Palestine Journal of Mathematics

Vol 13(1)(2024) , 310–315

On generalizations of the Wiener and Lévy theorems

H. El Atef, L. Bouchikhi and A. El Kinani

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: 46J10, 46H30.

Keywords and phrases: Hermitian p-Banach algebra, Real analytic function, Functional calculus for several variables, Fourier series, Weighted algebra,Wiener theorem, Lévy theorem.

The authors would like to thank the reviewers and the editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abdellah El Kinani is thankful to the NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for the support.

Abstract. We obtain muti-dimensional generalizations of the N. Wiener and P. Lévy theorems. We also give two applications of the real analytic functional calculus for several variables in p-Banach algebras ($0 < p \leq 1$). The first one consists of giving the real-analytic versions of P. Lévy theorem for several functions of a single variable. The second consists of giving a generalization of N. Wiener and P. Lévy theorems for functions of several variables.

1 Preliminaries and introduction

Let A be a complex algebra. A linear p-norm $\|.\|_p$, $0 < p \leq 1$, on A is called an algebra p-norm if it is submultiplicative. A complete p-normed algebra will be called p-Banach algebra. Let A be a complex p-Banach algebra with unit e and involution $x \longmapsto x^*$. The spectrum of an element x of A will be denoted by $Sp(x)$. An element h of A such that $h^* = h$ is called hermitian, $H(A)$ denotes the set of hermitian elements of A. The real and imaginary parts of an element x of A are denoted by $Re\,(x)$ and $Im\,(x)$, respectively; i.e., $Re\,(x) = (x + x^*)\,/2$ and $Im\,(x) = (x - x^*)\,/2i$. We say that the p-Banach algebra is hermitian if the spectrum of every element of $H(A)$ is real. In the sequel, $Sp(A)$ denotes the Gelfand spectrum of A, that is, (the set of non-zero characters of A). Let n be a positive integer, and let A^n denote the cartesian product of n copies of A. For $\mathbf{a} = (a_1, ..., a_n) \in A^n$, let $\widehat{\mathbf{a}}$ denote the mapping, of $Sp(A)$ into $\mathbb{C}^n$, defined by: $\widehat{\mathbf{a}}(\chi) = (\chi(a_1), ..., \chi(a_n))$, for every $\chi \in Sp(A)$. The image $\widehat{\mathbf{a}}\,(Sp(A)) \subset \mathbb{C}^n$ is therefore a nonempty compact of $\mathbb{C}^n$. It is called the simultaneous spectrum of $\mathbf{a}$ and denoted by $Sp\,(\mathbf{a})$.

Let A be a commutative p-Banach algebra, $0 < p \leq 1$, with unit and let $\mathbf{a} = (a_1, ..., a_n)$ be an n-tuple of elements of A. Denote by $\mathcal{O}\,(Sp(\mathbf{a}))$ the algebra of germs of holomorphic functions near $Sp(\mathbf{a})$ endowed with its natural topology of inductive limit and, for every $i = 1, 2, ..., n$, put z_i the i-th-coordinate function on $\mathbb{C}^n$. So, there is a holomorphic functional calculus [13] given by a continuous unital homomorphism $\Theta_{\mathbf{a}} : \mathcal{O}\,(Sp(\mathbf{a})) \longrightarrow A$ such that $\Theta_{\mathbf{a}}\,(z_i) = a_i$, for every $i = 1, 2, ..., n$ and $\Theta_{\mathbf{a}}(f)\hat{}\,|_{Sp(A)} = f \circ (\widehat{a_1}, ..., \widehat{a_n})|_{Sp(A)}$, for each $f \in \mathcal{O}\,(Sp(\mathbf{a}))$, where $\Theta_{\mathbf{a}}(f)\hat{}\,|_{Sp(A)}$ designates the restriction from $\Theta_{\mathbf{a}}(f)\hat{}$ to $Sp\,(A)$.

