Palestine Journal of Mathematics

Vol 13(1)(2024) , 291–301

On Coupled Systems of Time-Fractional Differential Problems by Using $\psi-$Caputo fractional derivative

M'hamed Elomari, Said Melliani, Khadija Oufkir and Hamid Sadiki

Communicated by Ayman Badawi

MSC 2010 Classifications: 20M99, 13F10, 13A15, 13M05.

Keywords: Semigroup, maximal subsemigroup, nongenerator, power-joined partition, special principal ideal ring, finite ring, prime ideal, nilpotent, unit, Frattini subgroup.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract In this paper, we introduce a novel theorem concerning the existence of solutions for Coupled Systems of Time-Fractional Differential Problems. This is achieved through the utilization of the ψ-Caputo fractional derivative, where the order is confined within the range of $0 < \alpha < 1$. The validation of the existence result is demonstrated by employing certain conditions, namely Lipschitz and Carathéodory conditions.

1 Introduction

Fractional calculus has emerged as a pivotal tool in contemporary research, greatly aiding researchers in tackling intricate problems. The methodologies and techniques within this field have undergone substantial refinement in recent years. An illustrative instance is the establishment of existence results for numerous linear or nonlinear fractional equations, each defined with specific initial conditions. Across various articles, scholars are delving into fractional differential problems concerning time. Each study introduces a fresh fractional derivative approach. For instance, in [10], the problem is examined utilizing the Caputo-Fabrizio fractional derivative.

Conversely, our article ventures into a similar problem domain but employs the ψ-Caputo fractional derivative. Furthermore, we incorporate the pantograph fractional equation key to enrich our investigation. Our research is centered around the examination of an equation presented in the following form:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi} x(\xi,t) = \varphi_1\left((\xi,\ t,\ x(\xi,\eta t),\ y(\xi,\delta t)\right), \\ {}^{C}D_{0^+}^{\beta,\psi} y(\xi,t) = \varphi_2\left(\xi,t,x(\xi,\eta t),y(\xi,\delta t)\right). \end{cases} \tag{1.1}$$

This portrays a system comprising of two nonlinear time-fractional differential equations, where the fractional derivative employed is the ψ-Caputo derivative.

$$\begin{cases} x(0,0) = \phi_1(x), \\ y(0,0) = \phi_2(y), \end{cases} \tag{1.2}$$

where $0 < \alpha < 1,\ 0 < \beta < 1,\ (x,t) \in [0,1] \times [0,1]$, $\eta, \delta > 0$, $\phi_1, \phi_2 \in C(X,\mathbb{R})$ and the mappings φ_1, φ_2 continuous functions of $[0,1]^2 \times \mathbb{R}^2$ with values in $\mathbb{R}$. Subsequent to investigating the aforementioned equation system, our focus will shift towards examining the same system, albeit with the inclusion of:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi} x(\xi,t) \in \mathcal{F}_1(\xi,\ t,\ x(\xi,\eta t),\ y(\xi,\delta t)), \\ {}^{C}D_{0^+}^{\beta,\psi} y(\xi,\ t) \in \mathcal{F}_2(\xi,t,x(\xi,\eta t),y(\xi,\delta t)). \end{cases} \tag{1.3}$$

with the follwing conditions,

$$\begin{cases} x(0,0) = \phi_1(x), \\ y(0,0) = \phi_2(y), \end{cases} \tag{1.4}$$

where $F_1, F_2 : [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are some multivalued maps.

This paper is structured in the following manner: firstly, we provide the essential preliminaries required to delve into the study of ψ-Caputo fractional calculus. Subsequently, a significant portion is dedicated to presenting the outcomes derived from our analysis of the problem. Finally, our discourse concludes with a succinct summary.

2 Preliminaries

In this section, we will commence by introducing all the essential outcomes requisite for our investigation into ψ-Caputo fractional calculus, encompassing both the derivative and the integral aspects. For a comprehensive understanding, we direct interested readers to references [2, 4, 6].

Definition 2.1. [3] The $\psi-$Caputo fractional derivative at order α of the function u is given by

$$ {}^C D_{0^+}^{\alpha,\psi} u(x,t) = \frac{1}{\Gamma(n-(\alpha))} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-(\alpha)-1} u_\psi^{[n]}(x,s)ds \tag{2.1} $$

Where

$$ u_\psi^{[n]}(x,s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n u(x,s) \quad and \quad n = [\alpha]+1. $$

And $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2. [3] The ψ-Riemann-Liouville fractional integral at order α of the function u is given by

$$ I_{0^+}^{\alpha,\psi} u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} u(x,s)ds. \tag{2.2} $$

Remark 2.3. In particular, if $\alpha \in]0,1[$, then we have

$$ {}^C D_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t)-\psi(s))^{\alpha-1} g'(s)ds. \tag{2.3} $$

and

$$ {}^C D_{0^+}^{\alpha,\psi} g(t) = I_{0^+}^{1-\alpha,\psi} \left(\frac{g'(t)}{\psi'(t)} \right) $$

