1-absorbing primary ideal (weakly 1-absorbing primary ideal) if whenever non-unit elements $a, b, c \in R$ and $(0 \neq abc)$ $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$.

Zhao in [16] introduced the concept of expansions of ideals, a function δ from $\mathcal{I}(R)$ to $\mathcal{I}(R)$ is an ideal expansion if it has the following properties: $I \subseteq \delta(I)$ and if $I \subseteq J$ for some ideals I, J of R, then $\delta(I) \subseteq \delta(J)$. For example, δ_0 is the identity function, where $\delta_0(I) = I$ for all ideals I of R, and δ_1 is defined by $\delta_1(I) = \sqrt{I}$. For other examples, consider the functions δ_+ and δ_* of $\mathcal{I}(R)$ defined with $\delta_+(I) = I + J$, where $J \in \mathcal{I}(R)$ and $\delta_*(I) = (I : P)$, where $P \in \mathcal{I}(R)$ for all $I \in \mathcal{I}(R)$, respectively. Also in [16] introduced the concept of δ-primary ideal. A proper ideal I of R is said to be a δ-primary ideal of R if whenever $a, b \in R$ with $ab \in I$, we have $a \in I$ or $b \in \delta(I)$, where δ is an expansion function of ideals of R. As a generalization of 1-absorbing primary ideal the authors of [13] introduce the notion of 1-absorbing δ-primary ideal. A proper ideal I of R is called 1-absorbing δ-primary ideal if whenever non-unit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \delta(I)$.
In this paper, we introduce and investigate a new concept of ideals that is closely related to the

class of δ-primary ideals. A proper ideal I of R is said to be a weakly 1-absorbing δ-primary ideal if whenever non-unit elements $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $c \in \delta(I)$. For example, let $\delta : \mathcal{I}(R) \longrightarrow \mathcal{I}(R)$ such that $\delta(I) = \sqrt{I}$ for each ideal I of R. Then δ is an expansion function of ideals of R, and hence a proper ideal I of R is a weakly 1-absorbing δ-primary ideal of R if and only if I is a weakly 1-absorbing primary ideal of R. Among many results in this paper are given to disclose the relations between this new class and others that already exist. The reader may find it helpful to keep in mind the implications noted in the following figures where all arrows are irreversible.

primary $\longrightarrow$ *weakly primary* $\longrightarrow$ *weakly 1-absorbing δ-primary*
$\downarrow$
δ primary $\longrightarrow$ *1-absorbing δ-primary* $\longrightarrow$ *weakly 1-absorbing δ-primary*

Let E be an A-module. the set $A \propto E = \{ (a, e) : r \in A,\ e \in E \}$ is called the trivial (ring) extension of A by E; is a commutative ring with coordinate-wise addition and the multiplication $(a, e)(b, f) := (ab, af + be)$ for each $a; b \in A$ and all $e; f \in E$. (This construction is also known by other terminology and other notation, such as the idealization $A(+)E$.) see [14, 15, 2, 12]

2 Main Results

We start this section with the following definition.

Definition 2.1. Let R be a commutative ring and I a proper ideal of R. We call I a weakly 1-absorbing δ-primary ideal of R if whenever non-unit elements $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $c \in \delta(I)$.

We can easily see that every weakly 1-absorbing δ-prime is weakly 1-absorbing δ-primary. Now we give an example that shows that the converse is not always true.

Example 2.2. Let $R := K[[X_1, X_2, X_3]]$ be a ring of formal power series where K is a field. Consider the expansion function $\delta : \mathcal{I}(\mathcal{R}) \longrightarrow \mathcal{I}(\mathcal{R})$ defined by $\delta(I) = I + M$ where $M = (X_1, X_2, X_3)$ is the maximal ideal of R. Let $I = (X_1X_2X_3)$ be an ideal of R. Thus, I is not a 1-absorbing prime ideal of R since $0 \neq X_1X_2X_3 \in I$ but neither $X_1X_2 \in I$ nor $X_3 \in I$. Now, let x, y, z be non-unit elements of R such that $0 \neq xyz \in I$. Clearly I is a weakly 1-absorbing δ-primary because $z \in \delta(I) = M$.

It is clear that every 1-absorbing δ-primary ideal of a ring R is a weakly 1-absorbing primary ideal of R, and $I = \{0\}$ is a weakly 1-absorbing δ-primary ideal of R. The following example shows that the converse is not true.

Example 2.3. (i) Let $R = \mathbb{Z}_6$ and $\delta_+(P) = P + J$ where $J = 2\mathbb{Z}_6$. It's clear that $I = \{0\}$ is a weakly 1-absorbing δ-primary ideal of R. But is not 1-absorbing δ primary of R. Indeed, $2.2.3 \in I$ but neither $2.2 \in I$ nor $3 \in \delta_+(I) = J$.

