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NEW BINOMIAL FIBONACCI SUMS

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Abstract. We present some new linear, quadratic, cubic, and quartic binomial Fibonacci, Lucas, and Fibonacci–Lucas summation identities.

1 Introduction

Our goal is to derive, from elementary identities, some presumably new Fibonacci and Lucas identities including binomial coefficients. The research is a continuation of the recent works by Adegoke [1, 2] and Adegoke et al. [4, 5, 6]. The results are similar to those found in the classical articles of Carlitz [7], Carlitz and Ferns [8], Hoggatt et al. [10], Layman [13], Long [14], and Zeitlin [18]. More recent results on finite sums involving Fibonacci numbers and their generalizations can be found in [3, 9, 11, 16], among others.

Recall that the Fibonacci numbers F_j and the Lucas numbers L_j are defined, for $j \in \mathbb{Z}$, through the recurrences $F_j = F_{j-1} + F_{j-2}$, $j \geq 2$, $F_0 = 0$, $F_1 = 1$ and $L_j = L_{j-1} + L_{j-2}$, $j \geq 2$, $L_0 = 2$, $L_1 = 1$, with $F_{-j} = (-1)^{j-1} F_j$ and $L_{-j} = (-1)^j L_j$. See sequences A000045 and A000032 in the On-Line Encyclopedia of Integer Sequences [15] and references contained therein.

Throughout this paper, we denote the golden ratio by $\alpha = \frac{1+\sqrt{5}}{2}$ and write $\beta = -\frac{1}{\alpha}$, so that $\alpha\beta = -1$ and $\alpha + \beta = 1$. Binet formulas for the Fibonacci and Lucas numbers are

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \quad L_j = \alpha^j + \beta^j, \quad j \in \mathbb{Z}. \tag{1.1}$$

Here are a couple of results to whet the reader's appetite for reading on:

$$\sum_{k=1}^{n} \binom{2n-1}{2k-1} F_{2k-1} = \begin{cases} \frac{1}{2} F_{2n-1} L_{n-1} L_n, & n \text{ odd;} \\ \frac{5}{2} F_{2n-1} F_{n-1} F_n, & n \text{ even,} \end{cases}$$

$$\sum_{k=1}^{n} \binom{2n-1}{2k-1} L_{2k-1} = \frac{1}{2}(L_{4n-2} - L_{2n-1}),$$

$$\sum_{k=0}^{n} \binom{2n}{2k} F_{3k+r} F_{3k+s} = 2^{2n-1} F_n F_{3n+r+s},$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{n-2k+1} F_{2k+s} = \begin{cases} (-1)^{s+1} F_{2n-s} + 5^{n/2} F_{n+s}, & n \text{ even;} \\ (-1)^{s+1} F_{2n-s} + 5^{(n-1)/2} L_{n+s}, & n \text{ odd,} \end{cases}$$

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{2k} L_{j(2rk+s)} = \begin{cases} \sqrt{5}^n F_{jr}^n L_{j(rn+s)} \cos\left(2n \arctan(\alpha^{jr})\right), & jr \text{ odd;} \\ L_{jr}^n L_{j(rn+s)} \cos\left(2n \arctan(\alpha^{jr})\right), & jr \text{ even, } n \text{ even;} \\ \sqrt{5} L_{jr}^n F_{j(rn+s)} \cos\left(2n \arctan(\alpha^{jr})\right), & jr \text{ even, } n \text{ odd.} \end{cases}$$

The organization of this paper is as follows. The next section is concerned with the preliminaries, while in Sections 3–5 we derive identities involving Fibonacci (Lucas) numbers and binomial coefficients. In the remaining sections, we will present results containing higher-order binomial Fibonacci and Lucas identities. For instance, in Section 6, using some of Vajda's formulas, we will derive new binomial identities involving products of Fibonacci and Lucas numbers. Finally, in Sections 7 and 8 we derive some binomial identities involving the product of three and four Fibonacci and/or Lucas numbers.

Note that our results can be applied more generally to broader classes of second-order linearly recurrent sequences with constant coefficients.

2 Required identities

Lemma 2.1 (K. Adegoke [1]). *For real or complex z, let a given well-behaved function $h(z)$ have in its domain the representation $h(z) = \sum_{k=c_1}^{c_2} g(k) z^{f(k)}$, where $f(k)$ and $g(k)$ are given real sequences and $-\infty \le c_1 \le c_2 \le \infty$. Let j be an integer. Then*

$$\sqrt{5} \sum_{k=c_1}^{c_2} g(k) z^{f(k)} F_{jf(k)} = h(\alpha^j z) - h(\beta^j z), \tag{2.1}$$

$$\sum_{k=c_1}^{c_2} g(k) z^{f(k)} L_{jf(k)} = h(\alpha^j z) + h(\beta^j z). \tag{2.2}$$

Lemma 2.2 (S. Vajda [17]). *For integers r and s,*

$$F_{r+s} + (-1)^s F_{r-s} = L_s F_r, \qquad F_{r+s} - (-1)^s F_{r-s} = F_s L_r, \tag{2.3}$$

$$L_{r+s} + (-1)^s L_{r-s} = L_s L_r, \qquad L_{r+s} - (-1)^s L_{r-s} = 5F_s F_r. \tag{2.4}$$

If u and v are integers having the same parity, then identities (2.3) and (2.4) can be put in the following useful versions:

$$F_u + (-1)^{\frac{u-v}{2}} F_v = L_{\frac{u-v}{2}} F_{\frac{u+v}{2}}, \qquad F_u - (-1)^{\frac{u-v}{2}} F_v = F_{\frac{u-v}{2}} L_{\frac{u+v}{2}},$$

$$L_u + (-1)^{\frac{u-v}{2}} L_v = L_{\frac{u-v}{2}} L_{\frac{u+v}{2}}, \qquad L_u - (-1)^{\frac{u-v}{2}} L_v = 5F_{\frac{u-v}{2}} F_{\frac{u+v}{2}},$$

and

$$F_u + F_v = \begin{cases} L_{\frac{u-v}{2}} F_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ even;} \\ F_{\frac{u-v}{2}} L_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ odd,} \end{cases} \qquad F_u - F_v = \begin{cases} L_{\frac{u-v}{2}} F_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ odd;} \\ F_{\frac{u-v}{2}} L_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ even,} \end{cases}$$

$$L_u + L_v = \begin{cases} L_{\frac{u-v}{2}} L_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ even;} \\ 5F_{\frac{u-v}{2}} F_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ odd,} \end{cases} \qquad L_u - L_v = \begin{cases} L_{\frac{u-v}{2}} L_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ odd;} \\ 5F_{\frac{u-v}{2}} F_{\frac{u+v}{2}}, & \frac{u-v}{2} \text{ even.} \end{cases}$$