Let $f(t) = \sum\limits_{n\in\mathbb{Z}} a_n e^{int}$, where $\mathbb{Z}$ is the set of all integers, be a periodic function such that $\sum\limits_{n\in\mathbb{Z}} |a_n| < +\infty$. If $f(t) \neq 0$ for all t, then the function $1/f$ can be developed in a trigonometric series $1/f(t) = \sum\limits_{n\in\mathbb{Z}} b_n e^{int}$ such that $\sum\limits_{n\in\mathbb{Z}} |b_n| < +\infty$, this is a famous theorem of N. Wiener [12]. A generalization of this result due to P. Lévy [10] states that if F is a holomorphic function on an open set containing the image of f, then $F(f)$ also can be developed in a trigonometric series $F(f)(t) = \sum\limits_{n\in\mathbb{Z}} c_n e^{int}$ such that $\sum\limits_{n\in\mathbb{Z}} |c_n| < +\infty$. The first proof of these two theorems was a simple classical analysis calculations. The new proof obtained by Gelfand [9] via Banach algebras constitute one of the nicest applications of the theory of Banach algebras to harmonic analysis. There are several generalizations of the above results based on the approach developed by I. M. Gelfand and by holomorphic, harmonic, and real analytic functional calculus. In [13], W.Żelazko extended this results to p-Banach algebras where $0 < p \leq 1$. He showed that if

$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}$ is a periodic function such that $\sum_{n\in\mathbb{Z}} |a_n|^p < +\infty$ and if $f(t) \neq 0$ for all t, then $1/f$ can be developed in a trigonometric series $1/f(t) = \sum_{n\in\mathbb{Z}} b_n e^{int}$ such that $\sum_{n\in\mathbb{Z}} |b_n|^p < +\infty$. S. Rolwicz ([11] replaced the function t^p by a function with suitable properties. In [3], S. J, Bhatt, H. V, Dedania considered a Beurling algebra analog of the classical theorems of Wiener and Levy on absolutely convergent Fourier series. In [1, 4, 5, 6, 7] and [8], the authors introduce several weighted algebras, which constitute the essential ingredients of the extensions of N. Wiener and P. Lévy theorems for p-ω-convergent Fourier series, where $p \in]1, +\infty[$ and ω suitably chosen weight and this is in the frame of Banach algebras. In this paper, we extend the above mentioned results to p-Banach algebras $(0 < p \leq 1)$. We obtain a holomorphic real analytic versions of P. Lévy theorem for several functions of a single variable. We also produce a generalization of N. Wiener and P. Lévy theorems for functions of several variables.

2 Real-analytic calculus for several variables in p-Banach algebras

The real analytic functional calculus for a single hermitian Banach algebra element is defined and studied in [2]. In [8], the authors extended this calculus to several hermitian Banach algebra elements. It is also valid in p-Banach algebras $(0 < p \leq 1)$. Its construction goes along the lines of [8]. To make the paper self-contained, we recall the fundamental properties of this calculus. Let U be an open subset of $\mathbb{R}^{2n}$ and $f : U \longrightarrow \mathbb{C}$ be a real analytic function. It is well known that there exists an open subset V of $\mathbb{C}^{2n}$, and an holomorphic function $F : V \longrightarrow \mathbb{C}$ such that $V \cap \mathbb{R}^{2n} = U$ and $F|_U = f$. In the sequel, we denote by $\mathcal{A}(U)$ the algebra of real analytic functions on U and we consider the mapping:

$$\Psi : \mathcal{A}(U) \longrightarrow \mathcal{O}(V) : f \longmapsto \Psi(f) = F.$$

By construction, Ψ is an algebra isomorphism. Now let A be a hermitian p-Banach algebra, $\mathbf{a} = (h_1 + ik_1, ..., h_n + ik_n) \in A^n$ and $\mathbf{a}' = (h_1, k_1 ..., h_n, k_n) \in A^{2n}$ with $h_j,\ k_j \in H(A)$, for every $j \in \{1, ..., n\}$. So, one has $Sp(h_1, k_1 ..., h_n, k_n) \subset \mathbb{R}^{2n}$. By the identification $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, we can identify $Sp(\mathbf{a})$ and $Sp(\mathbf{a}')$. This and the map $\Theta_{\mathbf{a}'}$ that defined the holomorphic functional calculus for $\mathbf{a}'$ motivate the following definition:

Definition 2.1. Let A be a commutative and unital hermitian p-Banach algebra, $0 < p \leq 1$, with continuous involution $x \longmapsto x^*$, $\mathbf{a} = (a_1, ..., a_n)$ be a family of elements of A, U be an open subset, of $\mathbb{R}^{2n}$, containing $Sp(\mathbf{a})$ and $f \in \mathcal{A}(U)$. We denote by $f(\mathbf{a})$ the element of A defined by:

$$f(\mathbf{a}) = \Theta_{\mathbf{a}'}(\Psi(f)) = \Psi(f)(h_1, k_1 ..., h_n, k_n),$$

where $\mathbf{a} = (h_1 + ik_1, ..., h_n + ik_n)$ and $\mathbf{a}' = (h_1, k_1 ..., h_n, k_n)$ with $h_j, k_j \in H(A)$, for every $j \in \{1, ..., n\}$.

If we denote by $\phi_{\mathbf{a}}(f)$ the element $f(\mathbf{a})$, one has a mapping of $\mathcal{A}(U)$ into A, noted by $\phi_{\mathbf{a}}$, given by:

$$\phi_{\mathbf{a}} : \mathcal{A}(U) \longrightarrow A : f \longmapsto f(\mathbf{a}).$$

The fundamental properties of this calculus ares contained in the following result:

Proposition 2.2. *(i) The mapping $\phi_{\mathbf{a}}$ is a unique algebra homomorphism of $\mathcal{A}(U)$ into A that extends the involutive homomorphism from $h(U)$ into A, where $h(U)$ is the set of all harmonic functions on U.*

(ii) "Spectral mapping theorem":

$$Spf(\mathbf{a}) = f(Sp(\mathbf{a})), \textit{ for every } f \in \mathcal{A}(\mathcal{U})$$

3 Generalizations of the Wiener and Lévy theorems

In this section, we obtain multi-dimensional generalizations of the N. Wiener and P. Lévy theorems and this in the frame of a variable. The second consists of a generalization of N. Wiener and P. Lévy theorems for functions of several variables. Let's first start by introducing weighted algebras, which constitute the essential ingredient of our generalizations of N. Wiener and P.

Lévy theorems. We say that ω is a weight on $\mathbb{Z}^k$ ($k \in \mathbb{N}^*$ fixed) if $\omega : \mathbb{Z}^k \longrightarrow [1, +\infty[$, is a map satisfying:

$$\omega(0) = 1, \ \ \omega(n+m) \leq \omega(n)\,\omega(m)\,, \text{for every } \ n, m \in \mathbb{Z}^k$$

and, for every $j = 1, ..., k$,

$$\lim_{|n| \longrightarrow +\infty} (\omega(n))^{\frac{1}{n_j}} = 1.$$

For $n = (n_1, ..., n_k) \in \mathbb{Z}^k$ and $t = (t_1, ..., t_k) \in \mathbb{R}^k$, we will use the notation $(n, t) = n_1 t_1 + \cdots + n_k t_k$. Now, for $0 < p \leq 1$, we consider the following weighted space:

$$\mathcal{A}_k^p(\omega) = \left\{ f : \mathbb{R}^k \longrightarrow \mathbb{C} : f(t) = \sum_{n \in \mathbb{Z}^k} a_n e^{i(n,t)} : (a_n)_{n \in \mathbb{Z}^k} \in l_\omega^p\left(\mathbb{Z}^k\right) \right\},$$

where

$$l_\omega^p\left(\mathbb{Z}^k\right) = \left\{ (a_n)_{n \in \mathbb{Z}^k} : \left|(a_n)_{n \in \mathbb{Z}^k}\right|_{\omega,p} = \sum_{n \in \mathbb{Z}^k} \omega(n)\,|a_n|^p < +\infty \right\}.$$