(ii) Let $R = \mathbb{Z}_{12} \propto J$ with $J = 6\mathbb{Z}_{12}$, we consider $\delta(K) = \sqrt{K}$. Let $I = 0 \propto J$ be ideal of R . Observe that $abc \in I$ for some $a, b, c \in R \backslash I$ if and only if $abc = (0, 0)$. which implies I is a weakly 1-absorbing δ-primary ideal of R. However, it is not a 1-absorbing primary ideal of R. Indeed; $(2, 0)(2, 0)(3, 0) \in I$, but neither $(2, 0)(2, 0) \in I$ nor $(3, 0) \in \delta(I)$.

Proposition 2.4. *If R/I is an integral domain, then I is a weakly 1-absorbing δ-primary ideal if and only if I is a 1-absorbing δ-primary ideal of R.*

Proof. Suppose that R/I is an integral domain and I is a weakly 1-absorbing δ-primary ideal. Let $abc \in I$ for some $a, b, c \in R$. If $abc \neq 0$ it's clear. Now let $abc = 0$, suppose that $ab \notin I$ and $c \notin \delta(I)$ then $c \notin I$. That implies $\overline{abc} = \overline{0}$ in R/I with $\overline{ab} \neq \overline{0}$ and $\overline{c} \neq \overline{0}$, and that contradicts the fact that R/I is an integral domain. For the converse, it's clear. □

Theorem 2.5. *Assume that R is not a quasilocal ring. Let I be a proper ideal of R such that for all $i \in I$ we have $(0 : i)$ is not a maximal ideal of R. Then, I is a weakly 1-absorbing δ-primary ideal of R if and only if I is a weakly δ-primary ideal of R.*

Proof. Suppose that I is a weakly 1-absorbing δ-primary ideal of R and suppose that $0 \neq ab \in I$ for some elements $a, b \in R$. We may assume that a, b are non-unit elements of R. Let $K = (0 : ab)$. Since $ab \neq 0, K$ is a proper ideal of R. Since K is not a maximal ideal, Then there exists L a maximal ideal of R such that $K \subset L$. since R is a non-quasilocal ring. Then there is a maximal ideal M of R such that $M \neq L$. Let $m \in M \backslash L$. Hence $m \notin K$ and $0 \neq mab \in I$. Since I is a weakly 1 -absorbing δ-primary ideal of R, we have $ma \in I$ or $b \in \delta(I)$. If $b \in \delta(I)$, then we are done. Hence assume that $b \notin \delta(I)$. Hence $ma \in I$. Since $m \notin L$ and L is a maximal ideal of R, we conclude that $m \notin J(R)$. Hence there exists an $r \in R$ such that $1 + rm$ is a non-unit element of R. Suppose that $1 + rm \notin K$. Hence $0 \neq (1 + rm)ab \in I$. Since I is a weakly 1-absorbing δ-primary ideal of R and $b \notin \delta(I)$, we conclude that $(1 + rm)a = a + rma \in I$. Since $rma \in I$, we have $a \in I$ and we are done. Suppose that $1 + rm \in K$. Since K is not a maximal ideal of R and $K \subset L$, there is a $w \in L \backslash K$. Hence $0 \neq wab \in I$. Since I is a weakly 1-absorbing δ-primary ideal of R and $b \notin \delta(I)$, we conclude that $wa \in I$. Since $1 + rm \in K \subset L$ and $w \in L \backslash K$, we have $1 + rm + w$ is a non-zero non-unit element of L. Hence $0 \neq (1 + rm + w)ab \in I$. Since I is a weakly 1-absorbing δ-primary ideal of R and $b \notin \delta(I)$, we conclude that $(1 + rm + w)a = a + rma + wa \in I$. Since $rma, wa \in I$, we get that $a \in I$. For the converse it's clear. □

Theorem 2.6. *Let I be a weakly 1-absorbing δ-primary ideal of R such that for every non-zero element $i \in I$, there exists a non-unit $w \in R$ such that $wi \neq 0$ and $w + u$ is a non-unit element of R for some unit $u \in R$. Then, I is a weakly δ-primary ideal of R.*

Proof. Suppose that $0 \neq ab \in I$ and $b \notin \delta(I)$ for some $a, b \in R$. We may assume that a, b are non-unit elements of R. Hence, there is a non-unit $w \in R$ such that $wab \neq 0$ and $w + u$ is a non-unit element of R for some unit $u \in R$. Since $0 \neq wab \in I$ and $b \notin \delta(I)$ and I is a weakly 1-absorbing δ-primary ideal of R, we conclude that $wa \in I$. Since $(w+u)ab \in I$ and I is a weakly 1-absorbing δ-primary ideal of R and $b \notin \delta(I)$, we conclude that $(w + u)a = wa + ua \in I$. Since $wa \in I$ and $wa + ua \in I$, we conclude that $ua \in I$. Since u is a unit, we have $a \in I$. □

