Palestine Journal of Mathematics

Vol 13(1)(2024) , 280–290

A LOGARITHMIC COUNTERPART TO A HIGHER ORDER SEMILINEAR ABSTRACT CAUCHY PROBLEM

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 34A08; Secondary 47D06, 47D03.

Keywords and phrases: logarithmic operator, fractional powers, semilinear Cauchy problem, linear semigroup.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper. The first author would like to thank the CNPq/Brazil project (# 303039/2021-3) for their necessary support and facility.

Abstract In this paper, we study a well-posed logarithmic counterpart of an ill-posed semilinear Cauchy problem associated with an abstract evolution equation of n-th order in time.

1 Introduction

In this paper, we present a result of local well-posedness for a logarithmic counterpart of an ill-posed semilinear problem associated with a higher-order abstract Cauchy problem. The notion of logarithmic operators under different spectral conditions is well-known in the literature, see e.g. [2, 3, 5, 9, 10, 11, 12, 16, 17, 19, 20, 22, 26]. In this sense, logarithm equations and models of evolution equations with logarithmic operators have attracted the attention of many researchers and appeared in the literature with increasing frequency, see e.g. [3, 4, 5, 9, 10, 11, 12, 16, 17, 24].

To better present our results, we introduce some notation. Initially, we consider the following abstract semilinear evolution equation of n-th order in time

$$\frac{d^n u}{dt^n} + Au = f(u), \quad t > 0, \tag{1.1}$$

with initial conditions given by

$$\frac{d^i u}{dt^i}(0) = u_i \in X^{\frac{n-(i+1)}{n}}, \quad i \in \{0, 1, \ldots, n-1\},\ n \geqslant 3, \tag{1.2}$$

where X is a separable Hilbert space and $A : D(A) \subset X \to X$ is an unbounded linear, closed, densely defined, self-adjoint, and positive definite operator. See [14] and the references therein for examples. We wish to study the fractional powers of Λ_n, the matrix operator obtained by rewriting (1.1)-(1.2) as a first-order abstract system. Before, we need to compile some basic facts and set up some terminologies.

Since A is a sectorial operator in the sense of [18, Definition 1.3.1], this allows us to define the fractional power $A^{-\alpha}$ of order $\alpha \in (0, 1)$ according to [2, Formula 4.6.9] and [18, Theorem 1.4.2], as a closed linear operator on its domain $D(A^{-\alpha})$ with inverse A^{α}.

Denote by $X^{\alpha} = D(A^{\alpha})$ for $\alpha \in [0, 1)$, taking $A^0 := I$ on $X^0 := X$ when $\alpha = 0$. Recall that X^{α} is dense in X for all $\alpha \in (0, 1]$, for details see [2, Theorem 4.6.5]. The fractional power space X^{α} endowed with the norm $\|\cdot\|_{X^{\alpha}} := \|A^{\alpha}\cdot\|_X$ is a Banach space. It is not difficult to show that A^{α} is the generator of a strongly continuous analytic semigroup on X for any $\alpha \in [0, 1]$, see [18]. With this notation, we have $X^{-\alpha} = (X^{\alpha})'$ for all $\alpha > 0$, see [2] for the characterization of the negative scale.

The nonlinearity f in (1.1) is defined in $X^{\frac{n-1}{n}}$ taking values in X and it is Lipschitz continuous in bounded subsets of $X^{\frac{n-1}{n}}$.

After that below, we rewrite (1.1)-(1.2) as a first-order abstract system; namely, we consider the phase space

$$Y = X^{\frac{n-1}{n}} \times X^{\frac{n-2}{n}} \times X^{\frac{n-3}{n}} \times \cdots \times X$$

which is a Banach space equipped with the norm

$$\|\cdot\|_Y^2 = \|\cdot\|^2_{X^{\frac{n-1}{n}}} + \|\cdot\|^2_{X^{\frac{n-2}{n}}} + \|\cdot\|^2_{X^{\frac{n-3}{n}}} + \cdots + \|\cdot\|^2_X.$$

We can write the problem (1.1)-(1.2) as a Cauchy problem on Y, letting $v_1 = u$, $v_2 = \frac{du}{dt}$, $v_3 = \frac{d^2u}{dt^2}, \ldots, v_n = \frac{d^{n-1}u}{dt^{n-1}}$ and

$$\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

and the initial value problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \Lambda \mathbf{u} = F(\mathbf{u}),\ t > 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1.3}$$

where the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ is defined by

$$D(\Lambda) = X^1 \times X^{\frac{n-1}{n}} \times X^{\frac{n-2}{n}} \times \cdots \times X^{\frac{1}{n}}, \tag{1.4}$$

and

$$\Lambda \mathbf{u} = \begin{bmatrix} 0 & -I & 0 & \cdots & 0 & 0 \\ 0 & 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -I \\ A & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} := \begin{bmatrix} -v_2 \\ -v_3 \\ \vdots \\ -v_n \\ Av_1 \end{bmatrix},\ \forall \mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in D(\Lambda). \tag{1.5}$$

The nonlinearity F in (1.3) is given by

$$F(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(v_1) \end{bmatrix} \tag{1.6}$$

for any $\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$ belonging to a suitable norm space.