Lemma 2.3 (K. Adegoke [1]). *For p and q integers,*

$$1 + (-1)^p \alpha^{2q} = \begin{cases} (-1)^p \alpha^q \sqrt{5} F_q, & p-q \text{ odd;} \\ (-1)^p \alpha^q L_q, & p-q \text{ even,} \end{cases}$$

$$1 - (-1)^p \alpha^{2q} = \begin{cases} (-1)^{p-1} \alpha^q L_q, & p-q \text{ odd;} \\ (-1)^{p-1} \alpha^q \sqrt{5} F_q, & p-q \text{ even.} \end{cases}$$

The following formulas can be easily derived from the Binet formulas (1.1).

Lemma 2.4. *For p and q integers,*

$$(-1)^q + \alpha^{2q} = \alpha^q L_q, \qquad (-1)^q - \alpha^{2q} = -\sqrt{5} \alpha^q F_q, \tag{2.5}$$

$$(-1)^q + \beta^{2q} = \beta^q L_q, \qquad (-1)^q - \beta^{2q} = \sqrt{5} \beta^q. \tag{2.6}$$

Lemma 2.5 (V. E. Hoggatt, Jr. et al. [10]). *For p and q integers,*

$$L_{p+q} - L_p\alpha^q = -\sqrt{5}\beta^p F_q, \qquad L_{p+q} - L_p\beta^q = \sqrt{5}\alpha^p F_q,$$
$$F_{p+q} - F_p\alpha^q = \beta^p F_q, \qquad F_{p+q} - F_p\beta^q = \alpha^p F_q.$$

Lemma 2.6. *We have*

$$1-\alpha=\beta, \quad 1-\beta=\alpha, \quad 1+\alpha=\alpha^2, \quad 1+\beta=\beta^2, \tag{2.7}$$
$$1-\alpha^3=-2\alpha, \quad 1-\beta^3=-2\beta, \quad 1+\alpha^3=2\alpha^2, \quad 1+\beta^3=2\beta^2, \tag{2.8}$$
$$1-2\alpha=-\sqrt{5}, \quad 1-2\beta=\sqrt{5}, \quad 1+2\alpha=\alpha^3, \quad 1+2\beta=\beta^3, \tag{2.9}$$
$$2-\alpha=\beta^2, \quad 2-\beta=\alpha^2, \quad 2+\alpha=\alpha\sqrt{5}, \quad 2+\beta=-\beta\sqrt{5}, \tag{2.10}$$
$$1-\alpha^3\sqrt{5}=-2\alpha^3, \ 1-\beta^3\sqrt{5}=4\beta^2, \ 1+\alpha^3\sqrt{5}=4\alpha^2, \ 1+\beta^3\sqrt{5}=-2\beta^3, \tag{2.11}$$
$$\sqrt{5}-\alpha^3=-2, \quad \sqrt{5}-\beta^3=-4\beta, \quad \sqrt{5}+\alpha^3=4\alpha, \quad \sqrt{5}+\beta^3=2, \tag{2.12}$$
$$3-\alpha^3=2\beta, \quad 3-\beta^3=2\alpha, \quad 3+\alpha^3=2\alpha\sqrt{5}, \quad 3+\beta^3=-2\beta\sqrt{5}, \tag{2.13}$$
$$1-3\alpha^3=-2\alpha^2\sqrt{5}, \quad 1-3\beta^3=2\beta^2\sqrt{5}, \quad 1+3\alpha^3=2\alpha^4, \quad 1+3\beta^3=2\beta^4. \tag{2.14}$$

Proof. Each identity is obtained by making appropriate substitutions for p and q in the identities given in Lemma 2.5. □

3 Binomial summation identities, Part 1

The first two key identities used frequently in this part of the paper are stated in the next fundamental lemma.

Lemma 3.1. *For integers j, r, s, and a non-negative integer n, we have*

$$\begin{aligned}
&2\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}x^{n-2k}z^{2k}F_{j(2rk+s)} \\
&\qquad = \alpha^{js}\big((x+\alpha^{jr}z)^n + (x-\alpha^{jr}z)^n\big) - \beta^{js}\big((x+\beta^{jr}z)^n + (x-\beta^{jr}z)^n\big),
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}x^{n-2k}z^{2k}L_{j(2rk+s)} \\
&\qquad = \alpha^{js}\big((x+\alpha^{jr}z)^n + (x-\alpha^{jr}z)^n\big) + \beta^{js}\big((x+\beta^{jr}z)^n + (x-\beta^{jr}z)^n\big).
\end{aligned} \tag{3.2}$$

Proof. In the identity

$$h(z) = 2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}x^{n-2k}z^{2rk+s} = z^s(x+z^r)^n + z^s(x-z^r)^n, \tag{3.3}$$

identify $g(k) = 2\binom{n}{2k}x^{n-2k}$, $f(k) = 2rk+s$, $c_1 = 0$, $c_2 = \lfloor n/2\rfloor$, and use these in (2.1) and (2.2). □

In our first main results, we state mixed Fibonacci–Lucas identities with additional parameters.

Theorem 3.2. *For a non-negative integer n and any integers s and j,*

$$2\sum_{k=0}^{n}\binom{2n}{2k}F_{j(4k+s)} = \big(L_j^{2n} + 5^n F_j^{2n}\big)F_{j(2n+s)}, \tag{3.4}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{j(4k+s)} = \big(L_j^{2n} + 5^n F_j^{2n}\big)L_{j(2n+s)}, \tag{3.5}$$

$$2\sum_{k=0}^{n-1}\binom{2n-1}{2k}F_{j(4k+s)}=(-1)^j\big(L_j^{2n-1}F_{j(2n+s-1)}-5^{n-1}F_j^{2n-1}L_{j(2n+s-1)}\big),\tag{3.6}$$

$$2\sum_{k=0}^{n-1}\binom{2n-1}{2k}L_{j(4k+s)}=(-1)^j\big(L_j^{2n-1}L_{j(2n+s-1)}-5^{n}F_j^{2n-1}F_{j(2n+s-1)}\big).\tag{3.7}$$

Proof. In (3.1), set $x=(-1)^j$, $z=1$, and $r=2$ and use (2.5) and (2.6) to obtain

$$2\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{j(4k+s)}=L_j^n\big(\alpha^{j(n+s)}-\beta^{j(n+s)}\big)+(-\sqrt{5})^nF_j^n\big(\alpha^{j(n+s)}-(-1)^n\beta^{j(n+s)}\big),$$

from which (3.4) and (3.6) follow from the parity of n and the Binet formulas (1.1). The proof of (3.5) and (3.7) is similar; use $x=(-1)^j$, $z=1$, and $r=2$ in (3.2). □

From Lemma 3.1 we can deduce the following Fibonacci and Lucas identities.