Theorem 2.7. *Let I be a weakly 1-absorbing δ-primary ideal of a ring R and let $d \in R \setminus I$ be a non-unit element of R. Then $(I : d) = \{x \in R \mid dx \in I\}$ is a weakly δ-primary ideal of R.*

Proof. Suppose that $0 \neq ab \in (I : d)$ for some elements $a, b \in R$. Without loss of generality, we may assume that a and b are non-unit elements of R. Suppose that $a \notin (I : d)$. Since dab in I and I is a 1-absorbing δ-primary ideal of R, we conclude that $b \in \delta(I)$. So, $b \in \delta((I : d))$ and this completes the proof. □

Proposition 2.8. *Let R be a ring, I a proper ideal of R and δ be an ideal expansion. Then I is a weakly 1-absorbing δ-primary ideal if and only if whenever $0 \neq I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2 and I_3 of R, then $I_1 I_2 \subseteq I$ or $I_3 \subseteq \delta(I)$*

Proof. It suffices to prove the "if" assertion. Suppose that I is a weakly 1-absorbing δ-primary ideal and let I_1, I_2 and I_3 be proper ideals of R such that $0 \neq I_1 I_2 I_3 \subseteq I$ and $I_3 \not\subseteq \delta(I)$. Thus $abc \in I$ for every $a \in I_1, b \in I_2$ and $c \in I_3 \backslash \delta(I)$. Since I is a weakly 1-absorbing δ-primary ideal, we then have $I_1 I_2 \subseteq I$, as desired. □

Definition 2.9. Let I be a weakly 1-absorbing δ-primary ideal of R and a, b, c be non-unit elements of R. We call (a, b, c) a 1-δ-triple-zero of I if $abc = 0, ab \notin I$, and $c \notin \delta(I)$.

Observe that if I is a weakly 1-absorbing δ-primary ideal of R that is not 1-absorbing δ-primary, then there exists a 1-δ-triple-zero (a, b, c) of I for some non-unit elements $a, b, c \in R$.

Theorem 2.10. *Let I be a weakly 1-absorbing δ-primary ideal of R, and (a, b, c) be a 1-δ-triple-zero of I. Then*

(i) $abI = 0$.

(ii) If $a, b \notin (I : c)$, *then* $abI = acI = aI^2 = bI^2 = cI^2 = 0$.

(iii) If $a, b \notin (I : c)$, *then* $I^3 = 0$.

Proof. (1) Suppose that $abI \neq 0$. Then $abx \neq 0$ for some non-unit $x \in I$. Hence $0 \neq ab(c+x) \in I$. Since $ab \notin I, (c+x)$ is a non-unit element of R. Since I is a weakly 1-absorbing δ-primary ideal of R and $ab \notin I$, we conclude that $(c+x) \in \delta(I)$. Since $x \in I$, we have $c \in \delta(I)$, a contradiction. Thus $abI = 0$.
(2) Suppose that $bcI \neq 0$. Then $bcy \neq 0$ for some non-unit element $y \in I$. Hence $0 \neq bcy = b(a+y)c \in I$. Since $b \notin (I : c)$, we conclude that $a + y$ is a non-unit element of R. Since I is a weakly 1-absorbing δ-primary ideal of R and $ab \notin I$ and $by \in I$, we conclude that $b(a+y) \notin I$, and hence $c \in \delta(I)$, a contradiction. Thus $bcI = 0$. We show that $acI = 0$. Suppose that $acI \neq 0$. Then $acy \neq 0$ for some non-unit element $y \in I$. Hence $0 \neq acy = a(b+y)c \in I$. Since $a \notin (I : c)$, we conclude that $b + y$ is a non-unit element of R. Since I is a weakly 1-absorbing δ-primary ideal of R and $ab \notin I$ and $ay \in I$, we conclude that $a(b+y) \notin I$, and hence $c \in \delta(I)$, a contradiction. Thus $acI = 0$. Now we prove that $aI^2 = 0$. Suppose that $axy \neq 0$ for some $x, y \in I$. Since $abI = 0$ by (1) and $acI = 0$ by $(2), 0 \neq axy = a(b+x)(c+y) \in I$. Since $ab \notin I$, we conclude that $c + y$ is a non-unit element of R. Since $a \notin (I : c)$, we conclude that $b + x$ is a non-unit element of R. Since I is a weakly 1-absorbing δ-primary ideal of R, we have $a(b+x) \in I$ or $(c+y) \in \delta(I)$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \delta(I)$, is a contradiction. Thus $aI^2 = 0$. We show $bI^2 = 0$. Suppose that $bxy \neq 0$ for some $x, y \in I$. Since $abI = 0$ by (1) and $bcI = 0$ by (2), $0 \neq bxy = b(a+x)(c+y) \in I$. Since $ab \notin I$, we conclude that $c + y$ is a non-unit element of R. Since $b \notin (I : c)$, we conclude that $a + x$ is a non-unit element of R. Since I is a weakly 1-absorbing δ-primary ideal of R, we have $b(a+x) \in I$ or $(c+y) \in \delta(I)$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \delta(I)$, is a contradiction. Thus $bI^2 = 0$. We show $cI^2 = 0$. Suppose that $cxy \neq 0$ for some $x, y \in I$. Since $acI = bcI = 0$ by $(2), 0 \neq cxy = (a+x)(b+y)c \in I$. Since $a, b \notin (I : c)$, we conclude that $a + x$ and b + y are non-unit elements of R. Since I is a weakly 1-absorbing δ-primary ideal of R, we have $(a+x)(b+y) \in I$ or $c \in \delta(I)$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \delta(I)$, is a contradiction. Thus $cI^2 = 0$.
(3) Assume that $xyz \neq 0$ for some $x, y, z \in I$. Then $0 \neq xyz = (a+x)(b+y)(c+ z) \in I$ by (1) and (2). Since $ab \notin I$, we conclude that $c + z$ is a non-unit element of R. Since $a, b \notin (I : c)$, we conclude that $a + x$ and $b + y$ are non-unit elements of R. Since I is a weakly 1-absorbing δ-primary ideal of R, we have $(a+x)(b+y) \in I$ or $c+z \in \delta(I)$. Since $x, y, z \in I$, we conclude that $ab \in I$ or $c \in \delta(I)$, is a contradiction. Thus $I^3 = 0$. □