The problem (1.1)-(1.2) with f is identically zero was studied in [14]. In [14] it is proved that the Cauchy problem with $n \geqslant 3$ is well-posed if and only if A is a bounded linear operator on X. It is also possible to find works involving Cauchy problems associated with an abstract evolution equation of n-th order in time in [6], [15], and [21].

To our best knowledge, there is no logarithmic counterpart of the semilinear Cauchy problem of n-the order in time, for $n \geqslant 4$ in the literature. The cases $n = 2$ and $n = 3$ can be found at [3] and [8], respectively. The most interesting here is that we have an ill-posed problem on Y via the theory of strongly continuous semigroup of linear bounded operators, whose logarithmic formulation is a well-posed problem on Y, as we will see later. To obtain the logarithm operator $\log \Lambda$ explicitly, we will first calculate the $-\alpha$-th fractional powers ($\alpha \in (0,1)$) of Λ, and then we will get a characterization of the logarithm operator of Λ.

We know that the unbounded linear operator $-\Lambda$ with $\Lambda : D(\Lambda) \subset Y \to Y$ defined in (1.4)-(1.5) is not the infinitesimal generator of a strongly continuous semigroup on Y, see [6, Theorem 1.1]. But, by [6, Lemma 2.13] the unbounded linear operator Λ defined in (1.4)-(1.5) is of positive type $K \geqslant 1$. This allows us to study logarithmic operators and its properties; namely, given $\Lambda : D(\Lambda) \subset Y \to Y$ defined in (1.4)-(1.5), we consider the family of fractional powers $\{\Lambda^\alpha = (\Lambda^{-\alpha})^{-1}; \alpha \in (0,1)\}$. We know that Λ is densely ranged and densely defined (because $0 \in \rho(\Lambda)$), and consequently, we can consider the analytic semigroup of linear bounded operators $\{\Lambda^{-t}; t \geqslant 0\}$ in Y and its infinitesimal generator denoted by $-\log \Lambda$ defined by

$$D(-\log \Lambda) = \left\{ \mathbf{u} \in Y; \exists \lim_{t \searrow 0} \frac{1}{t}(\Lambda^{-t} - I)\mathbf{u} \right\}$$

and for any $\mathbf{u} \in D(-\log \Lambda)$

$$(-\log \Lambda)\mathbf{u} = \lim_{t \searrow 0} \frac{1}{t}(\Lambda^{-t} - I)\mathbf{u}.$$

We can also consider the logarithmic operator $\log \Lambda : D(\log \Lambda) \subset Y \to Y$ defined by

$$D(\log \Lambda) = D(-\log \Lambda)$$

and for any $\mathbf{u} \in D(\log \Lambda)$

$$\log \Lambda \mathbf{u} := -(-\log \Lambda)\mathbf{u}.$$

With this, we consider the logarithmic counterpart to (1.3) in Y; namely, the semilinear Cauchy problem

$$\begin{cases} \dfrac{d\mathbf{u}_\ell}{dt} + (\log \Lambda)\mathbf{u}_\ell = F(\mathbf{u}_\ell), \ t > 0, \\ \mathbf{u}_\ell(0) = \mathbf{u}_{\ell_0}, \end{cases} \tag{1.7}$$

where F is given by (1.6).

We also consider the following notion of mild solution for (1.7). Given $\mathbf{u}_0 \in Y$ we say that $\mathbf{u}_\ell$ is a mild solution of (1.7) provided $\mathbf{u}_\ell \in C([0, t_0(\mathbf{u}_0)), Y)$ for $t_0(\mathbf{u}_0)) > 0$ and $\mathbf{u}_\ell$ satisfies for $t \in (0, t_0(\mathbf{u}_0))$ the variation of constants formula

$$\mathbf{u}_\ell(t) = \Lambda^{-t}\mathbf{u}_0 + \int_0^t \Lambda^{-(t-s)} F(\mathbf{u}_\ell(s))ds. \tag{1.8}$$

This paper is organized as follows. In Section 2 we study the logarithm operator defined by Λ given in (1.4)-(1.5). Finally, in Section 3 we study the semilinear Cauchy problem given in (1.7).

2 Logarithmic operators

In this section, we study the spectral properties of the unbounded linear operator that we will understand as being the logarithm operator of Λ. The following lemma is a key result in our analysis, a proof of this result is given in [1].