Endowed with the p-norm $|.|_{\omega,p}$ and the convolution as a multiplication, the space $\left(l_\omega^p\left(\mathbb{Z}^k\right), |.|_{\omega,p}\right)$ becomes a non-normed commutative p-Banach algebra. If $a = (a_n)_{n \in \mathbb{Z}^k} \in l_\omega^p\left(\mathbb{Z}^k\right)$, we write $\widehat{a}$ for the Fourier transform of a defined by:

$$\widehat{a}(t) = \sum_{n \in \mathbb{Z}^k} a_n e^{i(n,t)}, \text{ for every } \ t \in \mathbb{R}^k.$$

Then one has $\widehat{a} \in \mathcal{A}_k^p(\omega)$ and the mapping $a \longmapsto \widehat{a}$ is an isomorphism of $l_\omega^p\left(\mathbb{Z}^k\right)$ onto $\mathcal{A}_k^p(\omega)$. It follows that $\mathcal{A}_k^p(\omega)$ is closed under pointwise multiplication and $\left(\mathcal{A}_k^p(\omega), \|.\|_{p,\omega}\right)$, where

$$\|f\|_{p,\omega} = \sum_{n \in \mathbb{Z}^k} |a_n|^p\,\omega(n)\,, \text{ for every } \ f \in \mathcal{A}_k^p(\omega)\,,$$

is a non normed commutative p-Banach algebra with unity element $\widehat{e}$ given by $\widehat{e}(t) = 1$, $(t \in \mathbb{R})$. Moreover, its characters are exactly the evaluations at some $t^0 \in \mathbb{R}^k$, where $t^0 = \left(t_1^0, ..., t_k^0\right)$ with $0 \leq t_j^0 < 2\pi$, for every $j = 1, ..., k$, and so, for every $f(t) = \sum_{n \in \mathbb{Z}^k} a_n e^{i(n,t)} \in \mathcal{A}_k^p(\omega)$, one has: $Sp(f) = \left\{ f(t) : t \in [0, 2\pi[^k \right\}$.

3.1 Multi-dimensional generalization Lévy theorem

Using the fact that the spectrum of $f \in \mathcal{A}_k^p(\omega)$ is nothing but the set of values of f, we obtain the following generalization of P. Lévy theorem for holomorphic functions of several variables.

Theorem 3.1 (Multi-dimensional holomorphic version of P. Lévy theorem). *Let $p \in]0, 1]$ and ω be a weight on $\mathbb{Z}$. Let $f(t) = (f_1(t), ..., f_k(t))$, where $f_j(t) = \sum_{n \in \mathbb{Z}} a_{n,j} e^{int}$, for $j = 1, ..., k$, is a periodic function such that:*

$$\|f_j\|_{p,\omega} = \sum_{n \in \mathbb{Z}} |a_{n,j}|^p\,\omega(n) < +\infty.$$

Let F be a holomorphic function of k real variables on an open set U containing the image of the function f. Then $F(f)$ also can be developed in a trigonometric series $F(f)(t) = \sum_{n \in \mathbb{Z}} b_n e^{int}$ such that:

$$\|F(f)\|_{p,\omega} = \sum_{n \in \mathbb{Z}} |b_n|^p\,\omega(n) < +\infty$$

and, for every $t \in \mathbb{R}$,

$$F(f_1(t), ..., f_k(t)) = \sum_{n \in \mathbb{Z}} b_n e^{int}.$$

Now we consider, in the algebra $\mathcal{A}_k^p(\omega)$, the algebra involution $f \longmapsto f^*$ defined by:

$$f^*(t) = \sum_{n \in \mathbb{Z}^k} \overline{a_{-n}} e^{i(n,t)}, \text{ for every } f \in \mathcal{A}_k^p(\omega).$$

This involution is continuous, for the algebra is semi-simple. Moreover $\left(\mathcal{A}_k^p(\omega), \|.\|_{p,\omega}\right)$ is a hermitian p-Banach algebra. This allows us to get the following generalization of the P. Lévy theorem for real analytic functions for several variables.