The solution of $({}^C D_{0^+}^{\alpha,\psi} x)(\xi,t) = g(\xi,t)$ is written in the form

$$ x(\xi,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(\xi,s)ds. \tag{2.4} $$

Before proceeding, we consider the following sets :
(X,d) : a metric space;
$\mathcal{P}(X)$: the class of all subsets of X:
2^X: the class of all nonempty subsets of X;
$\mathcal{P}_{\text{cl}}(X)$: the class of all closed subsets of X;
$\mathcal{P}_{\text{bd}}(X)$:the class ofall bounded subsets of X;
$\mathcal{P}_{\text{cv}}(X)$: the class of all convex subsets of X;
$\mathcal{P}_{\text{cp}}(X)$: the class of all compact subsets of X;
$\mathcal{P}_{\text{cp,cv}}(X)$: the class of all compact and convex subsets of X.

Definition 2.4. [1, 9]

i- Let the function $\mathcal{F} : X \to 2^X$, $x \in X$ is said to be a fixed point of $\mathcal{F}$ when : $x \in \mathcal{F}x$.

ii- The function $\mathcal{F} : [0,1] \times [0,1] \to \mathcal{P}_{\text{cl}}(\mathbb{R})$ is called measurable if for all $w \in \mathbb{R}$:

$$ (\xi,t) \mapsto d(w, \mathcal{F}(\xi,\ t)) = \inf\{\|w-x\| \ : \ x \in \mathcal{F}(\xi,\ t)\}. $$

iii- The Pompeiu-Hausdorffmetric H_d: $2^X \times 2^X \to [0, \infty)$ is defined by :

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

such that $d(A,b) = \inf_{a\in A} d(a, b)$.

Let $CB(X)$ the set of closed and bounded subsets of X, and $C(X)$ the set of closed subsets of X.

Proposition 2.5. *[9, 1]*

1- $(CB(X), H_d)$ is a metric space.

2- $\mathcal{F}$ is convex-valued, if for all $x \in X$, $\mathcal{F}x$ is convex .

3- $\mathcal{F}$ is compact-valued, if for all $x \in X$, $\mathcal{F}x$ is compact .

4- $\mathcal{F} : X \to C(X)$ is contracting if there is a constant $\gamma \in (0,1)$ such that : $H_d(\mathcal{F}x, \mathcal{F}y) \leq \gamma d(x, y)$, $\forall x, y \in X$.

Definition 2.6. Let $\mathcal{F} : [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to 2^{\mathrm{R}}$ is a multifunction .

i- We say that $\mathcal{F}$ is a Caratheodory multifunction if for all $(\xi,t) \in [0,1]^2$ and $(x_1,x_2) \in \mathbb{R}$: $(\xi,t) \longmapsto \mathcal{F}(\xi,t,x_1,x_2)$ is mesurable, and $(x_1,x_2) \longmapsto \mathcal{F}(\xi,t,x_1,x_2)$ is upper semicontinuous .

ii- We say that $\mathcal{F}$ is a L^1-Caratheodory when for each $\rho > 0$ there exists $\beta_\rho \in L^1([0,1] \times [0,1], \mathbb{R}^+)$ such that :

$$\|\mathcal{F}(\xi,t,x_1,x_2)\| = \sup_{(\xi,t)\in[0,1]\mathrm{x}[0,1]} \{|s| : s \in \mathcal{F}(\xi,t,x_1,x_2)\} \leq \beta_\rho(\xi,t),$$

for all $|x_i| \leq \rho$ and $(\xi, t) \in [0,1] \times [0,1]$.

We pass to define some notions related to multifunction.

Definition 2.7. 1- The set of selections of $\mathcal{F}_i$ at x_i is defined by :

$$S_{\mathcal{F}_i(x_i)} = \left\{ \begin{array}{ll} w_i \in L^1([0,1]\times[0,1], \mathbb{R}) & : w_i(\xi, t) \in \mathcal{F}(\xi, t, x_i(\xi, t), x_i'(\xi, t)) \\ \text{for almost all } (\xi,t) \in [0,1]\times[0,1]\} & , \end{array} \right\} \quad (2.5)$$

for all $x_i, x_i' \in C_{\mathrm{R}}([0,1] \times [0,1])$ for $i = 1, 2$.

2- The graph of the multifunction $\mathcal{F} : X \to Y$ is defined by the set :

$$\mathrm{Gr}(\mathcal{F}) = \{(x,y) \in X \times Y : y \in F(x)\}.$$

Proposition 2.8. *1- For all $x_i \in C_K([0,1] \times [0,1])$ when $\dim K < \infty$, we have $S_{\mathcal{F}_i(x_i)}$ are nonempty.*

2- Let $\mathcal{F} : X \to Y$ is a multifunction, we say that $\mathrm{Gr}(\mathcal{F})$ is a closed subset of $X \times Y$, when for all sequences $\{x_n\}_{n\in\mathbb{N}} \in X$ with $x_n \to x_0$ and $\{y_n\}_{n\in\mathbb{N}} \in Y$ with $y_n \to y_0$, such that $y_n \in \mathcal{F}(x_n)$ we have $y_0 \in \mathcal{F}(x_0)$.