Theorem 2.11. *(i) Let I be a weakly 1-absorbing δ-primary ideal of a reduced ring R. Suppose that I is not a 1-absorbing ideal δ-primary ideal of R and (a, b, c) is a 1-δ-triple-zero of I such that $a, b \notin (I : c)$. Then $I = 0$.*

(ii) Let I be a non-zero weakly 1-absorbing δ-primary ideal of a reduced ring R. Suppose that I is not a 1-absorbing ideal δ-primary ideal of R and (a, b, c) is a 1-δ-triple-zero of I. Then $ac \in I$ or $bc \in I$.

Proof. (1) Since $a, b \in (I : c)$, then $I^3 = 0$ by Theorem 2.10(3). Since R is reduced, we conclude that $I = 0$.
(2) Suppose that neither $ac \in I$ nor $bc \in 0$. Then $I = 0$ by (1), a contradiction since I is a non-zero ideal of R by hypothesis. Hence if (a, b, c) is a 1-δ-triple-zero of I, then $ac \in I$ or $bc \in I$. □

Theorem 2.12. *Let I be a weakly 1-absorbing δ-primary ideal of R. If I is not a weakly δ-primary ideal of R, then there exists an irreducible element $x \in R$ and a non-unit element $y \in R$ such that $xy \in I$, but neither $x \in I$ nor $y \in \delta(I)$. Furthermore, if $ab \in I$ for some non-unit elements $a, b \in R$ such that neither $a \in I$ nor $b \in \delta(I)$, then a is an irreducible element of R.*

Proof. Suppose that I is not a weakly δ-primary ideal of R. Then there exist non-unit elements $x, y \in R$ such that $0 \neq xy \in I$ with $x \notin I, y \notin \delta(I)$. Suppose that x is not an irreducible element of R. Then $x = cd$ for some non-unit elements $c, d \in R$. Since $0 \neq xy = cdy \in I$ and I is weakly 1-absorbing δ-primary and $y \notin \delta(I)$, we conclude that $cd = x \in I$, a contradiction. Hence x is an irreducible element of R. □

Proposition 2.13. *Let $\{I_i : i \in \Lambda\}$ be a collection of weakly 1-absorbing δ-primary ideals of R such that $Q = \delta(I_i) = \delta(I_j)$ for every distinct $i, j \in \Lambda$. Then $I = \bigcap_{i\in\Lambda} I_i$ is a weakly 1-absorbing δ-primary ideal of R.*

Proof. Suppose that $0 \neq abc \in I = \cap_{i\in\Lambda} I_i$ for non-unit elements a, b, c of R and $ab \notin I$. Then for some $k \in \Lambda, 0 \neq abc \in I_k$ and $ab \notin I_k$. It implies that $c \in \delta(I_k) = Q = \delta(I)$. □

Proposition 2.14. *Let $\{J_i \mid i \in D\}$ be a directed set of weakly 1-absorbing δ primary ideals of R, where δ is an ideal expansion. Then the ideal $J = \cup_{i\in D} J_i$ is a weakly 1-absorbing δ-primary ideal of R.*

Proof. Let $0 \neq abc \in J$, then $0 \neq abc \in J_i$ for some $i \in D$. Since J_i is a weakly 1-absorbing δ-primary ideal of $R, ab \in J_i$ or $c \in \delta(J_i) \subseteq \delta(J)$. Hence, J is a weakly 1-absorbing δ-primary ideal of R. □

