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SURFACE PENCIL WITH A COMMON TIMELIKE ADJOINT CURVE

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Abstract. *The adjoint curve of a Frenet curve r=r(s) is defined as the unit speed curve tangent to the principal normal vector field of r. We show that the adjoint curve of a spacelike curve with timelike binormal is a timelike curve. We obtain some relationships between a Frenet curve and its adjoint in Minkowski 3-space. For a given spacelike curve with timelike binormal, we obtain conditions on surfaces that possess the adjoint curve as a common asymptotic, geodesic or curvature line in Minkowski 3-space. We also give examples confirming our theory.*

1 Introduction

The study of curves and surfaces has wide application areas such as computer aided design, architectural design, astronomy, astrophysics and genetics,[1, 2, 3, 4, 5, 6], [11, 12, 13, 14, 15, 16],. Finding and classifying special curves, such as geodesic, line of curvature, asymptotic curve etc., on a given surface has been a long-term research topic in differential geometry, [1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15]. In these studies, it is seen that the results depend on the marching-scale functions of the surface in order to be a common special curve by taking the given curve as parameter curve. The concept of family of surfaces having a given characteristic curve was first introduced by Wang et al in Euclidean 3-space, [1]. They parameterized the surface by using the Frenet frame of the given curve and gave the necessary and sufficient condition to satisfy the geodesic requirement. Kasap and Akyıldız studied spacelike surfaces with a common spacelike geodesic and timelike surfaces with a common spacelike or timelike geodesic, [2]. Li et al defined the surface pencil with a common line of curvature, [3]. Recently, Guler et. al. presented conditions for offset surfaces with a common asymptotic curve, [4]. Bayram constructed surfaces with constant mean curvature along a timelike curve, [5]. In [6], the authors studied the surface family by using the vector elements of alternative frame.

Another important characteristic curve is adjoint curve which is defined in [7] as the integral of the binormal vector of a curve $r = r(s)$. Adjoint curves play an important role in various areas of mathematics such as number theory, coding theory, algebraic geometry, etc., [8]. There are a lot of work on adjoint curves. In [9], they investigate generalhyperplane sections and study embeddings of adjoint line bundles. In [10], authors gave some characterizations of parallel curves of adjoint curves. Bayram [11] studied surfaces possessing an adjoint curve of a given curve as an asymptotic curve, geodesic curve or line of curvature on a surface family in Euclidean space. Güler [12] researched surfaces possessing an spacelike adjoint curve of a given curve as an asymptotic curve, geodesic curve or line of curvature on a surface family in Minkowski space.

In this paper, we analyzed the problem of finding a surface pencil through the timelike adjoint curve of a given spacelike curve with timelike binormal as an asymptotic, geodesic or a line of curvature in Minkowski 3-space. We obtain conditions for a surface pencil possessing the timelike adjoint curve as a common asymptotic, geodesic or a line of curvature. We give *examples confirming this method.*

2 Preliminaries

The Lorentzian inner product of $X = (x_1, x_2, x_3)$ *and* $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ *in Minkowski 3-space* $\mathbb{R}^3_1 = \left[\mathbb{R}^3, (-,+,+)\right]$ *is given by*

$$\langle X, Y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3. \tag{2.1}$$

A vector $X \in \mathbb{R}^3_1$ *is called a spacelike vector when* $\langle X, X \rangle > 0$ *or* $X = 0$. *It is called a timelike or a null (lightlike) vector in case of* $\langle X, X \rangle < 0$, $\langle X, X \rangle = 0$ *for* $X \neq 0$*.respectively. The vector product of vectors* $X = (x_1, x_2, x_3)$ *and* $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ *is defined by*

$$X \times Y = (x_2 y_3 - x_3 y_2,\ x_1 y_3 - x_3 y_1,\ x_2 y_1 - x_1 y_2). \tag{2.2}$$

Let $r = r(s)$ *be a unit speed curve in* $\mathbb{R}^3_1$. *By* $\kappa(s)$ *and* $\tau(s)$ *we denote the curvature and torsion of* $r(s)$*, respectively. Consider the Frenet frame* $\{T, N, B\}$ *associated with the curve* $r = r(s)$ *such that* $T = T(s), N = N(s)$ *and* $B = B(s)$ *are the unit tangent, the principal normal and the binormal vector fields, respectively.*

If $r = r(s)$ *is a spacelike curve, then the structural equations (or Frenet formulas) of this frame are given as*

$$\dot{T}(s) = \kappa(s) N(s),\ \dot{N}(s) = \varepsilon\kappa(s) T(s) + \tau(s) B(s),\ \dot{B}(s) = \tau(s) N(s) \tag{2.3}$$

where

$$\varepsilon = \begin{cases} +1, B \text{ is spacelike}, \\ -1, B \text{ is timelike}. \end{cases}$$