We provide several crucial results that will be utilized in the subsequent proofs.

Theorem 2.9. *Let $\mathcal{T} : X \to X$ the completely continuous operator with X is a Banach space. The set $\mathcal{K} = \{x \in X, x = \lambda\mathcal{T}x \ , \ \lambda \in [0,1]\}$ is bounded. Then, the operator $\mathcal{T}$ has a fixed point.*

Lemma 2.10. *Let $\mathcal{F}: X \to \mathcal{P}_{cl}(Y)$,*

i- $\mathrm{Gr}(\mathcal{F})$ *is closed subset of $X \times Y$ if $\mathcal{F}$ is upper semicontinuous.*

ii- $\mathcal{F}$ *is upper semicontinuous if $\mathcal{F}$ is completely continuous and* $\mathrm{Gr}(\mathcal{F})$ *is closed.*

Lemma 2.11. *Let $\mathcal{F}: [0,1] \times [0,1] \times X \times X \to \mathcal{P}_{cp,cv}(X)$ an L^1-Caratheodory function with X is a separable Banach space, and let $\Theta : L^1([0,1] \times [0,1],\ X) \to C_X([0,1] \times [0,1])$ a linear continuous mapping. Then, the operator :*

$$\Theta \cdot S_{\mathcal{F}} : \begin{cases} C_X([0,1] \times [0,1]) \to \mathcal{P}_{cp,cv}(C_X([0,1] \times [0,1]) \\ \qquad\qquad x \mapsto (\Theta \cdot S_{\mathcal{F}})(x) = \Theta(S_{F,x}), \end{cases}$$

is a closed graph operator.

Theorem 2.12. *Let E be a Banach space C a dosed convex subset of E, U an open subset of C and $0 \in U$. Let us suppose that $\mathcal{F} : \overline{U} \to \mathcal{P}_{cp,cv}(C)$ depicts an upper semicontinuous compact map, such that $\mathcal{P}_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either $\mathcal{F}$ admits a fixed point in U or there exist $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda \mathcal{F}(x)$.*

3 Main result

We consider the following system:

$$\begin{cases} {}^C D_{0^+}^{\alpha,\psi} x(\xi,t) = \varphi_1\left(\xi,\ t,\ x(\xi,\eta t),\ y(\xi,\delta t)\right) \\ {}^C D_{0^+}^{\beta,\psi} y(\xi,t) = \varphi_2\left(\xi,t,x(\xi,\eta t),y(\xi,\delta t)\right) \end{cases} \tag{3.1}$$

with initial conditions $x(0,0) = \phi_1(x)$ and $y(0,0) = \phi_2(y)$.
Where: $\varphi_1, \varphi_2 : [0,1]^2 \times X^2 \to X$ are continuous mappings, $\phi_1, \phi_2 \in C(X, \mathbb{R})$, $\alpha, \beta \in (0,1)$, $(\xi, t) \in [0,1]$, and ${}^C D_{0^+}^{\alpha,\psi}$ and ${}^C D_{0^+}^{\beta,\psi}$ are the fractional derivatives of $\psi-$Caputo of order respectively α, β.
Consider $X = \{x : x \in C_{\mathrm{R}}([0,1] \times [0,1])\}$ is the Banach space with the following norm:

$$\|x\|_X = \sup_{(\xi,t)\in[0,1]\mathrm{x}[0,1]} |x(\xi,t)|.$$

And the space $(X \times X, \|.\|_{X\times X})$ is a Banach space via the product norm :

$$\|(x,\ y)\|_{X\times X} = \|x\|_X + \|y\|_X.$$

We proceed to show the following lemma concerning the integral solution.

Lemma 3.1. *Let $\varphi \in L^1_X([0,1] \times [0,1])$ and $\alpha \in (0,1)$, the fractional differential equation ${}^C D_{0^+}^{\alpha,\psi} x(\xi,\ t) = \varphi(\xi,t)$, with $x(0,0) = \phi_1(x)$, admits a unique integral solution $x_0 \in C_X([0,1] \times [0,1])$ of the form :*

$$x_0(\xi,t) = \phi_1(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(\xi,s) ds.$$