Theorem 3.2 (Multi-dimensional real analytic version of Lévy theorem). *Let $p \in]0,1]$ and ω be a weight on $\mathbb{Z}$ satisfying. Let $f(t) = (f_1(t), ..., f_k(t))$, where $f_j(t) = \sum_{n \in \mathbb{Z}} a_{n,j} e^{int}$, for $j = 1, ..., k$, is a periodic function such that:*

$$\|f_j\|_{p,\omega} = \sum_{n \in \mathbb{Z}} |a_{n,j}|^p \omega(n) < +\infty.$$

Let Φ be an analytic function in $2k$ real variables on an open set U containing the image of the function f. Then $\Phi(f)$ also can be developed in a trigonometric series $\Phi(f)(t) = \sum_{n \in \mathbb{Z}} b_n e^{int}$ such that:

$$\|\Phi(f)\|_{p,\omega} = \sum_{n \in \mathbb{Z}} |b_n|^p \omega(n) < +\infty$$

and, for every $t \in \mathbb{R}$,

$$\Phi(Ref_1(t),\ Imf_1(t), ..., Ref_k(t),\ Imf_k(t)) = \sum_{n \in \mathbb{Z}} b_n e^{int}.$$

3.2 Another generalization of Wiener and Lévy theorems

We will now consider complex functions of several variables and real analytic functional calculus for a single variable in p-Banach algebras to give a generalization of N. Wiener and P. Lévy theorems. So, we obtain the following multi-dimensional generalization of the N. Wiener theorem.

Theorem 3.3 (Multi-dimensional generalization N. Wiener theorem). *Let $p \in]0,1]$ and ω be a weight on $\mathbb{Z}^k$. Let $f(t) = f(t_1, ..., t_k)$ be a 2π-periodic function concerning each variable, represented by a series*

$$f(t) = \sum_{n \in \mathbb{Z}^k} a_n e^{i(n,t)}$$

such that

$$\|f\|_{p,\omega} = \sum_{n \in \mathbb{Z}^k} |a_n|^p \omega(n) < +\infty.$$

If $f(t) \neq 0$, for all $t \in \mathbb{R}^k$, then the function $\frac{1}{f}$ can be developed in a trigonometric series $\frac{1}{f(t)} = \sum_{n \in \mathbb{Z}^k} b_n e^{i(n,t)}$ such that:

$$\left\|\frac{1}{f}\right\|_{p,\omega} = \sum_{n \in \mathbb{Z}^k} |b_n|^p \omega(n) < +\infty.$$

Using the holomorphic functional calculus and the fact that the spectrum of $f \in \mathcal{A}_k^p(\omega)$ is nothing but the set of values of f, we also obtain the following multi-dimensional generalization of the P. Lévy theorem.

Theorem 3.4 (Multi-dimensional generalization of P. Lévy theorem). *Let $p \in]0,1]$ and ω be a weight on $\mathbb{Z}^k$. Let $f(t) = f(t_1, ..., t_k)$ be a 2π-periodic function for each variable, represented by a series*

$$f(t) = \sum_{n \in \mathbb{Z}^k} a_n e^{i(n,t)}$$

such that

$$\|f\|_{p,\omega} = \sum_{n\in\mathbb{Z}^k} |a_n|^p \omega(n) < +\infty.$$

Let F be a holomorphic function defined on an open set containing the image of the function f. Then $F(f)$ is a periodic function with each variable, defined on $\mathbb{R}^k$ of the form:

$$F(f) = \sum_{n\in\mathbb{Z}^k} b_n e^{i(n,t)}$$

such that:

$$\|F(f)\|_{p,\omega} = \sum_{n\in\mathbb{Z}^k} |b_n|^p \omega(n) < +\infty.$$

Now, using real analytic functional of a single variable and the fact that the algebra $\left(\mathcal{A}_k^p(\omega), \|.\|_{p,\omega}\right)$ is a hermitian, we obtain the following generalization of Lévy's theorem for real analytic functions of several variables.