Theorem 3.3. *For non-negative integer n and any integer s, we have*

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+s}=F_{2n+s}-(-1)^sF_{n-s},\tag{3.8}$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{2k+s}=L_{2n+s}+(-1)^sL_{n-s}.\tag{3.9}$$

Proof. Set $x=z=j=r=1$ in (3.1) and (3.2), and then use (2.7). This gives

$$2\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+s}=\alpha^{2n+s}-\beta^{2n+s}-(\alpha\beta)^s(\alpha^{n-s}-\beta^{n-s}),$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{2k+s}=\alpha^{2n+s}+\beta^{2n+s}+(\alpha\beta)^s(\alpha^{n-s}+\beta^{n-s}),$$

from which the stated identities follow immediately from the Binet formulas. □

We proceed with some corollaries.

Corollary 3.4. *For a non-negative integer n and any integer s,*

$$2\sum_{k=0}^{n}\binom{2n}{2k}F_{2k+s}=\begin{cases}L_{n+s}F_{3n}, & n \text{ odd};\\ F_{n+s}L_{3n}, & n \text{ even},\end{cases}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{2k+s}=\begin{cases}5F_{n+s}F_{3n}, & n \text{ odd};\\ L_{n+s}L_{3n}, & n \text{ even}.\end{cases}$$

Proof. Write $2n$ for n in each of the identities (3.8) and (3.9) and use Lemma 2.2. □

A variant of the Fibonacci (Lucas) sums with even subscripts is stated as the next corollary.

Corollary 3.5. *For a positive integer n,*

$$2\sum_{k=0}^{n-1}\binom{2n-1}{2k}F_{2k}=\begin{cases}5F_{n-1}F_nF_{2n-1}, & n \text{ odd};\\ L_{n-1}L_nF_{2n-1}, & n \text{ even},\end{cases}$$

$$2\sum_{k=0}^{n-1}\binom{2n-1}{2k}L_{2k}=L_{4n-2}+L_{2n-1}.$$

Proof. Replace n by $2n-1$ in each of the identities (3.8) and (3.9) and simplify. □

Theorem 3.6. *For a non-negative integer n and any integer s,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{6k+s}=2^{n-1}\big(F_{2n+s}+(-1)^nF_{n+s}\big),\tag{3.10}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{6k+s}=2^{n-1}\big(L_{2n+s}+(-1)^nL_{n+s}\big).\tag{3.11}$$

Proof. Set $x=z=j=1$ and $r=3$ in (3.1) and (3.2), then use (2.8) and simplify. □

As special cases of Theorem 3.6 obtained so far, we have the following mixed Fibonacci–Lucas identities, depending on an additional parameter.

Corollary 3.7. *For a non-negative integer n and any integer s, we have*

$$\sum_{k=0}^{n}\binom{2n}{2k}F_{6k+s}=2^{2n-1}\begin{cases}L_nF_{3n+s}, & n\text{ even};\\ F_nL_{3n+s}, & n\text{ odd},\end{cases}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}L_{6k+s}=2^{2n-1}\begin{cases}L_nL_{3n+s}, & n\text{ even};\\ 5F_nF_{3n+s}, & n\text{ odd}.\end{cases}\tag{3.12}$$

Proof. Write $2n$ for n in each of the identities (3.10), (3.11) and use Lemma 2.2. □

Corollary 3.8. *For a positive integer n,*

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}F_{6k}=4^{n-1}\begin{cases}5F_{n-1}F_nF_{2n-1}, & n\text{ odd};\\ L_{n-1}L_nF_{2n-1}, & n\text{ even},\end{cases}$$

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}L_{6k}=4^{n-1}(L_{4n-2}-L_{2n-1}).$$

Proof. Replace n by $2n-1$ in each of the identities (3.10), (3.11) and simplify. □

Theorem 3.9. *For a non-negative integer n and any integer s,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^kF_{6k+s}=2^{n-1}\big(2^nF_{2n+s}+(-1)^nF_{3n+s}\big),$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^kL_{6k+s}=2^{n-1}\big(2^nL_{2n+s}+(-1)^nL_{3n+s}\big).\tag{3.13}$$

Proof. Setting $x=j=1$, $z=\sqrt{5}$, and $r=3$ in (3.1) and (3.2) while making use of the identities (2.11) gives

$$2\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^kF_{6k+s}=4^n\big(\alpha^{2n+s}-\beta^{2n+s}\big)+(-2)^n\big(\alpha^{3n+s}-\beta^{3n+s}\big),$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^kL_{6k+s}=4^n\big(\alpha^{2n+s}+\beta^{2n+s}\big)+(-2)^n\big(\alpha^{3n+s}+\beta^{3n+s}\big),$$

from which the stated identities follow. □

Corollary 3.10. *For a non-negative integer n and any integer s, we have*

$$\sum_{k=0}^{n}\binom{2n}{2k}5^kF_{6k+s}=2^{2n-1}\big(4^nF_{4n+s}+F_{6n+s}\big),$$

$$\sum_{k=0}^{n}\binom{2n}{2k}5^kL_{6k+s}=2^{2n-1}\big(4^nL_{4n+s}+L_{6n+s}\big).$$