Proof. we apply the $\psi-$Caputo fractional integral in two sides and find that :

$$x_0(\xi,t) - x_0(0,0) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(\xi,s) ds$$

so, $x_0(\xi,\ t) = x_0(0,0) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(\xi,s) ds.$
This shows the general form of the integral solution, we go on to show the uniqueness of this solution.
Let x_1, x_2 two integral solutions of the problem, we have :

$${}^C D_{0+}^{\alpha,\psi} x_1(\xi,t) -{}^C D_{0+}^{\alpha,\psi} x_2(\xi,\ t) = [{}^C D_{0+}^{\alpha,\psi}(x_1 - x_2)](\xi,t) = 0 \ and \ (x_1 - x_2)(0,0) = 0.$$

By the property of the $\psi-$Caputo fractional derivative, we get $u_1 = u_2$.
Hence, u_0 is a unique solution of initial value problem .
Then, we conclude that :

$$x_0(\xi,\ t) = \phi_1(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(\xi, s) ds.$$

With : $x_0(0, 0) = \phi_1(x)$. □

We consider two operators $T_1, T_2 : X \to X$ defined as follows :

$$(T_1 x)(\xi,\ t) = \phi_1(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(\xi, s, x(\xi, \eta s), y(\xi, \delta s)) ds$$

$$(T_2 y)(\xi,\ t) = \phi_2(y) + \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \varphi(\xi, s, x(\xi, \eta s), y(\xi, \delta s)) ds$$

and put

$$N_1 = \frac{(\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)}, N_2 = \frac{(\psi(T) - \psi(0))^{\beta}}{\Gamma(\beta + 1)} \tag{3.2}$$

Before completing, we consider the following hypotheses:

(H_1) $\varphi_1, \varphi_2 : [0, 1]^2 \times X^2 \to X$ are continuous mappings, and there exist two constant $L_1, L_2 > 0$ such that :
$| \varphi_1(\xi, t, x_1, x_2) | \leq L_1$ and $| \varphi_2(\xi, t, x_1, x_2) | \leq L_2, \forall (\xi, t) \in [0, 1]^2$, $\forall x_1, x_2 \in X$.

(H_2) There two constants L_x , $L_y > 0$ such that :
$| \phi_1(x) | \leq L_x$ and $| \phi_2(y) | \leq L_y$, for all $x, y \in X$.

(H_3) Suppose there are two positive constants K_x , K_y such that for all $x_1, x_2, y_1, y_2 \in X$:

$$| \phi_1(x_1) - \phi_1(x_2) | \leq K_x \parallel x_1 - x_2 \parallel,$$

$$| \phi_2(y_1) - \phi_2(y_2) | \leq K_y \parallel y_1 - y_2 \parallel,$$

(H_4) There exist a nondecreasing bounded continuous map $\chi : [0,\ \infty) \to (0,\ \infty)$ and a continuous function $p : [0, 1] \times [0, 1] \to (0,\ \infty)$ such that :

$$\|\mathcal{F}_i(\xi,\ t,\ x_i(\xi,\ t), y_i(\xi,\ t))\| \leq p(\xi,\ t)\chi(\|x_i\|)\ ,\ \forall (\xi,\ t) \in [0, 1] \times [0, 1], x_i, y_i \in X.$$

Theorem 3.2. *Suppose that the two hypotheses (H_1) and (H_2) are verified then the system admits at least one solution.*

Proof. let $T : X \times X \to X \times X$ be an operator defined by:

$$T(x,\ y)(\xi,\ t) := ((T_1 x)(\xi,\ t),\ (T_2 y)(\xi,\ t))\ ,\ \forall (\xi,\ t) \in [0, 1] \times [0, 1].$$

According to the hypothesis (H_1) , T is a continuous operator. We prove that the operator T maps bounded sets into the bounded subsets of $X \times X$.
Let Ω be a bounded subset of$X \times X, (x,\ y) \in \Omega$, and $(\xi,\ t) \in [0, 1] \times [0, 1]$. Then, we have :

$$\begin{aligned}
|(T_1 x)(\xi,\ t)| &= |\phi_1(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(\xi, s, x(\xi, \eta s), y(\xi, \delta s)) ds| \\
&\leqslant L_x + \frac{(\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)} L_1 \\
&\leqslant L_x + N_1 L_1,
\end{aligned}$$

hince, $\|(T_1 x)(\xi,\ t)\|_X \leq L_x + L_1 N_1$.
Also, we have :

$$\begin{aligned}|(T_2 y)(\xi,\ t)| &= |\phi_2(y) + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\varphi(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds| \\ &\leqslant L_y + \frac{(\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}L_2 \\ &\leqslant L_y + N_2 L_2,\end{aligned}$$

hince, $\|(T_2 y)(\xi,\ t)\|_X \leq L_y + L_2 N_2$.
Thus,

$$\|T(x,\ y)(\xi,\ t)\|_{X\times X} \leq L_x + L_1 N_1 + L_y + L_2 N_2.$$

This shows that the operator T maps bounded sets into the bounded sets of $X \times X$.
We go on to show that T is an equicontinuous operator. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, we have :

$$\begin{aligned}|(T_1 x)(\xi,\ t_1) - (T_1 x)(\xi,\ t_2)| &= |\left(\phi_1(x) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\varphi_1(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds\right) \\ &- \left(\phi_1(x) + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi_1(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds\right)| \\ &\leqslant |\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\varphi_1(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi_1(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds| \\ &+ |\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi_1(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds| \\ &- \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi_1(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds| \\ &\leqslant \frac{L_1}{\Gamma(\alpha+1)}\left(|\psi^\alpha(t_2)-\psi^\alpha(t_1)-(\psi(t_2)-\psi(t_1))^\alpha| - (\psi(t_2)-\psi(t_1))^\alpha\right)\end{aligned}$$