Theorem 3.5 (Real analytic multi-dimensional generalization of P. Lévy). *Let Let $p \in]0,1]$ and ω be a weight on $\mathbb{Z}^k$. Let $f(t) = f(t_1, ..., t_k)$ be a 2π-periodic function for each variable, represented by a theorem] series:*

$$f(t) = \sum_{n\in\mathbb{Z}^k} a_n e^{i(n,t)}$$

such that

$$\|f\|_{p,\omega} = \sum_{n\in\mathbb{Z}^k} |a_n|^p \omega(n) < +\infty.$$

Let F be an analytic function in two real variables on an open set containing the image of the function f. Then $F(f)$ is a 2π-periodic function with each variable, defined on $\mathbb{R}^k$ of the form:

$$F(f)(t) = \sum_{n\in\mathbb{Z}^k} b_n e^{i(n,t)}, \text{for every } t \in \mathbb{R}^k,$$

such that:

$$\|F(f)\|_{p,\omega} = \sum_{n\in\mathbb{Z}^k} |b_n|^p \omega(n) < +\infty.$$

Conflict of interest statement: On behalf of all authors, there is no conflict of interest.

References

[1] I. Abudulaï and M. Yaogan, Multipliers for Gelfand Pairs. Palestine Journal of Mathematics, Vol. 10 (1) (2021), 151-159.

[2] M. Akkar, A. El Kinani and M. Oudadess, Calculs fonctionnels harmonique et analytique réel, Ann. Sci. Math. Québec 12 (1988), no. 2, 151-169.

[3] S. J. Bhatt and H. V. Dedania, Beurling algebra analogues of the classical theorems of Wiener and Lévy on absolutely convergent Fourier series. Proc. Indian Acad. Sci. Math. Sci. 113 (2003), no. 2, 179-182.

[4] A. El Kinani, Version of Wiener's and Lévy's theorems. Rend. Circ. Mat. Palermo (2) 57 (2008), no. 3, 343-352.

[5] A. El Kinani and L. Bouchikhi, Corrigendum and addendum to "Wiener's and Lévy's theorems for some weighted power series". Rend. Circ. Mat. Palermo (2) 66 (2017), no. 3, 429-437.

[6] A. El Kinani and L. Bouchikhi, Wiener's and Lévy's theorems for some weighted power series. Rend. Circ. Mat. Palermo (2) 63 (2014), no. 2, 301-309.

[7] A. El Kinani and L. Bouchikhi, A weighted algebra analogues of Wiener's and Lévy's theorems. Rend. Circ. Mat. Palermo (2) 61 (2012), no. 3, 331-341.

[8] H. El Atef and A. El Kinani, Real analytic calculus for several variables and applications. Rend. Circ. Mat. Palermo (2) 70 (2021), no. 2, 653-663

[9] I. Gelfand, Normierte Ringe. (German) Rec. Math. [Mat. Sbornik] N. S. 9 (51), (1941). 3–24.

[10] P. Lévy, Sur la convergence absolue des séries de Fourier. Compositio Math. 1 (1935), 1-14.

[11] S. Rolewicz, On multi-dimensional generalizations of the Wiener-Żelazko and Lévy-Żelazko theorems. Topological algebras, their applications, and related topics, 283-291, Banach Center Publ., 67, Polish Acad. Sci. Inst. Math., Warsaw, 2005.

[12] N. Wiener, Tauberian theorems. Ann. Math. 33 (1932), 1-100.

[13] W. Żelazko, Selected topics in topological algebras. Lectures 1969/1970. Lecture Notes Series, No. 31. Matematisk Institut, Aarhus Universitet, Aarhus, 1971.

Author information

H. El Atef, Université Mohammed V de Rabat, E.N.S de Rabat, B. P. 5118, 10105, Rabat., Maroc.
E-mail: hamidatef1996master@gmail.com

L. Bouchikhi, Université Mohammed V de Rabat, E.N.S de Rabat, B. P. 5118, 10105, Rabat., Maroc.
E-mail: bouchikhi.maths@gmail.com

A. El Kinani, Université Mohammed V de Rabat, E.N.S de Rabat, B. P. 5118, 10105, Rabat., Maroc.
E-mail: abdellah.elkinani@um5.ac.ma

Received: 2022-09-08.
Accepted: 2023-04-11.