Theorem 3.11. *For a non-negative integer n and any integer s,*

$$2\sum_{k=0}^{n}\binom{2n}{2k}\frac{F_{6k+s}}{5^k}=\left(\frac{4}{5}\right)^n\left(4^nF_{2n+s}+F_s\right), \tag{3.14}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}\frac{L_{6k+s}}{5^k}=\left(\frac{4}{5}\right)^n\left(4^nL_{2n+s}+L_s\right), \tag{3.15}$$

$$8\sum_{k=0}^{n}\binom{2n-1}{2k}\frac{F_{6k+s}}{5^k}=\left(\frac{4}{5}\right)^n\left(4^nL_{2n-1+s}-2L_s\right), \tag{3.16}$$

$$2\sum_{k=0}^{n}\binom{2n-1}{2k}\frac{L_{6k+s}}{5^k}=\left(\frac{4}{5}\right)^{n-1}\left(4^nF_{2n-1+s}-2F_s\right). \tag{3.17}$$

Proof. Setting $z=j=1$, $x=\sqrt{5}$, and $r=3$ in (3.1) and (3.2) while making use of the identities (2.12) gives

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{F_{6k+s}}{5^{k-(n+1)/2}}=4^n\left(\alpha^{n+s}-(-1)^n\beta^{n+s}\right)+(-2)^n\left(\alpha^s-(-1)^n\beta^s\right), \tag{3.18}$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{L_{6k+s}}{5^{k-n/2}}=4^n\left(\alpha^{n+s}+(-1)^n\beta^{n+s}\right)+(-2)^n\left(\alpha^s+(-1)^n\beta^s\right). \tag{3.19}$$

Writing $2n$ for n in (3.18) and (3.19) produces identities (3.14), (3.15) while writing $2n-1$ for n yields identities (3.16) and (3.17). □

Theorem 3.12. *For a non-negative integer n and any integer s,*

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}4^kF_{2k+s}=\begin{cases}F_{3n+s}+5^{n/2}F_s, & n \text{ even};\\ F_{3n+s}-5^{(n-1)/2}L_s, & n \text{ odd},\end{cases} \tag{3.20}$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}4^kL_{2k+s}=\begin{cases}L_{3n+s}+5^{n/2}L_s, & n \text{ even};\\ L_{3n+s}-5^{(n+1)/2}F_s, & n \text{ odd}.\end{cases}$$

Proof. Set $x=j=r=1$ and $z=2$ in (3.1), (3.2) and make use of (2.9). The calculations are straightforward and omitted. □

Theorem 3.13. *For a non-negative integer n and any integer s,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{F_{2k+s}}{4^k}=\frac{1}{2^{n+1}}\begin{cases}(-1)^{s+1}F_{2n-s}+5^{n/2}F_{n+s}, & n \text{ even};\\ (-1)^{s+1}F_{2n-s}+5^{(n-1)/2}L_{n+s}, & n \text{ odd},\end{cases}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{L_{2k+s}}{4^k}=\frac{1}{2^{n+1}}\begin{cases}(-1)^{s}L_{2n-s}+5^{n/2}L_{n+s}, & n \text{ even};\\ (-1)^{s}L_{2n-s}+5^{(n+1)/2}F_{n+s}, & n \text{ odd}.\end{cases}$$

Proof. Set $x=2$, $z=-1$, and $r=j=1$ in (3.1), (3.2) and make use of (2.10). The calculations are straightforward and omitted. □

Theorem 3.14. *For a non-negative integer n and any integer s,*

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{F_{6k+s}}{9^k}=\left(\frac{2}{3}\right)^n\begin{cases}(-1)^{s+1}F_{n-s}+5^{n/2}F_{n+s}, & n \text{ even};\\ (-1)^{s+1}F_{n-s}+5^{(n-1)/2}L_{n+s}, & n \text{ odd},\end{cases}$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{L_{6k+s}}{9^k}=\left(\frac{2}{3}\right)^n\begin{cases}(-1)^{s}L_{n-s}+5^{n/2}L_{n+s}, & n \text{ even};\\ (-1)^{s}L_{n-s}+5^{(n+1)/2}F_{n+s}, & n \text{ odd}.\end{cases}$$

Proof. Set $x = r = 3$, $z = -1$, and $j = 1$ in (3.1) and (3.2) and make use of (2.13). The calculations are omitted. □

We conclude with the following evaluation.

Theorem 3.15. *For a non-negative integer n and any integer s,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}9^k F_{6k+s} = 2^{n-1}\begin{cases} F_{4n+s} + 5^{n/2}F_{2n+s}, & n \text{ even};\\ F_{4n+s} - 5^{(n-1)/2}L_{2n+s}, & n \text{ odd},\end{cases}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}9^k L_{6k+s} = 2^{n-1}\begin{cases} L_{4n+s} + 5^{n/2}L_{2n+s}, & n \text{ even};\\ L_{4n+s} - 5^{(n+1)/2}F_{2n+s}, & n \text{ odd}.\end{cases}$$

Proof. Set $x = j = 1$ and $z = r = 3$ in (3.1), (3.2) and make use of (2.14). The calculations are omitted. □

4 Binomial summation identities, Part 2

This section is based on the following fundamental lemma.

Lemma 4.1. *For integers j, r, s, and n with n non-negative, we have*

$$\begin{aligned} &2\sqrt{5}\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}x^{n-2k+1}z^{2k-1}F_{j(2rk+s)} \\ &\quad = \alpha^{j(r+s)}\big((x+\alpha^{jr}z)^n - (x-\alpha^{jr}z)^n\big) - \beta^{j(r+s)}\big((x+\beta^{jr}z)^n - (x-\beta^{jr}z)^n\big), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}x^{n-2k+1}z^{2k-1}L_{j(2rk+s)} \\ &\quad = \alpha^{j(r+s)}\big((x+\alpha^{jr}z)^n - (x-\alpha^{jr}z)^n\big) + \beta^{j(r+s)}\big((x+\beta^{jr}z)^n - (x-\beta^{jr}z)^n\big). \end{aligned} \tag{4.2}$$

Proof. In the identity

$$h(z) = 2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}x^{n-2k}z^{2rk+s} = z^{r+s}(x+z^r)^n - z^{r+s}(x-z^r)^n,$$

identify $g(k) = 2\binom{n}{2k-1}x^{n-2k}$, $f(k) = 2rk+s$, $c_1 = 1$, $c_2 = \lceil n/2\rceil$, and use these in (2.1) and (2.2). □

The next achievement of the paper is the following statement.