When ψ is a continuous function, then we obtain : $|(T_1 x)(\xi,\ t_2) - (T_1 x)(\xi,\ t_1)| \to 0$ whenever $t_2 \to t_1$.
By utilizing the Arzela-Ascoli theorem, T_1 is completely continuous.
Similarly we have :

$$\begin{aligned}|(T_2 y)(\xi,\ t_1) - (T_2 y)(\xi,\ t_2)| &= |\left(\phi_2(y) + \frac{1}{\Gamma(\beta)}\int_0^{t_1}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\varphi_2(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds\right) \\ &- \left(\phi_2(y) + \frac{1}{\Gamma(\beta)}\int_0^{t_2}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\varphi_2(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds\right)| \\ &\leqslant |\frac{1}{\Gamma(\beta)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\beta-1}\varphi_2(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds \\ &- \frac{1}{\Gamma(\beta)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\beta-1}\varphi_2(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds| \\ &+ |\frac{1}{\Gamma(\beta)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\beta-1}\varphi_2(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds| \\ &- \frac{1}{\Gamma(\beta)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi_2(\xi,s,x(\xi,\eta s),y(\xi,\delta s))ds| \\ &\leqslant \frac{L_2}{\Gamma(\beta+1)}\left(\left|\psi^\beta(t_2)-\psi^\beta(t_1)-(\psi(t_2)-\psi(t_1))^\beta\right| - (\psi(t_2)-\psi(t_1))^\beta\right)\end{aligned}$$

by utilizing the Arzela-Ascoli theorem we observe that T_2 is completely continuous. Therefore, we get $\|T(x,\ y)(\xi,\ t_2) - T(x,\ y)(\xi,\ t_2)\|_{X\times X} \ \to 0$ whenever t_2 tends to t_1 . Thus, T is completely continuous.
Now show that the set, $\Omega = \{(u,\ \nu) \in X \times X : (u,\ \nu) = \lambda T(u,\ \nu)\ for\ some \lambda \in [0, 1]\}$ is bounded.
Let $x, y \in \Omega$ and $\lambda \in [0, 1]$ such that: $(x,\ y) = \lambda T(x,\ y)$.
Hence, $x(\xi,\ t) = \lambda(T_1 x)(\xi,\ t)$ and $y(\xi, t) = \lambda(T_2 y)(\xi, t)$ for all $(\xi,\ t) \in [0, 1] \times [0, 1]$.
We have :

$$\frac{1}{\lambda} \mid x(\xi,t) \mid = \mid (T_1 x)(\xi,t) \mid \leq L_x + L_1 N_1,$$

$$\frac{1}{\lambda} \mid y(\xi,t) \mid = \mid (T_2 y)(\xi,t) \mid \leq L_y + L_2 N_2,$$

so we get : $\mid x(\xi,t) \mid \leq \lambda(L_x + L_1 N_1)$ and $\mid y(\xi,t) \mid \leq \lambda(L_y + L_2 N_2)$.
Then,

$$\| (x, y) \|_{X\times X} \leq \lambda(L_x + L_1 N_1 + L_y + L_2 N_2).$$

which implies that Ω is bounded. Hence, T has a fixed point which is a solution for the coupled system of the time-fractional differential equations. □

Now, we study the existence of solution for the coupled system of time-fractional differential inclusions

$$\begin{cases} ({}^C D^{\alpha,\psi}_{0^+} x)(\xi,t) \in \mathcal{F}_1(\xi,\ t,\ x(\xi,\eta t),\ y(\xi,\delta t)), \\ ({}^C D^{\beta,\psi}_{0^+} y)(\xi,\ t) \in \mathcal{F}_2(\xi, t, x(\xi,\eta t), y(\xi,\delta t)) \end{cases} \tag{3.3}$$

with :

$$x(0, 0) = \phi_1(x), y(0, 0) = \phi_2(y), \tag{3.4}$$

Definition 3.3. Let $(x_1,\ x_2) \in C([0, 1] \times [0, 1],\ X) \times C([0, 1] \times [0, 1],\ X)$, we say that $(x_1,\ x_2)$ is a solution of system (3.3) with the conditions (3.4) if there is $(w_1,\ w_2) \in L^1([0, 1] \times [0, 1]) \times L^1([0, 1] \times [0, 1])$ such that $w_i(\xi,\ t) \in \mathcal{F}_i(\xi,\ t,\ x(\xi,\ t),\ y(\xi,\ t))$ for all $(\xi,\ t) \in [0, 1] \times [0, 1]$ and $i = 1, 2$ and xe have :

$$x_i(\xi,\ t) = \phi_i(x_i) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} w_i(\xi, s, x(\xi, \eta s), y(\xi, \delta s))ds.$$

Theorem 3.4. *Let $\mathcal{F}_1, \mathcal{F}_2 : [0, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ be L^1- Caratheodory multifunctions, and suppose that (H_4) is verifed .Then the system (3.3) − (3.4) has at least one solution* .