Lemma 2.1. *Let U_n be nth degree Chebyshev polynomial of the second kind defined in Theorem 2.2. Then*

$$U_n(\cos\theta)\sin\theta = \sin((n+1)\theta), \tag{2.1}$$

and

$$U_n(-x) = (-1)^n U_n(x), \tag{2.2}$$

for all $\theta \in \mathbb{R}$ and $n \geqslant 0$.

The following theorem is one of the main results of this paper.

Theorem 2.2. *If A and Λ are as in (1.4)-(1.5), respectively, then we have all the following.*

$i)$ $-\alpha$-th fractional power $\Lambda^{-\alpha}$ can be defined for $\alpha \in (0,1)$ through

$$\Lambda^{-\alpha} = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + \Lambda)^{-1} d\lambda. \tag{2.3}$$

$ii)$ Given any $\alpha \in (0,1)$ we have $\Lambda^{-\alpha} : Y \to Y$ is given by

$$\Lambda^{-\alpha} = \left[\frac{(-1)^{i-j}}{n} U_{n-1}\left(\cos\left(\frac{\alpha - i + j}{n}\pi\right)\right) A^{-\frac{\alpha-i+j}{n}} \right] \tag{2.4}$$

where $U_n : \mathbb{C} \to \mathbb{C}$ is the nth degree Chebyshev polynomial of the second kind defined by the recurrence relation for every $x \in \mathbb{C}$ and $n \geqslant 2$

$$U_0(x) = 1,$$
$$U_1(x) = 2x,$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

Proof. Part $i)$ is a consequence of the fact that Λ is of positive type operator, see Pazy [23, Theorem 2.6.9]. For the part $ii)$ note that

$$(\lambda I + \Lambda)^{-1} = (\lambda^n I + A)^{-1} \begin{bmatrix} \lambda^{n-1}I & \lambda^{n-2}I & \lambda^{n-3}I & \cdots & \lambda I & I \\ -A & \lambda^{n-1}I & \lambda^{n-2}I & \cdots & \lambda^2 I & \lambda I \\ -\lambda A & -A & \lambda^{n-1}I & \cdots & \lambda^3 I & \lambda^2 I \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda^{n-3}A & -\lambda^{n-4}A & -\lambda^{n-5}A & \cdots & \lambda^{n-1}I & \lambda^{n-2}I \\ -\lambda^{n-2}A & -\lambda^{n-3}A & -\lambda^{n-4} & \cdots & -A & \lambda^{n-1}I \end{bmatrix}, \text{ for } \lambda \in \rho(-\Lambda). \quad (2.5)$$

Letting $(\lambda I + \Lambda)^{-1} = [a_{ij}]$, with

$$a_{ij} = \begin{cases} -\lambda^{i-j-1}A(\lambda^n I + A)^{-1}, & \text{if } i > j; \\ \lambda^{n+i-j-1}(\lambda^n I + A)^{-1}, & \text{if } i \leqslant j. \end{cases}$$

Now considering the above characterization, using (2.3), applying in each entry of the matrix (2.5) the fractional formula for A,

$$A^{-\alpha} = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1} d\lambda,$$

and after the change of variable $s = \lambda^n$ we obtain the following for $i > j$:

$$\frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha}(-\lambda^{i-j-1}A(\lambda^n I + A)^{-1})d\lambda$$
$$= -A\frac{1}{n}\frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{-\frac{\alpha-i+j+n}{n}}(sI + A)^{-1}ds$$
$$= -A\frac{(-1)^{i-j-n}}{n} \frac{\sin\left(\left(\frac{\alpha-i+j+n}{n}\right)n\pi\right)}{\pi} \int_0^\infty (sI + A)^{-1}ds.$$

Notice that we can rewrite the expression of the sin in terms of

$$\sin\left(\left(\frac{\alpha-i+j+n}{n}\right)n\pi\right) = U_{n-1}\left(\cos\left(\left(\frac{\alpha-i+j+n}{n}\right)\pi\right)\right)\sin\left(\left(\frac{\alpha-i+j+n}{n}\right)\pi\right),$$

where $U_n : \mathbb{C} \to \mathbb{C}$ is the nth degree Chebyshev polynomial of second kind. Thus, we have

$$\frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha}(-\lambda^{i-j-1}A(\lambda^n I + A)^{-1}d\lambda$$
$$= -\frac{(-1)^{i-j-n}}{n}U_{n-1}\left(\cos\left(\left(\frac{\alpha-i+j+n}{n}\right)\pi\right)\right)A^{-\frac{\alpha-i+j}{n}}$$
$$= -\frac{(-1)^{i-j}}{n}U_{n-1}\left(\cos\left(\left(\frac{\alpha-i+j}{n}\right)\pi\right)\right)A^{-\frac{\alpha-i+j}{n}}.$$