Theorem 4.2. *For a non-negative integer n and any integer s,*

$$2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{2k+s} = F_{2n+s+1} - (-1)^s F_{n-s-1}, \tag{4.3}$$

$$2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}L_{2k+s} = L_{2n+s+1} + (-1)^s L_{n-s-1}.$$

Proof. Set $x = z = j = r = 1$ in (4.1) and (4.2) to obtain

$$2\sqrt{5}\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{2k+s} = \alpha^{2n+s+1} - \beta^{2n+s+1} + (\alpha\beta)^{s+1}(\alpha^{n-s-1} - \beta^{n-s-1}),$$

$$2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}L_{2k+s} = \alpha^{2n+s+1} + \beta^{2n+s+1} - (\alpha\beta)^{s+1}(\alpha^{n-s-1} + \beta^{n-s-1}),$$

and hence the stated identities. □

From Theorem 4.2 we can immediately obtain the following binomial identities.

Corollary 4.3. *For a non-negative integer n and any integer s,*

$$2\sum_{k=1}^{n}\binom{2n}{2k-1}F_{2k+s}=\begin{cases}L_{n+s+1}F_{3n}, & n \text{ even};\\ F_{n+s+1}L_{3n}, & n \text{ odd},\end{cases} \tag{4.4}$$

$$2\sum_{k=1}^{n}\binom{2n}{2k-1}L_{2k+s}=\begin{cases}5F_{n+s+1}F_{3n}, & n \text{ even};\\ L_{n+s+1}L_{3n}, & n \text{ odd}.\end{cases}$$

Proof. We prove (4.4). From (4.3), using (2.3) we have

$$\begin{aligned}2\sum_{k=1}^{n}\binom{2n}{2k-1}F_{2k+s}&=F_{4n+s+1}-(-1)^{s}F_{2n-s-1}\\&=F_{3n+(n+s+1)}+(-1)^{s+1}F_{3n-(n+s+1)}\\&=\begin{cases}L_{n+s+1}F_{3n}, & n \text{ even};\\ F_{n+s+1}L_{3n}, & n \text{ odd}.\end{cases}\end{aligned}$$

□

Corollary 4.4. *For a positive integer n,*

$$2\sum_{k=1}^{n}\binom{2n-1}{2k-1}F_{2k-1}=\begin{cases}F_{2n-1}L_{n-1}L_{n}, & n \text{ odd};\\ 5F_{2n-1}F_{n-1}F_{n}, & n \text{ even},\end{cases}$$

$$2\sum_{k=1}^{n}\binom{2n-1}{2k-1}L_{2k-1}=L_{4n-2}-L_{2n-1}.$$

Theorem 4.5. *For a non-negative integer n and any integer s,*

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{6k+s}=2^{n-1}\big(F_{2n+3+s}-(-1)^{n}F_{n+3+s}\big),$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}L_{6k+s}=2^{n-1}\big(L_{2n+3+s}-(-1)^{n}L_{n+3+s}\big).$$

Proof. Set $x=z=j=1$ and $r=3$ in (4.1) and (4.2), use (2.8) and simplify. □

Corollary 4.6. *For a positive integer n and any integer s,*

$$\sum_{k=1}^{n}\binom{2n-1}{2k-1}F_{6k+s}=4^{n-1}\big(F_{4n+1+s}+F_{2n+2+s}\big),$$

$$\sum_{k=1}^{n}\binom{2n-1}{2k-1}L_{6k+s}=4^{n-1}\big(L_{4n+1+s}+L_{2n+2+s}\big).$$

Theorem 4.7. *For a non-negative integer n and any integer s,*

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}4^{k}F_{2k+s}=\begin{cases}F_{3n+1+s}-5^{n/2}F_{s+1}, & n \text{ even};\\ F_{3n+1+s}+5^{(n-1)/2}L_{s+1}, & n \text{ odd},\end{cases}$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}4^{k}L_{2k+s}=\begin{cases}L_{3n+1+s}-5^{n/2}L_{s+1}, & n \text{ even};\\ L_{3n+1+s}+5^{(n+1)/2}F_{s+1}, & n \text{ odd}.\end{cases}$$

Proof. Set $x=j=r=1$ and $z=2$ in (4.1) and (4.2), use (2.9) and simplify. □

Theorem 4.8. *For a non-negative integer n and any integer s,*

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{F_{2k+s}}{4^k}=\frac{1}{2^{n+2}}\begin{cases}(-1)^{s+1}F_{2n-1-s}+5^{n/2}F_{n+s+1}, & n \text{ even};\\ (-1)^{s+1}F_{2n-1-s}+5^{(n-1)/2}L_{n+s+1}, & n \text{ odd},\end{cases}$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{L_{2k+s}}{4^k}=\frac{1}{2^{n+2}}\begin{cases}(-1)^{s}L_{2n-1-s}+5^{n/2}L_{n+s+1}, & n \text{ even};\\ (-1)^{s}L_{2n-1-s}+5^{(n+1)/2}F_{n+s+1}, & n \text{ odd}.\end{cases}$$

Proof. Set $x=2$, $z=-1$, $j=r=1$ in (4.1) and (4.2), use (2.10) and simplify. □

Theorem 4.9. *For a non-negative integer n and any integer s,*

$$6\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{F_{6k+s}}{9^k}=\left(\frac{2}{3}\right)^n\begin{cases}(-1)^{s+1}F_{n-3-s}+5^{n/2}F_{n+s+3}, & n \text{ even};\\ (-1)^{s+1}F_{n-3-s}+5^{(n-1)/2}L_{n+s+3}, & n \text{ odd},\end{cases}$$

$$6\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{L_{6k+s}}{9^k}=\left(\frac{2}{3}\right)^n\begin{cases}(-1)^{s}L_{n-3-s}+5^{n/2}L_{n+s+3}, & n \text{ even};\\ (-1)^{s}L_{n-3-s}+5^{(n+1)/2}F_{n+s+3}, & n \text{ odd}.\end{cases}$$

Proof. Set $x=r=3$, $z=-1$, and $j=1$ in (4.1) and (4.2), use (2.13) and simplify. □

Theorem 4.10. *For a non-negative integer n and any integer s,*

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}9^kF_{6k+s}=2^{n-1}\begin{cases}F_{4n+3+s}-5^{n/2}F_{2n+s+3}, & n \text{ even};\\ F_{4n+3+s}+5^{(n-1)/2}L_{2n+s+3}, & n \text{ odd},\end{cases}$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}9^kL_{6k+s}=2^{n-1}\begin{cases}L_{4n+3+s}-5^{n/2}L_{2n+s+3}, & n \text{ even};\\ L_{4n+3+s}+5^{(n+1)/2}F_{2n+s+3}, & n \text{ odd}.\end{cases}$$

Proof. Set $x=j=1$ and $z=r=3$ in (4.1), and (4.2), use (2.14) and simplify. □

5 Binomial summation identities, Part 3

In this section, we introduce the following results.