Proof. Let the operator $N : X \times X \to 2^{X\times X}$ define by $N(x_1,\ x_2) = \binom{N_1(x_1,x_2)}{N_2(x_1,x_2)}$, where:

$$N_1(x_1,\ x_2) = \{h_1 \in X\times X : there\ exists\ z_1 \in S_{\mathcal{F}_1,x_1}\ such\ that\ h_1(\xi,\ t) = z_1(\xi,\ t)\ \forall(\xi,\ t) \in [0, 1]\times[0, 1]\},$$

and,

$$N_2(x_1,\ x_2) = \{h_2 \in X\times X :\ there\ exists z_2 \in S_{\mathcal{F}_2,x_2}\ such\ that\ h_2(\xi,\ t) = z_2(\xi,\ t) \forall(\xi,\ t) \in [0, 1]\times[0, 1]\},$$

we have:

$$h_1(\xi,\ t) = \phi_1(x_1) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} z_1(\xi, s)ds.$$

$$h_2(\xi,\ t) = \phi_2(x_2) + \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} z_2(\xi, s)ds.$$

then, each fixed point of the operator N is a solution for system of time-fractional differential inclusions (3.3).
Step1 : Prove that N is convex-valued.
Let $(x_1,\ x_2) \in X \times X, (h_1,\ h_2), (h'_1,\ h'_2) \in N(u_1,\ u_2)$ and $z_i, z'_i \in S_{\mathcal{F}_i(x_1,x_2)}$ such that :

$$\begin{cases} h_1(\xi,\ t) = \phi_1(x_1) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} z_1(\xi, s)ds, \\ (h'_1(\xi,\ t) = \phi_1(x'_1) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} z'_1(\xi, s)ds \end{cases} \tag{3.5}$$

$$\begin{cases} h_2(\xi,\ t) = \phi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} z_2(\xi,s)ds, \\ (h_2'(\xi,\ t) = \phi_2(x_2') + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} z_2'(\xi,s)ds \end{cases} \tag{3.6}$$

Let $\lambda \in [0,1]$,then, we have :

$$[\lambda h_1 + (1-\lambda)h_1'](\xi,\ t) = (\lambda\phi_1(x_1) + (1-\lambda)\phi_1(x_1')) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[\lambda z_1(\xi,s) + (1-\lambda)z_1'(\xi,s)]ds.$$

and,

$$[\lambda h_2 + (1-\lambda)h_2'](\xi,\ t) = (\lambda\phi_2(x_2) + (1-\lambda)\phi_2(x_2')) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[\lambda z_2(\xi,s) + (1-\lambda)z_2'(\xi,s)]ds.$$

we have $\mathcal{F}_i$ has convex values, $S_{\mathcal{F}_i(x_i)}$ is a convex set and $[\lambda h_i + (1-\lambda)h_i'] \in N_i(x_1,\ x_2)$ for $i = 1, 2$.
This implies that the operator N has convex values.
Step2 : We go on to show that N maps bounded sets of X into bounded sets.
Let a positive constant a and the set $B_a = \{(x_1,\ x_2) \in X \times X : \|(x_1, x_2)\| \leq a\}$ be a bounded subset of $X \times X$,let $(h_1,\ h_2) \in N(x_1, x_2)$, and $(x_1,\ x_2) \in B_a$.
by the hypothesis (H_4), we obtain :

$$\begin{aligned} |(h_1)(\xi,t)| &= |\phi_1(x_1) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} z_1(\xi,s)ds| \\ &\leq L_x + \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}. \int_0^t |z_1(\xi,s)|ds \\ &\leq L_x + \|p\|_\infty \chi(\|x_1\|)N_1, \end{aligned}$$