Similarly, for $i \leqslant j$, we obtain the following

$$\frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda^{n+i-j-1}(\lambda^n I + A)^{-1})d\lambda$$
$$= \frac{(-1)^{i-j}}{n}U_{n-1}\left(\cos\left(\left(\frac{\alpha-i+j}{n}\right)\pi\right)\right)A^{-\frac{\alpha-i+j}{n}}.$$

□

To better show the next results we will use the following notations for the coefficients of matricial representation $\Lambda^{-\alpha}$:

$$C_{(\alpha,n,\text{differ})} = \frac{(-1)^{i-j}}{n} U_{n-1}\left(\cos\left(\left(\frac{\alpha-i+j}{n}\right)\pi\right)\right); \tag{2.6}$$

and

$$C_{(\alpha,n,\text{equal})} = \frac{1}{n} U_{n-1}\left(\cos\left(\frac{\alpha\pi}{n}\right)\right). \tag{2.7}$$

The terms 'differ' and 'equal' subscripts above refer to the entry of matricial representation of $\Lambda^{-\alpha}$ for $i \neq j$ and $i = j$, respectively. Therefore, we have the following results.

Proposition 2.3. *Let $C_{(\alpha,n,\mathit{differ})}$ and $C_{(\alpha,n,\mathit{equal})}$ be the coefficients defined in (2.6) and (2.7), respectively. Then*

$$\lim_{\alpha\searrow 0} \frac{C_{(\alpha,n,\mathit{differ})}}{\alpha} = \frac{1}{n}\frac{\pi}{\sin\left(\left(\frac{-i+j}{n}\right)\pi\right)},$$

and

$$\lim_{\alpha\searrow 0} \frac{C_{(\alpha,n,\mathit{equal})}-1}{\alpha} = 0.$$

Proof. Using (2.1), we have

$$U_{n-1}\left(\cos\left(\left(\frac{\alpha-i+j}{n}\right)\pi\right)\right) = \frac{\sin((\alpha-i+j)\pi)}{\sin\left(\left(\frac{\alpha-i+j}{n}\right)\pi\right)}.$$

Then

$$\begin{aligned}
\lim_{\alpha\searrow 0} \frac{C_{(\alpha,n,\text{differ})}}{\alpha} &= \lim_{\alpha\searrow 0} \frac{(-1)^{i-j}}{n\sin\left(\left(\frac{\alpha-i+j}{n}\right)\pi\right)} \frac{\sin((\alpha-i+j)\pi)}{\alpha} \\
&= \lim_{\alpha\searrow 0} \frac{(-1)^{i-j}}{n\sin\left(\left(\frac{\alpha-i+j}{n}\right)\pi\right)} \frac{\sin(\alpha\pi)}{\alpha\pi}\pi \\
&= \frac{1}{n}\frac{\pi}{\sin\left(\left(\frac{-i+j}{n}\right)\pi\right)}.
\end{aligned}$$

Now, we will calculate $\lim_{\alpha\searrow 0} \frac{C_{(\alpha,n,\text{equal})}-1}{\alpha}$. Note that as before, we have

$$U_{n-1}\left(\cos\left(\frac{\alpha\pi}{n}\right)\right) = \frac{\sin(\alpha\pi)}{\sin\left(\frac{\alpha\pi}{n}\right)},$$

and

$$\frac{1}{\alpha}\left[\frac{1}{n}\frac{\sin(\alpha\pi)}{\sin\left(\frac{\alpha\pi}{n}\right)} - 1\right] = -\frac{n-\sin(\alpha\pi)\csc\left(\frac{\alpha\pi}{n}\right)}{n\alpha}.$$

Thus,

$$\lim_{\alpha\searrow 0} \frac{C_{(\alpha,n,\text{equal})}-1}{\alpha} = \lim_{\alpha\searrow 0} -\frac{n-\sin(\alpha\pi)\csc\left(\frac{\alpha\pi}{n}\right)}{n\alpha}.$$

Note that this limit does not exist as $\alpha \searrow 0$. For the L'Hôspital rule, we have

$$\lim_{\alpha\searrow 0} -\frac{n-\sin(\alpha\pi)\csc\left(\frac{\alpha\pi}{n}\right)}{n\alpha} = \frac{\pi}{n^2}\lim_{\alpha\searrow 0}\csc\left(\frac{\alpha\pi}{n}\right)\left(\sin(\alpha\pi)\cot\left(\frac{\alpha\pi}{n}\right) - n\cos(\alpha\pi)\right)$$

$$= \frac{\pi}{n^2}\lim_{\alpha\searrow 0}\frac{\sin(\alpha\pi)\cos\left(\frac{\alpha\pi}{n}\right)-\cos(\alpha\pi)\sin\left(\frac{\alpha\pi}{n}\right)}{\sin^2\left(\frac{\alpha\pi}{n}\right)}$$

$$= \frac{\pi}{2n^2}\lim_{\alpha\searrow 0}\frac{(n^2-1)\sin(\alpha\pi)}{\cos\left(\frac{\alpha\pi}{n}\right)}$$

$$= 0.$$

In the third equality above, note that the limit also does not exist, and we use L'Hôspital's rule again to conclude. □