If $r = r(s)$ *is a timelike curve, then above equations are given as [17]*

$$\dot{T}(s) = \kappa(s) N(s),\ \dot{N}(s) = \kappa(s) T(s) + \tau(s) B(s),\ \dot{B}(s) = -\tau(s) N(s). \tag{2.4}$$

The norm of a vector X *is defined by*

$$\|X\| = \sqrt{|\langle X, X \rangle|}. \tag{2.5}$$

Definition 2.1. A surface curve is called an asymptotic curve provided its velocity always points in an asymptotic direction, which is the direction in which the normal curvature is zero [18].

According to this definition the curve is an asymptotic curve on the surface $P(s,t)$ *if and only if*

$$\left\langle \frac{\partial n(s,t_0)}{\partial s}, T(s) \right\rangle = 0, \tag{2.6}$$

where $n(s,t_0)$ *is the normal vector of* $M(s,t)$ *along the curve* $r(s)$.

If the curve $r(s)$ *is a parametric curve on a surface* $M = M(s,t)$ *in* IR^3_1 *that it has a constant* s *or* t *parameter value, then the parametric curve* $r(s)$ *is* $r(s) = M(s,t_0)$ *or* $r(t) = M(s_0,t)$.

3 The Timelike Adjoint Curve

Definition 3.1. Let r be a unit speed non null curve in Minkowski space with torsion $\tau \neq 0$ and the Frenet frame of r be $\{T_r, N_r, B_r\}$ .The adjoint curve of r defined as

$$\beta(s) = \int_{s_0}^{s} B_r(s)\, ds \tag{3.1}$$

In this section, our aim is to investigate the adjoint curve $\beta(s)$of a given unit speed spacelike curve with timelike binormal $r(s)$ in IR_1^3. Let the curvature and torsion of the spacelike curve with timelike binormal $r(s)$ be κ_r and τ_r respectively. Then, the Frenet frame is given as

$$T_r(s) = r^{'}(s), \quad N_r(s) = \frac{r^{''}(s)}{\|r''(s)\|}, \quad B_r(s) = T_r(s) \times N_r(s). \tag{3.2}$$

Let the adjoint curve of r be β and the Frenet frame of r and β are $\{T_r, N_{r,}B_r\}$ and $\{T_\beta, N_{\beta,}B_\beta\}$, respectively. By differentiating Eq. (3.1), we write

$$\frac{d}{ds}\beta(s) = B_r(s) \quad and \quad \frac{d}{ds}\beta(s) = T_\beta(s). \tag{3.3}$$

We have the following result.

Corollary 3.2. *If r be a unit speed spacelike curve with timelike binormal, then adjoint curve of r is a timelike curve.*

Since the adjoint curve β of r is timelike curve, the Frenet frame is

$$T_\beta(s) = \beta^{'}(s), \quad N_\beta(s) = \frac{\beta^{''}(s)}{\|\beta''(s)\|}, \quad B_\beta(s) = -T_\beta(s) \times N_\beta(s). \tag{3.4}$$

Using Eqs. (3.3) and (3.4) we have

$$T_\beta = B_r. \tag{3.5}$$

The following result can be given.

Corollary 3.3. *Let r be a spacelike curve with timelike binormal with arc length s and β be adjoint curve of r. If the Frenet vectors of r and β are $\{T_r, N_{r,}B_r\}$ and $\{T_\beta, N_{\beta,}B_\beta\}$, the curvature and torsion are κ_r, τ_r and κ_β, τ_β respectively, then the following relations hold:*

$$T_\beta = B_r, \; N_\beta = N_r, \; B_\beta = -T_r \quad and \quad \kappa_\beta = \tau_r, \quad \tau_\beta = \kappa_r. \tag{3.6}$$

4 Surface pencil with a common timelike adjoint curve

In this section, we analyze the conditions on surfaces that possess the timelike adjoint curve as a common parametric and asymptotic, goeodesic or curvature line for a given spacelike curve with timelike binormal.

Let $M = M(s,t)$ be a parametric surface and $\beta = \beta(s)$ be the timelike adjoint curve of a spacelike curve with timelike binormal $r = r(s)$. So, the surface is defined by

$$M(s,t) = \beta(s) + (f_1(s,t), f_2(s,t), f_3(s,t)) \begin{pmatrix} T_\beta(s) \\ N_\beta(s) \\ B_\beta(s) \end{pmatrix}, \tag{4.1}$$

$S_1 \le s \le S_2, T_1 \le t \le T_2$, where $\{T_\beta(s), N_\beta(s), B_\beta(s)\}$ is the Frenet frame associated with the curve $\beta(s)$ and functions $f_i(s,t)$, $i = 1,2,3$, are C^1 called marching scale functions.