Proposition 2.15. *Let I be a weakly 1-absorbing δ-primary ideal of R and c be a non-unit element of $R\backslash I$. Then $(I : c)$ is a weakly δ-primary ideal of R.*

Proof. Suppose that $0 \neq ab \in (I : c)$ for some non-unit $c \in R\backslash I$ and assume that $a \notin (I : c)$. Hence b is a non-unit element of R. If a is unit, then $b \in (I : c) \subseteq \delta((I : c))$ and we are done. So assume that a is a non-unit element of R. Since $0 \neq abc = acb \in I$ and $ac \notin I$ and I is a weakly 1-absorbing δ-primary ideal of R, we conclude that $b \in \delta(I) \subseteq \delta((I : c))$. Thus $(I : c)$ is a weakly δ-primary ideal of R. □

Let R_1 and R_2 be two rings, let δ_i be an expansion function of $\mathcal{I}(R_i)$ for each $i \in \{1, 2\}$ and $R = R_1 \times R_2$: For a proper ideal $I_1 \times I_2$, the function $\delta_\times$ defined by $\delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ is an expansion function of I(R).
The following result characterizes the 1-absorbing $\delta_\times$-primary ideals of the direct product of rings.

Theorem 2.16. *Let R_1 and R_2 be commutative rings with identity that are not fields, $R = R_1 \times R_2$, and I be a a non-zero proper ideal of R. Then the following statements are equivalent.*

(i) I is a weakly 1-absorbing $\delta_\times$-primary ideal of R.

(ii) Either $I = I_1 \times R_2$, where $I = I_1 \times R_2$ for some δ_1-primary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some δ_2-primary ideal I_2 of R_2, or $I = I_1 \times I_2$, where I_1 and I_2 are proper ideals of R_1, R_2, respectively with $\delta_1(I_1) = R_1$ and $\delta_2(I_2) = R_2$.

(iii) I is a 1-absorbing $\delta_\times$-primary ideal of R.

(iv) I is a $\delta_\times$-primary ideal of R_1.

Proof. $(1) \Rightarrow (2)$. Suppose that I is a weakly 1-absorbing $\delta_\times$-primary ideal of R. Then I is of the form $I_1 \times I_2$ for some ideals I_1 and I_2 of R_1 and R_2, respectively. Assume that $I = I_1 \times R_2$ for some proper ideal I_1 of R_1. We show that I_1 is a δ-primary ideal of R_1. Let $ab \in I_1$ for some $a, b \in R_1$. We can assume that a and b are non-unit elements of R_1. Since R_2 is not a field, there exists a non-unit non-zero element $x \in R_2$. Then $0 \neq (a, 1)(1, x)(b, 1) \in I_1 \times R_2$ which implies that either $(a, 1)(1, x) \in I_1 \times R_2$ or $(b, 1) \in \delta_\times(I_1 \times R_2) = \delta_1(I_1) \times R_2$; i.e, $a \in I_1$ or $b \in \delta_1(I_1)$. Similarly, If $I = R_1 \times I_2$ for some proper ideal I_2 of R_2.
Now suppose that both I_1 and I_2 are proper. Since I is a non-zero ideal of R, we conclude that $I_1 \neq 0$ or $I_2 \neq 0$. We may assume that $I_1 \neq 0$. Let $0 \neq c \in I_1$. Then $0 \neq (1, 0)(1, 0)(c, 1) = (c, 0) \in I_1 \times I_2$. It implies that $(1, 0)(1, 0) \in I_1 \times I_2$ or $(c, 1) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$, since I_1 is proper then $\delta_2(I_2) = R_2$. Now show that case 1: If $I_2 \neq 0$ Let $0 \neq b \in I_2$. Then $0 \neq (0, 1)(0, 1)(1, b) = (0, b) \in I_1 \times I_2$. It implies that $(0, 1)(0, 1) \in I_1 \times I_2$ or $(1, b) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$, since I_2 is proper then $\delta_1(I_1) = R_1$. Now show that case 2: If $I_2 = 0$. Let $e \in R_2$ non-unit, then $(c, 1)(1, e)(1, 0) = (c, 0) \in I$. That implies $(c, 1)(1, e) \in I_1 \times I_2$ or $(1, 0) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ since $I_2 = 0$ then $\delta_1(I_1) = R_1$.
$(2) \Rightarrow (3) \Rightarrow (4)$. by [13, Theorem 2.28].
$(4) \Rightarrow (1)$. Clear. □

Let R and S be commutative rings with non-zero identity, and $f : R \to S$ be a homomorphism of the ring. f is called $\delta\gamma$-homomorphism if δ, γ be two expansion functions of $\mathcal{I}(R)$ and $\mathcal{I}(S)$, respectively, and $\delta\left(f^{-1}(I)\right) = f^{-1}(\gamma(I))$ for all ideals I of S. Additionally, if f is a $\delta\gamma$-epimorphism, then for each I is an ideal of R containing Ker(f) *we get $\gamma(f(I)) = f(\delta(I))$.*