Theorem 2.4. *If A and $\varLambda$ are as in (1.4)-(1.5), respectively, then*

$$\log \varLambda = [\ell_{i,j}] \tag{2.8}$$

where

$$\ell_{i,j} = \begin{cases} \log A^{\frac{1}{n}}, & i = j, \\ -\dfrac{1}{n}\dfrac{\pi}{\sin\left(\dfrac{i-j}{n}\pi\right)}A^{\frac{i-j}{n}}, & i \neq j. \end{cases}$$

Moreover

$$D(\log \varLambda) = (X^{\frac{n-1}{n}} \cap D(\log A)) \times \cdots \times (X^{\frac{1}{n}} \cap D(\log A)) \times D(\log A). \tag{2.9}$$

Proof. Let $\alpha > 0$ and $\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \in D(\log \varLambda)$, then by Theorem 2.2, we have

$$\frac{1}{\alpha}(\varLambda^{-\alpha} - I)\mathbf{u}$$

$$= \begin{bmatrix} \frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I) & \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha+1}{n}} & \cdots & \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-1}{n}} \\ \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-1}{n}} & \frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I) & \cdots & \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha+n-2}{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-n+1}{n}} & \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-n+2}{n}} & \cdots & \frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I)v_1+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha+1}{n}}v_2+\cdots+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-1}{n}}v_n \\ \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-1}{n}}v_1+\frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I)v_2+\cdots+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha+n-2}{n}}v_n \\ \vdots \\ \frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-n+1}{n}}v_1+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-n+2}{n}}v_2+\cdots+\frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I)v_n \end{bmatrix}.$$

We can write

$$\frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}} - I) = \frac{1}{\alpha}(C_{(t,n,equal)} - 1)A^{-\frac{\alpha}{n}} + \frac{1}{\alpha}(A^{-\frac{\alpha}{n}} - I) \tag{2.10}$$

If we compute the limit at each entry of the matrix representation, thanks to Proposition 2.3 and (2.10), we have the following convergences

$$\frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I)v_1+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha+1}{n}}v_2+\cdots+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-1}{n}}v_n$$

converges to

$$-\log A^{\frac{1}{n}}v_1+\frac{1}{n}\frac{\pi}{\sin\left(\frac{\pi}{n}\right)}A^{-\frac{1}{n}}v_2+\cdots\frac{1}{n}\frac{\pi}{\sin\left(\left(\frac{n-1}{n}\right)\pi\right)}A^{-\frac{n}{-1}}v_n$$

in $X^{\frac{n-1}{n}}$, as $\alpha\searrow 0$;

$$\frac{1}{\alpha}(C_{(t,n,differ)}A^{-\frac{\alpha-1}{n}}v_1+\frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I)v_2+\cdots+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-2}{n}}v_n$$

converges to

$$-\frac{1}{n}\frac{\pi}{\sin\left(\left(\frac{n-1}{n}\right)\pi\right)}A^{\frac{1}{n}}+(-\log A^{\frac{1}{n}}v_1+\frac{1}{n}\frac{\pi}{\sin\left(\frac{\pi}{n}\right)}A^{-\frac{1}{n}}v_2+\cdots\frac{1}{n}\frac{\pi}{\sin\left(\left(\frac{n-1}{n}\right)\pi\right)}A^{-\frac{n-2}{n}}v_n$$

in $X^{\frac{n-2}{n}}$, as $\alpha\searrow 0$;

$$\vdots$$

$$\frac{1}{\alpha}(C_{(t,n,differ)}A^{-\frac{\alpha-n+1}{n}}v_1+\frac{1}{\alpha}C_{(t,n,differ)}A^{-\frac{\alpha-n+2}{n}}v_2+\cdots+\frac{1}{\alpha}(C_{(t,n,equal)}A^{-\frac{\alpha}{n}}-I)v_n$$

converges to

$$-\frac{1}{n}\frac{\pi}{\sin\left(\left(\frac{n-1}{n}\right)\pi\right)}A^{\frac{1}{n}}v_1+\left(-\frac{1}{n}\right)\frac{\pi}{\sin\left(\left(\frac{n-1}{n}\right)\pi\right)}A^{-\frac{n-2}{n}}v_2+\cdots+(-\log A^{\frac{1}{n}})v_n$$

in X, as $\alpha\searrow 0$.