Lemma 5.1. *For integers j, r, s, and a non-negative integer n, we have*

$$\begin{aligned}\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}x^{n-2k}z^{2k}F_{j(2rk+s)}&\\ =\alpha^{js}\sqrt{(x^2+\alpha^{2jr}z^2)^n}\cos\left(n\arctan\left(\frac{\alpha^{jr}z}{x}\right)\right)&\\ -\beta^{js}\sqrt{(x^2+\beta^{2jr}z^2)^n}\cos\left(n\arctan\left(\frac{\beta^{jr}z}{x}\right)\right),&\end{aligned}\tag{5.1}$$

$$\begin{aligned}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}x^{n-2k}z^{2k}L_{j(2rk+s)}&\\ =\alpha^{js}\sqrt{(x^2+\alpha^{2jr}z^2)^n}\cos\left(n\arctan\left(\frac{\alpha^{jr}z}{x}\right)\right)&\\ +\beta^{js}\sqrt{(x^2+\beta^{2jr}z^2)^n}\cos\left(n\arctan\left(\frac{\beta^{jr}z}{x}\right)\right),&\end{aligned}\tag{5.2}$$

$$\begin{aligned}\sqrt{5}\sum_{k=1}^{\lceil n/2\rceil}(-1)^{k-1}\binom{n}{2k-1}x^{n-2k+1}z^{2k-1}F_{j(2rk+s)}\\ &= \alpha^{js}\sqrt{(x^2+\alpha^{2jr}z^2)^n}\,\sin\left(n\arctan\Big(\frac{\alpha^{jr}z}{x}\Big)\right)\\ &\quad-\beta^{js}\sqrt{(x^2+\beta^{2jr}z^2)^n}\,\sin\left(n\arctan\Big(\frac{\beta^{jr}z}{x}\Big)\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{\lceil n/2\rceil}(-1)^{k-1}\binom{n}{2k-1}x^{n-2k+1}z^{2k-1}L_{j(2rk+s)}\\ &= \alpha^{js}\sqrt{(x^2+\alpha^{2jr}z^2)^n}\,\sin\left(n\arctan\Big(\frac{\alpha^{jr}z}{x}\Big)\right)\\ &\quad+\beta^{js}\sqrt{(x^2+\beta^{2jr}z^2)^n}\,\sin\left(n\arctan\Big(\frac{\beta^{jr}z}{x}\Big)\right).\end{aligned}$$

Proof. In the identity

$$z^s\sqrt{(x^2+z^{2r})^n}\,\cos\left(n\arctan\Big(\frac{z^r}{x}\Big)\right)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}x^{n-2k}z^{2rk+s},$$

identify $h(z)=z^s\sqrt{(x^2+z^{2r})^n}\cos\big(n\arctan(\frac{z^r}{x})\big)$, $g(k)=(-1)^k\binom{n}{2k}x^{n-2k}$, $f(k)=2rk+s$, $c_1=0$, $c_2=\lfloor n/2\rfloor$, and use these in (2.1) and (2.2). This proves (5.1) and (5.2). For the other two identities use

$$z^{s+r}\sqrt{(x^2+z^{2r})^n}\,\sin\left(n\arctan\Big(\frac{z^r}{x}\Big)\right)=\sum_{k=1}^{\lceil n/2\rceil}(-1)^{k-1}\binom{n}{2k-1}x^{n-2k+1}z^{2rk+s},$$

and identify $h(z)=z^{s+r}\sqrt{(x^2+z^{2r})^n}\sin\big(n\arctan(\frac{z^r}{x})\big)$, $f(k)=2rk+s$, $c_1=1$, $c_2=\lceil n/2\rceil$, and $g(k)=(-1)^{k-1}\binom{n}{2k-1}x^{n-2k+1}$. □

We give an example. The next lemma proves useful.

Lemma 5.2. *For integers r and n,*

$$\begin{aligned}\cos\big(2n\arctan(\alpha^r)\big)&=(-1)^n\cos\big(2n\arctan(\beta^r)\big),\\ \cos\big((2n-1)\arctan(\alpha^r)\big)&=(-1)^{n+r+1}\sin\big((2n-1)\arctan(\beta^r)\big),\\ \sin\big(2n\arctan(\alpha^r)\big)&=(-1)^{n+r+1}\sin\big(2n\arctan(\beta^r)\big),\\ \sin\big((2n-1)\arctan(\alpha^r)\big)&=(-1)^{n-1}\cos\big((2n-1)\arctan(\beta^r)\big).\end{aligned}$$

Proof. On account of identities $\arctan x+\arctan(\frac{1}{x})=\frac{\pi}{2}$, if $x>0$, and $\alpha^r=(-1)^r\beta^{-r}$, we have $\arctan(\alpha^r)=\frac{\pi}{2}-(-1)^r\arctan(\beta^r)$, from which the identities follow upon applying addition formulas for trigonometric functions. □

The main result of this section is the following theorem in which we derive Fibonacci–Lucas identities with two parameters involving the golden ratio.

Theorem 5.3. *For a non-negative integer n and integers j, r, s,*

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}F_{j(2rk+s)}=\begin{cases}\sqrt{5^n}F_{jr}^nF_{j(rn+s)}\cos\big(2n\arctan(\alpha^{jr})\big), & jr\text{ odd};\\ L_{jr}^nF_{j(rn+s)}\cos\big(2n\arctan(\alpha^{jr})\big), & jr\text{ and }n\text{ even};\\ \frac{1}{\sqrt5}L_{jr}^nL_{j(rn+s)}\cos\big(2n\arctan(\alpha^{jr})\big), & jr\text{ even, }n\text{ odd,}\end{cases}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}L_{j(2rk+s)}=\begin{cases}\sqrt{5^n}F_{jr}^nL_{j(rn+s)}\cos\big(2n\arctan(\alpha^{jr})\big), & jr\text{ odd};\\ L_{jr}^nL_{j(rn+s)}\cos\big(2n\arctan(\alpha^{jr})\big), & jr\text{ and }n\text{ even};\\ \sqrt5L_{jr}^nF_{j(rn+s)}\cos\big(2n\arctan(\alpha^{jr})\big), & jr\text{ even, }n\text{ odd.}\end{cases}\quad(5.3)$$

Proof. The choice $x = z = 1$ in (5.1) and (5.2), noting also identities (2.5) and (2.6) with $q = jr$, jr odd, gives

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}F_{j(2rk+s)} = 5^{(n-1)/2}F_{jr}^n\left(\alpha^{j(rn+s)} - \beta^{j(rn+s)}\right)\cos\left(2n\arctan(\alpha^{jr})\right),$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}L_{j(2rk+s)} = 5^{n/2}F_{jr}^n\left(\alpha^{j(rn+s)} + \beta^{j(rn+s)}\right)\cos\left(2n\arctan(\alpha^{jr})\right),$$

from which the stated identities for jr odd follow.