Since the adjoint curve $\beta(s)$ of $r(s)$ is a parametric curve on the surface $M(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that $M(s,t_0) = \beta(s)$ $S_1 \le s \le S_2$, that is ,

$$f_1(s,t_0) = f_2(s,t_0) = f_3(s,t_0) \equiv 0, \; L_1 \le s \le L_2. \tag{4.2}$$

The normal vector can be expressed as

$$\frac{\partial M}{\partial s} \times \frac{\partial M}{\partial t} = \left(\frac{\partial f_3}{\partial t}\left(f_1\kappa_\beta + \frac{\partial f_2}{\partial s} - f_3\tau_\beta \right) - \frac{\partial f_2}{\partial t}\left(f_2\tau_\beta + \frac{\partial f_3}{\partial s} \right) \right) T_\beta$$

$$+\left(\frac{\partial f_3}{\partial t}\left(1+\frac{\partial f_1}{\partial s}+f_2\kappa_\beta\right)-\frac{\partial f_2}{\partial t}\left(f_2\tau_\beta+\frac{\partial f_2}{\partial s}\right)\right)N_\beta \tag{4.3}$$

$$+\left(\frac{\partial f_1}{\partial t}\left(f_1\kappa_\beta+\frac{\partial f_2}{\partial s}-f_3\tau_\beta\right)-\frac{\partial f_2}{\partial t}\left(1+\frac{\partial f_1}{\partial s}+f_2\kappa_\beta\right)\right)B_\beta.$$

Thus, we get

$$n\left(s,t_0\right)=\varphi_1\left(s,t_0\right)T_\beta+\varphi_2\left(s,t_0\right)N_\beta\left(s\right)+\varphi_3\left(s,t_0\right)B_\beta\left(s\right), \tag{4.4}$$

where

$$\varphi_1\left(s,t_0\right)=\left(\frac{\partial f_3\left(s,t_0\right)}{\partial t}\frac{\partial f_2\left(s,t_0\right)}{\partial s}-\frac{\partial f_2\left(s,t_0\right)}{\partial s}\frac{\partial f_3\left(s,t_0\right)}{\partial t}\right)$$

$$\varphi_2\left(s,t_0\right)=\left(\left(1+\frac{\partial f_1\left(s,t_0\right)}{\partial s}\right)\frac{\partial f_3\left(s,t_0\right)}{\partial t}-\frac{\partial f_1\left(s,t_0\right)}{\partial t}\frac{\partial f_3\left(s,t_0\right)}{\partial s}\right) \tag{4.5}$$

$$\varphi_3\left(s,t_0\right)=\left(\frac{\partial f_1\left(s,t_0\right)}{\partial t}\frac{\partial f_2\left(s,t_0\right)}{\partial s}-\frac{\partial f_2\left(s,t_0\right)}{\partial t}\left(1+\frac{\partial f_1\left(s,t_0\right)}{\partial s}\right)\right).$$

Theorem 4.1. *The timelike adjoint curve $\beta\left(s\right)$ is a parametric and an asymptotic curve on the surface $M\left(s,t\right)$ if*

$$f_1\left(s,t_0\right)=f_2\left(s,t_0\right)=f_3\left(s,t_0\right)\equiv 0.$$

$$\frac{\partial f_3\left(s,t_0\right)}{\partial t}=0,\quad \frac{\partial f_2\left(s,t_0\right)}{\partial t}\neq 0 \tag{4.6}$$

Proof. By Eq. (4.2), we have

$$f_1\left(s,t_0\right)=f_2\left(s,t_0\right)=f_3\left(s,t_0\right)\equiv 0$$

By the definition of partial differentiation and using Eq. (4.2) , we obtain

$$\phi_1\left(s,t_0\right)=0,\ \phi_2\left(s,t_0\right)=\frac{\partial f_3\left(s,t_0\right)}{\delta t},\phi_3\left(s,t_0\right)=\frac{\partial f_2\left(s,t_0\right)}{\delta t}, \tag{4.7}$$

From Eq. (2.6), we know that adjoint curve $\beta\left(s\right)$ is an asymptotic curve if

$$\frac{\partial \phi_1}{\partial s}\left(s,t_0\right)+\varepsilon\kappa_\beta\left(s\right)\phi_2\left(s,t_0\right)=0. \tag{4.8}$$