By the previous analysis, we conclude that (2.8) and (2.9) hold. □

3 Logarithmic equations

One of the main results of this work is based on the fact that it is well known that the unbounded linear operator $-\log\Lambda$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on Y, see e.g. [2]. In other words, the semilinear Cauchy problem (1.7) is a well-posed problem on Y via the theory of strongly continuous semigroup of bounded linear operators. Namely, we have the following results.

If we propose the semilinear Cauchy problem (1.7) on Y and we write $\mathbf{u}_\ell=[v_{i\ell}]_{i\in\{1,2,\dots,n\}}$ then thanks to our previous results, the remarks above and [7, Theorem 2.2], we can rewrite the semilinear differential equation in (1.7) as follows:

$$\frac{d^n v_{1\ell}}{dt^n}+A_{n-1}\frac{d^{n-1}v_{1\ell}}{dt^{n-1}}+\cdots+A_1\frac{dv_{1\ell}}{dt}+A_0v_{1\ell}=\text{The } 1,1 \text{ entry of the matrix } [p(\log\Lambda)f(\mathbf{u}_\ell)],$$

where

$$p(\log\Lambda)=(\log\Lambda)^{n-1}+A_{n-1}(\log\Lambda)^{n-2}+\cdots+A_2\log\Lambda+A_1.$$

subject to initial conditions given by

$$\partial_t^j v_{1\ell}(0)=w_j,\quad j=0,1,\dots,n-1.$$

The following results deal with the well-posedness of problem (1.7) with f identically zero.

Theorem 3.1. *There exists a unique mild solution to the linear Cauchy problem*

$$\begin{cases}\dfrac{d\mathbf{u}_\ell}{dt}+(\log\Lambda)\mathbf{u}_\ell=0,\ t>0,\\ \mathbf{u}_\ell(0)=\mathbf{u}_{\ell_0},\end{cases}\tag{3.1}$$

given by

$$\mathbf{u}_\ell(t)=\Lambda^{-t}\mathbf{u}_0$$

for any $t\geqslant 0$ such that if $\mathbf{u}_{\ell_0}\in Y$ then $\mathbf{u}_\ell\in C([0,\infty),Y)\cap C^1((0,\infty),Y)$.

Proof. Since Λ is of positive type $K \geq 1$, the linear operator $-\log \Lambda$ is the infinitesimal generator of $\{\Lambda^{-t} : t \geq 0\}$ which is an analytic semigroup of angle $\pi/2$, see [2, Theorem 4.6.4]. □

Moreover, we can rewrite the linear differential equation in (3.1) as follows

$$\frac{d^n}{dt^n}v_{1\ell} + \log \Lambda \frac{d^{n-1}}{dt^{n-1}}v_{1\ell} + \sum_{k=1}^{n-1}(-1)^k \text{tr}(\Lambda^k(\log \Lambda))\frac{d^{n-k}}{dt^{n-k}}v_{1\ell} + \det(\log \Lambda)v_{1\ell} = 0$$

for $t > 0$, with initial conditions given by

$$\partial_t^j v_{1\ell}(0) = w_j, \quad j = 0, 1, \ldots, n-1.$$

In a suitable space such that $\mathbf{u}_{\ell_0} \in Y$, and

$$\text{tr}(\Lambda^k M) = \frac{1}{k!}\det([t_{i,j}]_k), \quad t_{i,j} = \begin{cases} 0, & j > i+1, \\ k-i, & j = i+1, \\ \text{tr}(M^{i-j+1}), & j \leqslant i. \end{cases} \tag{3.2}$$

Proposition 3.2. *Let f be a function defined in $X^{\frac{1}{n}}$ taking values in X such that it is Lipschitz continuous in bounded subsets of $X^{\frac{1}{n}}$. Then, the function F defined as in (1.6) on Y taking values in Y is Lipschitz continuous in bounded subsets of Y.*

Proof. Let B be a bounded subset of Y. If $\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in Y$, then for some $c_B > 0$ we have

$$\left\| F\left(\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}\right) - F\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right)\right\|_Y^2 = \|f(w_1) - f(x_1)\|_X^2 \leqslant c_B\|w_1 - x_2\|_X^2.$$

□

Theorem 3.3. *Let f, F as in Proposition 3.2. Given a bounded subset B of Y and $\mathbf{u}_0 \in B$. Then the semilinear Cauchy problem (1.7) is a locally well-posed problem on Y; that is, for every $\mathbf{u}_0 \in B$ there exists a $t_0(\mathbf{u}_0) > 0$ such that (1.7) has a unique mild solution on $[0, t_0(\mathbf{u}_0))$ given by (1.8). Moreover, if $t_0(\mathbf{u}_0) < \infty$ then*

$$\lim_{t \nearrow t_0(\mathbf{u}_0)} \|\mathbf{u}(t)\| = \infty.$$

Proof. The result is a consequence of our previous analysis jointly with [23, Theorem 1.4]. □