Similarly, $x = z = 1$ in (5.1) and (5.2), with jr even, gives

$$\sqrt{5}\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}F_{j(2rk+s)} = L_{jr}^n\left(\alpha^{j(rn+s)} - (-1)^n\beta^{j(rn+s)}\right)\cos\left(2n\arctan(\alpha^{jr})\right),$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}L_{j(2rk+s)} = L_{jr}^n\left(\alpha^{j(rn+s)} + (-1)^n\beta^{j(rn+s)}\right)\cos\left(2n\arctan(\alpha^{jr})\right),$$

and hence the stated identities for jr even. □

So far in this paper, we have been concerned with identities that are linear in the Fibonacci and Lucas numbers. In the remaining three sections we will present results containing higher-order binomial Fibonacci identities.

6 Quadratic binomial summation identities

Here we will derive a pair of binomial identities involving products of Fibonacci and Lucas numbers. We require the results stated in the next two lemmas.

Lemma 6.1. *If k, r, and s are integers, then*

$$5F_{k+r}F_{k+s} = L_{2k+r+s} - (-1)^{k+s}L_{r-s}, \tag{6.1}$$

$$L_{k+r}F_{k+s} = F_{2k+r+s} - (-1)^{k+s}F_{r-s}, \tag{6.2}$$

$$L_{k+r}L_{k+s} = L_{2k+r+s} + (-1)^{k+s}L_{r-s}. \tag{6.3}$$

Proof. These are variations on [17, Identities (15b), (17a), (17b)]. □

Lemma 6.2. *If n is a positive integer, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(-1)^k = (\sqrt{2})^n\cos\left(\frac{n\pi}{4}\right), \tag{6.4}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k} = 2^{n-1}, \tag{6.5}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-1}{2k} = 2^{n-2}, \quad n \geq 2, \tag{6.6}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(-4)^k = (\sqrt{5})^n\cos(n\arctan 2), \tag{6.7}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(-5)^k = (\sqrt{6})^n\cos\left(n\arctan\sqrt{5}\right). \tag{6.8}$$

Proof. Setting $x = 1$, $z = i$ in (3.3) produces (6.4), while $x = z = 1$ gives $2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k} = 2^n$, from which (6.5) and (6.6) follow. Use of $x = \frac{1}{2}$, $z = i$, $r = 1$, $s = 0$ in (3.3) proves (6.7) while $x = \frac{1}{\sqrt{5}}$, $z = i$, $r = 1$, and $s = 0$ produces (6.8). □

We present our next findings in the next theorem which provides some summation formulas involving Fibonacci (Lucas) numbers and binomial coefficients.

Theorem 6.3. *If n is a non-negative integer and r, s are integers, then*

$$10\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{k+r}F_{k+s}=L_{2n+r+s}+(-1)^{r+s}L_{n-r-s}-(-1)^{s}2^{n/2+1}\cos\left(\frac{n\pi}{4}\right)L_{r-s},\tag{6.9}$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{k+r}F_{k+s}=F_{2n+r+s}-(-1)^{r+s}F_{n-r-s}-(-1)^{s}2^{n/2+1}\cos\left(\frac{n\pi}{4}\right)F_{r-s},\tag{6.10}$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{k+r}L_{k+s}=L_{2n+r+s}+(-1)^{r+s}L_{n-r-s}+(-1)^{s}2^{n/2+1}\cos\left(\frac{n\pi}{4}\right)L_{r-s}.\tag{6.11}$$

Proof. From (6.1), we get

$$5\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{k+r}F_{k+s}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{2k+r+s}-(-1)^{s}L_{r-s}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{k}\binom{n}{2k}$$

and hence (6.9), upon use of (3.9) and (6.4). The proof of (6.10) and (6.11) is similar. □

Theorem 6.4. *If n is a positive integer and r, s are any integers, then*

$$10\sum_{k=0}^{n}\binom{2n}{2k}4^{k}F_{k+r}F_{k+s}=L_{6n+r+s}+5^{n}L_{r+s}-(-1)^{s}5^{n}2L_{r-s}\cos\big(2n\arctan 2\big),\tag{6.12}$$

$$\begin{aligned}10\sum_{k=0}^{n-1}\binom{2n-1}{2k}4^{k}F_{k+r}F_{k+s}\\ =L_{6n+r+s-3}-5^{n}F_{r+s}-(-1)^{s}5^{n-1/2}2L_{r-s}\cos\big((2n-1)\arctan 2\big),\end{aligned}\tag{6.13}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}4^{k}L_{k+r}F_{k+s}=F_{6n+r+s}+5^{n}F_{r+s}-(-1)^{s}5^{n}2F_{r-s}\cos\big(2n\arctan 2\big),$$

$$\begin{aligned}2\sum_{k=0}^{n-1}\binom{2n-1}{2k+1}4^{k}L_{k+r}F_{k+s}\\ =F_{6n+r+s-3}-5^{n-1}L_{r+s}-(-1)^{s}5^{n-1/2}2F_{r-s}\cos\big((2n-1)\arctan 2\big),\end{aligned}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}4^{k}L_{k+r}L_{k+s}=L_{6n+r+s}+5^{n}L_{r+s}+(-1)^{s}5^{n}2L_{r-s}\cos\left(2n\arctan 2\right),$$

$$\begin{aligned}2\sum_{k=0}^{n-1}\binom{2n-1}{2k}4^{k}L_{k+r}L_{k+s}\\ =L_{6n+r+s-3}-5^{n}F_{r+s}+(-1)^{s}5^{n-1/2}2L_{r-s}\cos\big((2n-1)\arctan 2\big).\end{aligned}$$

Proof. Using (6.1) we have

$$10\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}4^{k}F_{k+r}F_{k+s}=2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}4^{k}L_{2k+r+s}-(-1)^{s}2L_{r-s}\sum_{k=0}^{\lfloor n/2\rfloor}(-4)^{k}\binom{n}{2k},$$

from which (6.12) and (6.13) now follow on account of (3.20) and (6.7). The remaining identities can be similarly proved, using (6.2), (6.3) and the identities stated in Theorem 3.12. □

Further interesting identities involving Fibonacci and Lucas numbers are stated in the next theorem.