Theorem 2.17. *Let R and S be commutative rings with $1 \neq 0$ and let $f : R \to S$ be a $\delta\gamma$-homomorphism such that $f(1) = 1$. Then the following statements hold:*

(i) Suppose that f is a monomorphism and $f(a)$ is a non-unit element of S for every non-unit element $a \in R$ (for example if $U(S)$ is a torsion group) and J is a weakly 1-absorbing γ-primary ideal of S. Then $f^{-1}(J)$ is a weakly 1-absorbing δ-primary ideal of R.

(ii) If f is an epimorphism and I is a weakly 1-absorbing δ-primary ideal of R such that $\mathrm{Ker}(f) \subseteq I$*, then $f(I)$ is a weakly 1-absorbing γ-primary ideal of S.*

Proof. (1) Let $0 \neq abc \in f^{-1}(J)$ for some non-unit elements $a, b, c \in R$. Since $\mathrm{Ker}(f) = 0$, we have $0 \neq f(abc) = f(a)f(b)f(c) \in J$, where $f(a), f(b), f(c)$ are non-unit elements of S by hypothesis. Hence $f(a)f(b) \in J$ or $f(c) \in \gamma(J)$. Hence $ab \in f^{-1}(J)$ or $c \in \delta(f^{-1}(J)) = f^{-1}(\gamma(J))$. Thus $f^{-1}(J)$ is a weakly 1-absorbing δ-primary ideal of R.
(2) Let $0 \neq xyz \in f(I)$ for some non-unit elements $x, y, z \in R$. Since f is epimorphism, there exists non-unit elements $a, b, c \in I$ such that $x = f(a), y = f(b), z = f(c)$. Then $f(abc) = f(a)f(b)f(c) = xyz \in f(I)$. Since $\mathrm{Ker}(f) \subseteq I$, we have $0 \neq abc \in I$. It follows $ab \in I$ or $c \in \delta(I)$. Thus $xy \in f(I)$ or $z \in f(\delta(I))$. Since f is epimorphism and $\mathrm{Ker}(f) \subseteq I$, we have $f(\delta(I)) = \gamma(f(I))$. Thus we are done. □

Let δ be an expansion function of $\mathcal{I}(R)$ and I a proper ideal of R. Then the function $\delta_q : R/I \to R/I$, defined by $\delta_q(J/I) = \delta(J)/I$ for all ideals $I \subseteq J$, becomes an expansion function of R/I. Consider the natural homomorphism $\pi : R \to R/J$. Then for ideals I of R with Ker $(\pi) \subseteq I$, we have $\delta_q(\pi(I)) = \pi(\delta(I))$.

Theorem 2.18. *Let I be a proper ideal of R. Then the following statements hold.*

(i) If J is a proper ideal of a ring R with $J \subseteq I$ and I is a weakly 1-absorbing δ-primary ideal of R, then I/J is a weakly 1-absorbing δ_q-primary ideal of R/J.

(ii) If J is a proper ideal of a ring R with $J \subseteq I$ such that $U(R/J) = \{a + J \mid a \in U(R)\}$. If J is a 1-absorbing δ-primary ideal of R and I/J is a weakly 1-absorbing δ_q-primary ideal of R/J, then I is a 1-absorbing δ-primary ideal of R.

(iii) If $\{0\}$ is a 1-absorbing δ-primary ideal of R and I is a weakly 1-absorbing δ-primary ideal of R, then I is a 1-absorbing δ-primary ideal of R.

(iv) If J is a proper ideal of a ring R with $J \subseteq I$ such that $U(R/J) = \{a + J \mid a \in U(R)\}$. If J is a weakly 1-absorbing δ-primary ideal of R and I/J is a weakly 1-absorbing δ_q-primary ideal of R/J, then I is a weakly 1-absorbing δ-primary ideal of R.

Proof. (1) Consider the natural epimorphism $\pi : R \to R/J$. Then $\pi(I) = I/J$. So we are done by Theorem 2.17 (2).
(2) Suppose that $abc \in I$ for some non-unit elements $a, b, c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \delta(J) \subseteq \delta(I)$ as J is a 1-absorbing δ-primary ideal of R. Now assume that $abc \notin J$. Then $J \neq (a + J)(b + J)(c + J) \in I/J$, where $a + J, b + J, c + J$ are non-unit elements of R/J by hypothesis. Thus $(a + J)(b + J) \in I/J$ or $(c + J) \in \delta_q(I/J) = \delta(I)/J$. Hence $ab \in I$ or $c \in \delta(I)$.
(3) The proof follows from (2).
(4) Suppose that $0 \neq abc \in I$ for some non-unit elements $a, b, c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \delta(J) \subseteq \delta(I)$ as J is a weakly 1-absorbing primary ideal of R. Now assume that $abc \notin J$. Then $J \neq (a+J)(b+J)(c+J) \in I/J$, where $a+J, b+J, c+J$ are non-unit elements of R/J by hypothesis. Thus $(a + J)(b + J) \in I/J$ or $(c + J) \in \delta_q(I/J) = \delta(I)/J$. Hence $ab \in I$ or $c \in \delta(I)$. □