4 Applications

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with with sufficiently smooth boundary $\partial\Omega$ with $N \in \{1, 2, 3\}$, and let $X = L^2(\Omega)$ be endowed with the standard inner product. In this section we consider the unbounded linear operators $A_D : D(A_D) \subset X \to X$ defined by linear $2m$-th order uniformly elliptic partial differential operator

$$A_D u = (-\Delta)^m u, \quad m \in \mathbb{N}, \tag{4.1}$$

with domain

$$D(A_D) = H^{2m}(\Omega) \cap H_0^m(\Omega), \tag{4.2}$$

and we also consider the linear evolution equations of n-th order in time with $n \geqslant \frac{4}{4-N}$ and $m \in \mathbb{N}$ ($m \geqslant 1$)

$$\partial_t^n u + (-\Delta)^m u = f(u), \tag{4.3}$$

subject to zero Dirichlet boundary conditions and initial conditions

$$\begin{cases} u(x,t) = \Delta^j u(x,t) = 0, & x \in \partial\Omega, \ t \geqslant 0, \ i \in \{j, \ldots, m-1\}, \\ u(x,0) = u_0(x), \ \partial_t^i u(x,0) = u_i(x), & x \in \partial\Omega, \ i \in \{1, \ldots, n-1\}. \end{cases} \tag{4.4}$$

The unbounded linear operator A_D defined in (4.1)-(4.2) is a closed, densely defined, self-adjoint, and positive definite operator. There exists $\zeta > 0$ such that $\mathrm{Re}\sigma(A_D) > \zeta$, that is, $\mathrm{Re}\lambda > \zeta$ for all $\lambda \in \sigma(A)$, and therefore, A_D is a sectorial operator in the sense of Henry [18, Definition 1.3.1], with the eigenvalues $\{\nu_j\}_{j\in\mathbb{N}}$:

$$0 < \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_j \leqslant \dots, \quad \nu_j \to +\infty \quad \text{as } j \to +\infty.$$

This allows us to define the fractional power $A_D^{-\alpha}$ of order $\alpha \in (0,1)$ according to [2, Formula 4.6.9] and [18, Theorem 1.4.2], as a closed linear operator on its domain $D(A_D^{-\alpha})$ with inverse A_D^{α}. Denote by $X^\alpha = D(A_D^\alpha)$ for $\alpha \in [0,1]$. The fractional power space X^α endowed with the graphic norm

$$\|\cdot\|_{X^\alpha} := \|A_D^\alpha \cdot\|_X$$

is a Banach space; namely, e.g., if $m\alpha$ is an integer, then

$$X^\alpha = D((-\Delta)^{m\alpha}) = H^{2m\alpha}(\Omega) \cap H_0^{m\alpha}(\Omega)$$

with equivalent norms, see [13, Page 29] and [18, Pages 29 and 30].

With this notation, we have $X^{-\alpha} = (X^\alpha)'$ for all $\alpha > 0$, see [2] and [25] for the characterization of the negative scale. The scale of fractional power spaces $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ associated with A_D safisty

$$X^\alpha \subset H^{2m\alpha}(\Omega), \quad \alpha \in [0,1],$$

where $H^{2m\alpha}(\Omega)$ are the potential Bessel spaces, see Cholewa and Dłotko [13, Page 48]. Moreover, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ in 4.3 is a continuously differentiable function satisfying for some $2 < \rho \leqslant \frac{nN}{nN-4m(n-1)}$ the growth condition

$$|f'(s)) \leqslant C(1 + |s|^{\rho-1}), \ s \in \mathbb{R}. \tag{4.5}$$

The following result is a direct consequence of (4.5) via Mean Value's Theorem.

Lemma 4.1. *Let f be a real function of one real variable such that (4.5) holds. Then*

$$|f(s_1) - f(s_2)| \leqslant 2^{\rho-1} c |s_1 - s_2| \big(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}\big),$$

for any $s_1, s_2 \in \mathbb{R}$.

Moreover, we have the following result.

Lemma 4.2. *Let f be a real function of one real variable such that the condition (4.5) holds. Then there exists $s \in (0, \frac{nN}{4})$ such that the Nemitskiĭ operator $f^e : X^{\frac{n-1}{n}} \to X^{-\frac{s}{n}}$ given by $f^e(u)(x) = f(u(x))$ for any $u \in X^{\frac{n-1}{n}}$ and $x \in \Omega$ is Lipschitz continuous in bounded subsets of $X^{\frac{n-1}{n}}$.*