Since $\kappa_\beta\left(s\right)=\left\|\beta''\left(s\right)\right\|\neq 0,\phi_2\left(s,t_0\right)=-\frac{\partial f_3\left(s,t_0\right)}{\partial t}$ and by Eq. (4.2) we have $\frac{\partial \phi_1}{\partial s}\left(s,t_0\right)=0$ Therefore Eq. (2.6) is simplified to

$$\frac{\partial f_3\left(s,t_0\right)}{\partial t}=0,\quad \frac{\partial f_2\left(s,t_0\right)}{\partial t}\neq 0 \tag{4.9}$$

which completes the proof. □

Theorem 4.2. *The timelike adjoint curve $\beta\left(s\right)$ is a parametric and a geodesic curve on the surface $M\left(s,t\right)$ if the followings are satisfied*

$$f_1\left(s,t_0\right)=f_2\left(s,t_0\right)=f_3\left(s,t_0\right)\equiv 0,$$

$$\frac{\partial f_3\left(s,t_0\right)}{\partial t}\neq 0,\quad \frac{\partial f_2\left(s,t_0\right)}{\partial t}=0. \tag{4.10}$$

Proof. The geodesic curvature $\kappa_g = \det(\beta', \beta'', n)$ vanishes along geodesics, [19]. This definition is equivalent to

$$n(s,t_0) // N_\beta(s), \tag{4.11}$$

where $n(s,t_0)$ is the surface normal along the adjoint curve $\beta(s)$ and N_β is the normal vector of adjoint curve $\beta(s)$.

By Eqs. (4.2), (4.11), we have

$$f_1(s,t_0) = f_2(s,t_0) = f_3(s,t_0) \equiv 0.$$

□

Theorem 4.3. *The timelike adjoint curve $\beta(s)$ is a line of curvature on the surface $M(s,t)$ if the followings are satisfied*

$$\alpha(s) = -\int_{s_0}^{s} \tau_\beta ds + \alpha(0),$$

$$f_1(s,t_0) = f_2(s,t_0) = f_3(s,t_0) \equiv 0, \tag{4.12}$$

$$\varphi_1(s,t_0) \equiv 0, \ \varphi_2(s,t_0) = \mu(s)\cos\alpha, \varphi_3(s,t_0) = -\mu(s)\sin\alpha.$$

Proof. Let the Lorentzian spacelike angle between $N_\beta$and n_1 be $\alpha = \alpha(s)$.

Let us take n_1 $(n_1 = \cos\alpha N_\beta + \sin\alpha B_\beta)$ be a vector orthogonal to the timelike adjoint curve $\beta(s)$. The timelike adjoint curve $\beta(s)$ is a line of curvature on the surface $M(s,t)$ if and only if n_1 is parallel to the normal vector $n(s,t)$ of the surface $M(s,t)$ and $S(T) = \omega T$, $\omega \neq 0$, where S is the shape operator of the surface.

Since n_1 to be parallel to the normal vector $n(s,t)$ of the surface $M(s,t)$ from Eq. (4.4) this follows that $n_1(s) // n(s,t_0)$, $S_1 \leq s \leq S_2$, if and only if there exits a function $\mu(s) \neq 0$ such that

$$\varphi_1(s,t_0) = 0, \ \varphi_2(s,t_0) = \mu(s)\cos\alpha, \ \varphi_3(s,t_0) = -\mu(s)\sin\alpha.$$

Also, since $S(T) = \omega T$, $\omega \neq 0$,

$$\alpha(s) = -\int_{s_0}^{s} \tau_\beta ds + \alpha_{0,}$$

where the starting value of arc length is s_0 and we assume $s_0 = 0$. □

Example 4.4. Let $r(s) = \left(\frac{4}{9}\sinh 3s, \frac{4}{9}\cosh 3s, \frac{5}{3}s\right)$ be an arc-length spacelike curve with timelike binormal. The Frenet frame of $r(s)$ is

$$T_r(s) = \left(\frac{4}{3}\cosh 3s, \frac{4}{3}\sinh 3s, \frac{5}{3}\right)$$

$$N_r(s) = (\sinh 3s, \cosh 3s, 0)$$

$$B_r(s) = \left(\frac{5}{3}\cosh 3s, \frac{5}{3}\sinh 3s, \frac{4}{3}\right)$$

and

$$\kappa_r = 4.$$

Using Eqs. (3.2), (3.5) and (3.6), the adjoint curve β of r is

$$\beta(s) = \left(\frac{5}{9}\sinh 3s, \frac{5}{9}\cosh 3s, \frac{4}{3}s\right)$$

and its Frenet frame is

$$T_\beta(s)=\left(\frac{5}{3}\cosh 3s,\frac{5}{3}\sinh 3s,\frac{4}{3}\right),$$

$$N_\beta(s)=(\sinh 3s,\cosh 3s,0),$$

$$B_\beta(s)=\left(-\frac{4}{3}\cosh 3s,-\frac{4}{3}\sinh 3s,-\frac{5}{3}\right).$$

The torsion of the adjoint curve β *is* $\tau_\beta=4$.