Then,

$$\|h_1\| \leq L_x + \|p\|_\infty \psi(\|x_1\|)N_1.$$

Similarly, we obtain:

$$\|h_2\| \leq L_y + \|p\|_\infty \psi(\|x_2\|)N_2.$$

Thus, $\|(h_1,\ h_2)\| \leq (L_u + L_\nu) + \|p\|_\infty \chi(\|(u_1, u_2)\|)(N_1 + N_2)$.
Step3 : Prove that N maps bounded sets into equicontinuous subsets of $X \times X$.
Let $(x_1,\ x_2) \in B_a$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, we have:

$$\begin{aligned} |(h_1)(\xi,t_2) - (h_1)(\xi,t_1)| &= |\left(\phi_1(x_1) + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} z_1(\xi,s)ds\right) \\ &\quad - \left(\phi_1(x_1) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} z_1(\xi,s)ds\right)| \\ &\leq |\frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} z_1(\xi,s)ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} z_1(\xi,s)ds| \\ &\quad + |\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} z_1(\xi,s)ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} z_1(\xi,s)ds| \\ &\leq \frac{\|p\|_\infty \chi(\|x_1)\|)}{\Gamma(\alpha+1)} (|\psi^\alpha(t_2) - \psi^\alpha(t_1) - (\psi(t_2)-\psi(t_1))^\alpha| - (\psi(t_2)-\psi(t_1))^\alpha) \end{aligned}$$

By using a similar method, we obtain :

$$|(h_2)(\xi,\ t_2){-}(h_2)(\xi,\ t_1)| \leq \frac{\|p\|_\infty\chi(\|x_2)\|)}{\Gamma(\beta+1)}\left(\left|\psi^\beta(t_2)-\psi^\beta(t_1)-(\psi(t_2)-\psi(t_1))^\beta\right|-(\psi(t_2)-\psi(t_1))^\beta\right)$$

Hence, $|h_i(\xi,\ t_2)-h_i(\xi,\ t_1)| \to 0$ as $(\xi,\ t_2) \to (\xi,\ t_1)$.
By using the Arzela-Ascoli theorem we get that N is completely continuous. Hence, N is upper semicontinuous.
$\mathbf{Step4}$: Show that N has a closed graph.
Let $\{(x_1^n,\ x_2^n)\}$ be a sequence in $X\times X$ with $(x_1^n,\ x_2^n) \to (x_1^0,\ x_2^0)$.
We get the following sequences:
$(h_1^n,\ h_2^n) \in N(u_1^n,\ u_2^n)$ with $(h_1^n,\ h_2^n) \to (h_1^0, h_2^0)$.
And, $(\mathrm{v}_1^n,\ \nu_2^n) \in S_{F_1(u_1^n)}\times S_{F_2(u_2^n)}$ with $(z_1^n,\ z_2^n) \to (z_1^0, z_2^0)$.
We have :

$$h_1^n(\xi,\ t) = \phi_1(x_1^n) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}z_1^n(\xi,s)ds\ ,\ \forall \xi, t\in[0,1]$$

$$h_2^n(\xi,t) = \phi_2(x_2^n) + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}z_2^n(\xi,s)ds\ ,\ \forall \xi, t\in[0,1]$$

with,

$$h_1^0(\xi,\ t) = \phi_1(x_1^0) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}z_1^0(\xi,s)ds,$$
$$h_2^0(\xi,\ t) = \phi_2(x_2^0) + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}z_2^0(\xi,s)ds,$$

show that there exists $(\nu_1^0,\ \nu_2^0) \in S_{F_1(u_1^0)}\times S_{F_2(u_2^0)}$.
Now, consider the linear operators $\Theta_1, \Theta_2 : L^1([0,1]\times[0,1], X) \to C([0,1]\times[0,1], X)$ defined by :

$$\Theta_1(z_1)(\xi,t) = \phi_1(x_1) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}z_1(\xi,s)ds,$$
$$\Theta_2(z_2)(\xi,\ t) = \phi_2(x_2) + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}z_2(\xi,s)ds.$$

Xe have :

$$\begin{aligned}\|h_1^n(\xi,\ t)-h_1^0(\xi,\ t)\| &= \|\left(\phi_1(x_1^n) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}z_1^n(\xi,s)ds\right)\\ &\quad -\left(\phi_1(x_1^0) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}z_1^0(\xi,s)ds\right)\|\\ &\leq \|\ \phi_1(x_1^n)-\phi_1(x_1^0)\ \| + \|\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[z_1^n(\xi,s)-z_1^0(\xi,s)]ds\|\\ &\leq K_x\ \|\ x_1^n-x_1^0\ \| + \|\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[z_1^n(\xi,s)-z_1^0(\xi,s)]ds\| \to 0,\end{aligned}$$

and,

$$\begin{aligned}\|h_2^n(\xi,\ t)-h_2^0(\xi,\ t)\| &= \|\left(\phi_2(x_2^n) + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}z_1^n(\xi,s)ds\right)\\ &\quad -\left(\phi_2(x_2^0) + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}z_1^0(\xi,s)ds\right)\|\\ &\leq \|\ \phi_2(x_2^n)-\phi_2(x_2^0)\ \| + \|\frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}[z_2^n(\xi,s)-z_2^0(\xi,s)]ds\|\\ &\leq K_y\ \|\ x_2^n-x_2^0\ \| + \|\frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}[z_2^n(\xi,s)-z_2^0(\xi,s)]ds\| \to 0.\end{aligned}$$