Corollary 2.19. *Let R be a ring and S a subring of R. If I is a weakly 1-absorbing δ-primary ideal of R with $S \nsubseteq I$, then $I \cap S$ is a weakly 1-absorbing δ-primary ideal of S.*

Proof. Suppose that S is a subring of R and I is a weakly 1-absorbing δ-primary ideal of R with $S \nsubseteq I$. Consider the injection $i : S \to R$. Note that $i^{-1}(I) = I \cap S$, so by Theorem 2.17 (1), $I \cap S$ is a weakly 1-absorbing δ-primary ideal of S. □

Let S be a multiplicatively closed subset of a ring R and δ an expansion function of $\mathcal{I}(R)$. Note that δ_S is an expansion function of $\mathcal{I}(S^{-1}R)$ such that $\delta_S(S^{-1}I) = S^{-1}\delta(I)$ for each ideal I of R.

Theorem 2.20. *Let S be a multiplicatively closed subset of R, and I is a weakly 1-absorbing δ-primary ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a weakly 1-absorbing δ_S-primary ideal of $S^{-1}R$.*

Proof. Suppose that $0 \neq \frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{-1}I$ for some non-unit $a, b, c \in R\backslash S$, $s_1, s_2, s_3 \in S$ and $\frac{a}{s_1}\frac{b}{s_2} \notin S^{-1}I$. Then $0 \neq uabc \in I$ for some $u \in S$. Since I is weakly 1-absorbing δ-primary and $uab \notin I$, we conclude $c \in \delta(I)$. Thus $\frac{c}{s_3} \in S^{-1}\delta(I) = \delta_s(S^{-1}I)$. Thus $S^{-1}I$ is a weakly 1-absorbing δ_s-primary ideal of $S^{-1}R$. □

Let A be a ring and E an A-module. Let I be an ideal of A and F be a submodule E. Then, $I \propto F$ is an ideal of $A \propto E$ if and only if $IE \subseteq F$. Moreover for an expansion function δ of A, it is clear that $\delta_\propto$ defined as $\delta_\propto(I \propto F) = \delta(I) \propto E$ is an expansion function of $A \propto E$.

Theorem 2.21. *Let A be a ring, E an A-module, and δ be an expansion function of $\mathcal{I}(A)$. Let I be an ideal of A and F a submodule of E such that $IE \subseteq F$. Then the following statement holds:*

(i) If $I \propto F$ is a weakly 1-absorbing $\delta_\propto$-primary ideal of $A \propto E$, then I is a weakly 1-absorbing δ-primary ideal of A.

(ii) $I \propto E$ is a weakly 1-absorbing $\delta_\propto$-primary ideal of $A \propto E$ if and only if I is a weakly 1-absorbing δ-primary ideal of A and for $a, b, c \in A$ with $abc = 0$, but $ab \notin I$ and $c \notin \delta(I)$,then $ab \in \mathrm{ann}(E)$ and $c \in \mathrm{ann}(E)$.

Proof. (1) Assume that $I \propto F$ is a weakly 1-absorbing $\delta_\propto$-primary ideal of $A \propto E$ and let a, b, c be non-unit elements of A such that $0 \neq abc \in I$. Thus $(0, 0) \neq (a, 0)(b, 0)(c, 0) = (abc, 0) \in I \propto F$ which implies that $(a, 0)(b, 0) \in I \propto F$ or $(c, 0) \in \delta_\propto(I \propto F) = \delta(I) \propto E$. Therefore $ab \in I$ or $c \in \delta(I)$ and so (1) holds.
(2) Let $I \propto E$ is a weakly 1-absorbing $\delta_\propto$-primary ideal of $A \propto E$. By (1) we have I is a weakly 1-absorbing δ-primary ideal of A. Suppose that $abc = 0$, but $ab \notin I$ and $c \notin \delta(I)$ $(a, b \in R)$. Assume, say, $ab \notin \mathrm{ann}(E)$. So there exists $e \in E$ with $abe \neq 0$. Then $(0, 0) \neq (a, 0)(b, 0)(c, e) \in I \propto E$, but $(a, 0), (b, 0) \notin I \propto E$ and $(c, e) \notin \delta(I) \propto E = \delta_\propto(I \propto F)$, a contradiction. ($\Leftarrow$) $(a, s), (b, t), (c, r)$ be non-unit elements of $A \propto E$ such that $(0, 0) \neq (a, s)(b, t)(c, r) = (abc, bcs + act + abr) \in I \propto E$. If $abc \neq 0$. Clearly, $abc \in I$ then $ab \in I$ or $c \in \delta(I)$ since I is a weakly 1-absorbing δ-primary ideal of A. Then $(a, s)(b, t) \in I \propto E$ or $(c, r) \in \delta(I) \propto E = \delta_\propto(I \propto E)$. So assume that $abc = 0$. Suppose $ab \notin I$ and $c \notin \delta(I)$. Then by hypothesis, $ab, c \in \mathrm{ann}(E)$. Then $(a, s), (b, t), (c, r) = (ab, at + bs)(c, r) = (0, 0)$ a contradiction. So (2) holds. □