Proof. Let B be a bounded subset of $X^{\frac{n-1}{n}}$ and $u_1, u_2 \in B$. Let $s \in (0, \frac{nN}{4})$ such that

$$\rho \leqslant \frac{nN + 4ms}{nN - 4ms}.$$

Since $X^\alpha \hookrightarrow H^{2m\alpha}(\Omega)$ for any $\alpha > 0$, we have

$$X^{\frac{n-1}{n}} \hookrightarrow X^{\frac{s}{n}} \hookrightarrow H^{\frac{2ms}{n}}(\Omega) \hookrightarrow L^{\frac{2nN}{nN-4ms}}(\Omega).$$

Therefore $L^{\frac{2nN}{nN+4ms}}(\Omega) \hookrightarrow X^{-\frac{s}{n}}$. Now, by Lemma 4.1 and Hölder's inequality, we obtain

$$\begin{aligned}
\|f^e(u_1) - f^e(u_2)\|_{X^{-\frac{s}{n}}} &\leqslant c_0 \|f^e(u_1) - f^e(u_2)\|_{L^{\frac{2nN}{nN+4ms}}(\Omega)} \\
&\leqslant c_0 \|2^{\rho-1} c |u_1 - u_2| (1 + |u_1|^{\rho-1} + |u_2|^{\rho-1})\|_{L^{\frac{2nN}{nN+4ms}}(\Omega)} \\
&\leqslant c_1 \|u_1 - u_2\|_{L^{\frac{2nN}{nN-4ms}}(\Omega)} \|1 + |u_1|^{\rho-1} + |u_2|^{\rho-1}\|_{L^{\frac{nN}{4ms}}(\Omega)} \\
&\leqslant c_2 \|u_1 - u_2\|_{L^{\frac{2nN}{nN-4ms}}(\Omega)} (1 + \|u_1\|^{\rho-1}_{L^{\frac{nN(\rho-1)}{4ms}}(\Omega)} + \|u_2\|^{\rho-1}_{L^{\frac{nN(\rho-1)}{4ms}}(\Omega)}),
\end{aligned}$$

where c_0 is the embedding constant from $L^{\frac{2nN}{nN+4ms}}(\Omega)$ to $X^{-\frac{s}{n}}$.

From Sobolev embeddings, we have

$$X^{\frac{n-1}{n}} \hookrightarrow X^{\frac{s}{n}} \hookrightarrow H^{\frac{2ms}{n}}(\Omega) \hookrightarrow L^{\frac{nN(\rho-1)}{4ms}}(\Omega),$$

for all $2 < \rho \leqslant \frac{nN+4ms}{nN-4ms}$, it follows that

$$\|f^e(u_1) - f^e(u_2)\|_{X^{-\frac{s}{n}}} \leqslant K\|u_1 - u_2\|_{X^{\frac{n-1}{n}}}(1 + \|u_1\|^{\rho-1}_{X^{\frac{n-1}{n}}} + \|u_2\|^{\rho-1}_{X^{\frac{n-1}{n}}}),$$

for some constant $K > 0$.

□

Remark 4.3. Since $L^{\frac{2nN}{(nN-4m(n-1))\rho}}(\Omega) \hookrightarrow L^2(\Omega)$ for all $2 < \rho \leqslant \frac{nN}{nN-4m(n-1)}$, it follows from the proof of Lemma 4.2 that $f^e : X^{\frac{n-1}{n}} \to L^2(\Omega)$ is Lipschitz continuous in bounded subsets; that is,

$$\|f^e(u_1) - f^e(u_2)\|_{L^2(\Omega)} \leqslant k\|f^e(u_1) - f^e(u_2)\|_{L^{\frac{2nN}{nN-4m(n-1)\rho}}(\Omega)} \leqslant k_1\|u_1 - u_2\|_{X^{\frac{n-1}{n}}},$$

where $k_1 = k_1(\|u_1\|_{X^{\frac{n-1}{n}}}, \|u_2\|_{X^{\frac{n-1}{n}}})$. The scheme below describes this situation:

$$X^{\frac{n-1}{n}} \hookrightarrow H^{\frac{2m(n-1)}{n}}(\Omega) \hookrightarrow L^{\frac{2nN}{nN-4m(n-1)}}(\Omega) \overset{f(u)\approx u^\rho}{\longrightarrow} L^{\frac{2nN}{(nN-4m(n-1))\rho}}(\Omega) \hookrightarrow L^2(\Omega),$$

with $1 < \rho \leqslant \frac{nN}{nN-4m(n-1)}$.

A direct consequence of Lemma 4.2 and Remark 4.3 is the following result.

Corollary 4.4. *If f is as in Lemma 4.2, then the function $F : Y \to Y$ given by (1.6) is Lipschitz continuous in bounded subsets of Y.*

Now, Theorem 3.3 and [23, Theorem 1.4] guarantee local well-posedness for the semilinear Cauchy problem (1.7) on Y with $A = (-\Delta_D)^m$ and f as in Lemma 4.2.

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Received: 2022-12-24
Accepted: 2023-09-25