So, $\Theta_i S_{F_i}$ is a closed graph operator for $i = 1, 2$.
Also, we get $h_i^n(x,\ t) \in \Theta_i(S_{F_i(u_i^n)})$ for all $n \in \mathbb{N}$.
Thus, N has a closed graph.
Step5 : Now, we prove that there is an open set $U \subseteq X$ with $(x_1,\ x_2) \notin N(x_1,\ x_2)$ for all $\lambda \in (0,1)$ and $(x_1,\ x_2) \in \partial U$.
Let $\lambda \in (0,1)$ and $(x_1,\ x_2) \in \lambda N(x_1,\ x_2)$. Then, there exists $z_i \in L^1([0,1] \times [0,1],\ \mathbb{R})$ with $z_i \in S_{\mathcal{F}_i(x_i)} (i = 1, 2)$ such that :

$$x_1(\xi,\ t) = \phi_1(x_1) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} z_1(\xi, s)ds\ ,\ \forall \xi, t \in [0,1]$$

$$x_2(\xi,\ t) = \phi_2(x_2) + \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} z_2(\xi, s)ds\ ,\ \forall \xi, t \in [0,1].$$

From the previous results we find that:

$$\|x_i\| \leq (L_x + L_y) + \|p\|_\infty \chi(\|x_i\|) \sum_{i=1}^{n} N_i.$$

Then,

$$\|x_i\|/(L_x + L_y) + \|p\|_\infty \chi(\|x_i\|) \sum_{i=1}^{n} N_i \leq 1$$

for $i = 1, 2$.
We take $M_i > 0$ with $\|x_i\| \neq M_i$ such that :

$$M_i/(L_x + L_y) + \|p\|_\infty \chi(\|x_i\|) \sum_{i=1}^{n} N_i > 1\ ,\ i = 1, 2.$$

Let $U = \{(x_1, x_2) \in X \times X : \|(x_1, x_2)\| < \min\{M_1, M_2\}\}$. The operator $N : \overline{U} \to \mathcal{P}(X)$ is upper semicontinuous and completely continuous. Also, we showed that there is no $(x_1,\ x_2) \in \partial U$ such that $(x_1,\ x_2) \in \lambda N(x_1,\ x_2)$ for some $\lambda \in (0,1)$.
Hence, the operator N has a fixed point $(x_1,\ x_2) \in \overline{U}$ which is a solution for time- fractional differential inclusion (3.3) − (3.4). □

4 Conclusion

In this article, we have examined the existence and uniqueness of solutions for a system of equations and inclusions. The system is characterized by differential fractional-in-time operators using the ψ-Caputo fractional derivative, along with a hint of the pantograph differential equation. Additionally, we incorporate L^1-Caratheodory multifunctions to analyze the problem further.

References

[1] S. M. Aleomraninejad, S. Rezapour, and N. Shahzad, *On fixed point generalizations of Suzuki's method*, Applied Mathematics Letters, **24** , 1037–1040, (2011).

[2] R. Almeida, *Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul., **44** , 460–481, (2017).

[3] R. Almeida, A.B. Malinowska, and M.T.T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Mathematical Methods in the Applied Sciences, **41** , 336–352, (2018).

[4] A. Belarbi, M. Benchohra, A. Ouahab, *Uniqueness results for fractional functional differential equations with infinite delay in Frechet spaces*, Appl. Anal., **58** , 1459–1470, (2006).

[5] H. Covitz and S. B. Nadler, *Multi-valued contraction mappings in generalized metric spaces*, Israel Journal ofMathematics, **8** , 5–11, (1970).

[6] T.L. Guo and M. Jiang, *Impulsive fractional functional differential equations*, Comput. Math. Appl., **64** , 3414–3424, (2012).

[7] I. Losada and I.I. Nieto, *Properties of a new fractional derivative without singular kernel*, Progress in Fractional Differentiation and Applications, **1** , 87–92, (2015).

[8] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Eqautions*, John Wiley & Sons,(1993).

[9] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, UK, (1980).

[10] A. Alsaedi, D. Baleanu,S. Etemad, and S. Rezapour, *On Coupled Systems of Time-Fractional Differential Problems by Using a New Fractional Derivative*, Hindawi Publishing Corporation , Journal of Function Spaces, 8, (2016).

Author information

M'hamed Elomari, Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, 23000, Beni Mellal, Morocco.
E-mail: m.elomari@usms.ma

Said Melliani, Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, 23000, Beni Mellal, Morocco.
E-mail: s.melliani@usms.ma

Khadija Oufkir, Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, 23000, Beni Mellal, Morocco.
E-mail: oufkirkhadijabzou@gmail.com

Hamid Sadiki, Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, 23000, Beni Mellal, Morocco.
E-mail: h.sadiki@usms.ma

Received: 2022-12-26
Accepted: 2023-08-21