If we choose $f_1(s,t)=0,\ \ f_2(s,t)=t,\ \ f_3(s,t)=0$ *and* $t_0=0$ *then Eq. (4.6) is satisfied. Thus, we obtain a member of the surface pencil with a common timelike adjoint asymptotic as*

$$M(s,t)=\left(\frac{5}{9}\sinh 3s+t\sinh 3s,\frac{5}{9}\cosh 3s+t\cosh 3s,\frac{4}{3}s\right)$$

where $-2<s<2,\ \ -2<t<2(Fig.1)$.

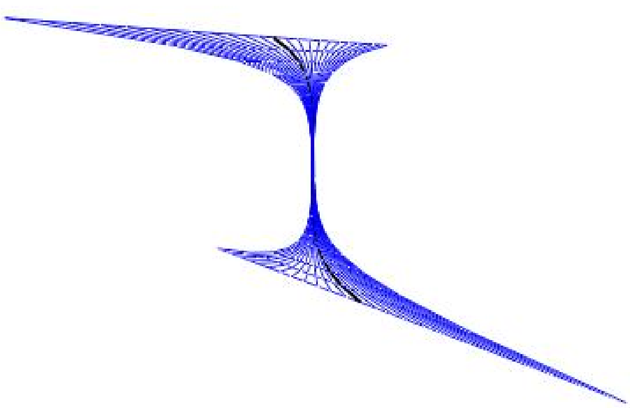

Figure 1. A member of the surface pencil with a common timelike adjoint asymptotic curve.

For the same curve, if we choose $f_1(s,t)=0,\ \ f_2(s,t)=0,\ \ f_3(s,t)=t$ *and* $t_0=0$ *then Eq. (4.10) is satisfied. Thus, we obtain a member of the surface pencil with a common timelike adjoint geodesic as*

$$M(s,t)=\left(\frac{5}{9}\sinh 3s-\frac{4}{3}t\cosh 3s,\frac{5}{9}\cosh 3s+\frac{4}{3}t\sinh 3s,\frac{4}{3}s+\frac{5}{3}t\right),$$

where $-5<s<5,\ \ -5<t<5(Fig.2)$.

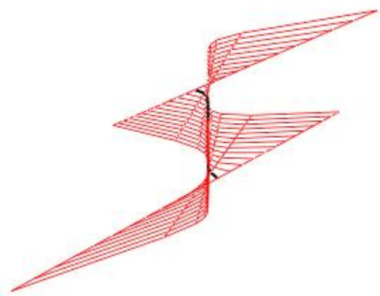

Figure 2. A member of the surface pencil with a common timelike adjoint geodesic curve.

For the same curve, if we let $\alpha(0) = 0$, *then* $\alpha(s) = -4s$. *By taking* $\mu(s) \equiv 1, t_0 = 0$ *and* $f_1(s,t) = 0,\;\; f_2(s,t) = t\sin\alpha(s),\;\; f_3(s,t) = t\;\cos\alpha(s)$ *, then Eq.* (4.12) *is satisfied. So, we have the following surface as a member of the surface pencil with a common timelike adjoint line of curvature as*

$$M(s,t) = \begin{pmatrix} \frac{5}{9}\sinh 3s + t\sin(-4s)\sinh 3s - \frac{4}{3}\cos(-4s)\cosh 3s, \\ \frac{5}{9}\cosh 3s + t\sin(-4s)\cosh 3s + \frac{4}{3}t\cos(-4s)\sinh 3s, \;, \\ \frac{4}{3}s + \frac{5}{3}t\cos(-4s) \end{pmatrix}$$

where $0 < s < 2,\;\; 0 < t < 2(Fig.3)$.

Figure 3. A member of the surface pencil with a common timelike adjoint line of curvature.

5 Conclusions

In this study, we research the problem of finding a surface pencil through the timelike adjoint curve in Minkowski space. Firstly, we obtain that the adjoint curve of a unit speed spacelike curve with timelike binormal is a timelike curve. Secondly, we analyze the conditions on surfaces that possess the timelike adjoint curve as a common parametric and asymptotic, goeodesic or curvature line for a given spacelike curve with timelike binormal. Finaly, examples supporting the theorems are given.

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