Let A and B be two rings with unity, let J be an ideal of B, and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$ defined by $A \bowtie^f J := \{(a, f(a) + j) \in A \times B \mid a \in A, j \in J\}$. is called the amalgamation of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal denoted $A \bowtie I$ (introduced and studied by D'Anna and Fontana in [10]). for more studies of this constriction see [7, 8, 9, 11].

Theorem 2.22. *Let A and B be commutative rings with $1 \neq 0$ and let $f : A \to B$ be a $\delta\gamma$-homomorphism such that $f(1_A) = 1_B$. Let $\delta_\bowtie(I \bowtie^f J) = \{(a, f(a) + j) \in A \bowtie^f J \mid a \in \delta(I)$ and $j \in f(\delta(I))\}$*

(i) if $I \bowtie^f J$ is weakly 1-absorbing $\delta_\bowtie$-primary, then I is weakly 1-absorbing δ-primary.

(ii) Let f is an epimorphism and $J \subseteq \gamma(f(I)) = f(\delta(I))$. if I is 1-absorbing δ-primary then $I \bowtie^f J$ is 1-absorbing $\delta_\bowtie$-primary.

(iii) Let f is an epimorphism and $\mathrm{Ker}(f) \subseteq I$. If $I \bowtie^f J$ is weakly 1-absorbing $\delta_\bowtie$-primary, then $f(I) + J$ is weakly 1-absorbing γ-primary of $f(A) + J$.

Proof. (1) Assume that $I \bowtie^f J$ is weakly 1-absorbing $\delta_{\bowtie}$-primary. Let a, b, c be non-unit elements of A such that $0 \neq ABC \in I$ with $ab \notin I$. Then, $(0,0) \neq (abc, f(abc)) \in I \bowtie^f J$ and $(ab, f(ab)) \notin I \bowtie^f J$ which implies $(c, f(c)) \in \delta_{\bowtie}(I \bowtie^f J)$. Thus, $c \in \delta(I)$ as desired.
(2) I is 1-absorbing δ-primary. Let $(0,0) \neq (a, f(a)+i)(b, f(b)+j)(c, f(c)+k) \in I \bowtie^f J$ with $(a, f(a)+i)(b, f(b)+j) \notin I \bowtie^f J$. Then, $abc \in I$ but $ab \notin I$ which implies $c \in \delta(I)$. Thus $f(c)+k \in \gamma(f(I))$ since $J \subseteq \gamma(f(I))$. Therefore $(c, f(c)+k) \in \delta_{\bowtie}(I \bowtie^f J)$.
(3) let $(f(x)+j), (f(y)+k), (f(z)+l)$ be non-unit elements of $f(A)+J$ such that $0 \neq (f(x)+j)(f(y)+k)(f(z)+l) \in f(I)+J$ with $(f(x)+j)(f(y)+k) \notin f(I)+J$. Then, the fact that Ker$(f) \subseteq I$ implies $(0,0) \neq (x, f(x)+j)(y, f(y)+k)(z, f(z)+l) \in I \bowtie^f J$ with $(x, f(x)+j)(y, f(y)+k) \notin I \bowtie^f J$. which implies $(z, f(z)+l) \in \delta_{\bowtie}(I \bowtie^f J)$. then $z \in \delta(I)$ and $l \in \gamma(f(I))$. which give $f(z) \in f(\delta(I)) = \gamma(f(I))$ since f is an epimorphism. Thus $(f(z)+l) \in \gamma(f(I)) \subseteq \gamma(f(I)+J)$ □

Corollary 2.23. *Let A be commutative ring with $1 \neq 0$ and let $\delta_{\bowtie}(I \bowtie J) = \{(a, a+j) \in A \bowtie J \mid a$ and $j \in \delta(I)$*

(i) if $I \bowtie J$ is weakly 1-absorbing $\delta_{\bowtie}$-primary, then I and $I+J$ are weakly 1-absorbing δ-primary ideals.

(ii) Let $J \subseteq \delta(I)$. if I is 1-absorbing δ-primary, then $I \bowtie J$ is 1-absorbing $\delta_{\bowtie}$-primary.

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