Theorem 6.5. *If n is a positive integer and r, s and j are any integers, then*

$$10\sum_{k=0}^{n}\binom{2n}{2k}F_{j(2k+r)}F_{j(2k+s)}=\left(L_j^{2n}+5^nF_j^{2n}\right)L_{j(2n+r+s)}-(-1)^{js}4^nL_{j(r-s)},$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{j(2k+r)}F_{j(2k+s)}=\left(L_j^{2n}+5^nF_j^{2n}\right)F_{j(2n+r+s)}-(-1)^{js}4^nF_{j(r-s)},\tag{6.14}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{j(2k+r)}L_{j(2k+s)}=\left(L_j^{2n}+5^nF_j^{2n}\right)L_{j(2n+r+s)}+(-1)^{js}4^nL_{j(r-s)}$$

and

$$\begin{aligned}10\sum_{k=0}^{n-1}&\binom{2n-1}{2k}F_{j(2k+r)}F_{j(2k+s)}\\&=(-1)^jL_j^{2n-1}L_{j(2n+r+s-1)}-(-1)^j5^nF_j^{2n-1}F_{j(2n+r+s-1)}-(-1)^{js}2^{2n-1}L_{j(r-s)},\\2\sum_{k=0}^{n-1}&\binom{2n-1}{2k}L_{j(2k+r)}F_{j(2k+s)}\\&=(-1)^jL_j^{2n-1}F_{j(2n+r+s-1)}-(-1)^j5^{n-1}F_j^{2n-1}L_{j(2n+r+s-1)}-(-1)^{js}2^{2n-1}F_{j(r-s)},\\2\sum_{k=0}^{n-1}&\binom{2n-1}{2k}L_{j(2k+r)}L_{j(2k+s)}\\&=(-1)^jL_j^{2n-1}L_{j(2n+r+s-1)}-(-1)^j5^nF_j^{2n-1}F_{j(2n+r+s-1)}+(-1)^{js}2^{2n-1}L_{j(r-s)}.\end{aligned}$$

Proof. We prove (6.14). The proof of each of the remaining identities is similar and requires the identities given in Theorem 3.2. In (6.1) write $2kj$ for k, rj for r and sj for s to obtain $5F_{j(2k+r)}F_{j(2k+s)}=L_{j(4k+r+s)}-(-1)^{js}L_{j(r-s)}$. Thus,

$$10\sum_{k=0}^{n}\binom{2n}{2k}F_{j(2k+r)}F_{j(2k+s)}=2\sum_{k=0}^{n}\binom{2n}{2k}L_{j(4k+r+s)}-(-1)^{js}2L_{j(r-s)}\sum_{k=0}^{n}\binom{2n}{2k},$$

from which (6.14) follows after using (3.5) and (6.5). □

Theorem 6.6. *If n is a positive integer and r and s are any integers, then*

$$\begin{aligned}5\sum_{k=0}^{\lfloor n/2\rfloor}&\binom{n}{2k}F_{3k+r}F_{3k+s}\\&=2^{n-1}\left(L_{2n+r+s}+(-1)^nL_{n+r+s}\right)-(-1)^s\sqrt{2^n}\cos\left(\frac{n\pi}{4}\right)L_{r-s},\end{aligned}\tag{6.15}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{3k+r}F_{3k+s}=2^{n-1}\left(F_{2n+r+s}+(-1)^nF_{n+r+s}\right)-(-1)^s\sqrt{2^n}\cos\left(\frac{n\pi}{4}\right)F_{r-s},$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{3k+r}L_{3k+s}=2^{n-1}\left(L_{2n+r+s}+(-1)^nL_{n+r+s}\right)+(-1)^s\sqrt{2^n}\cos\left(\frac{n\pi}{4}\right)L_{r-s}.$$

Proof. We prove only (6.15). Using (6.1), write

$$5F_{3k+r}F_{3k+s}=L_{6k+r+s}-(-1)^{k+s}L_{r-s}.\tag{6.16}$$

Thus,

$$5\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{3k+r}F_{3k+s}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{6k+r+s}-L_{r-s}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{k+s}\binom{n}{2k}$$

and hence (6.15), using (3.11) and (6.4). □

Using (3.12), we have the following binomial Fibonacci identity.

Corollary 6.7. *If n is a positive odd integer and r and s are any integers, then*

$$\sum_{k=0}^{n}\binom{2n}{2k}F_{3k+r}F_{3k+s}=2^{2n-1}F_nF_{3n+r+s}.$$

We conclude the analysis in this section with the following result.

Theorem 6.8. *If n is a non-negative integer and r and s are any integers, then*

$$\begin{aligned}&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^{k-1}F_{3k+r}F_{3k+s}\\&\qquad=2^{n-1}\big(2^nL_{2n+r+s}+(-1)^nL_{3n+r+s}\big)-(-1)^s\sqrt{6^n}\cos\big(n\arctan\sqrt{5}\big)L_{r-s},\end{aligned}\tag{6.17}$$

$$\begin{aligned}&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^{k}L_{3k+r}F_{3k+s}\\&\qquad=2^{n-1}\big(2^nF_{2n+r+s}+(-1)^nF_{3n+r+s}\big)-(-1)^s\sqrt{6^n}\cos\big(n\arctan\sqrt{5}\big)F_{r-s},\\&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^{k}L_{3k+r}L_{3k+s}\\&\qquad=2^{n-1}\big(2^nL_{2n+r+s}+(-1)^nL_{3n+r+s}\big)+(-1)^s\sqrt{6^n}\cos\big(n\arctan\sqrt{5}\big)L_{r-s}.\end{aligned}$$

Proof. We prove (6.17). Using (6.16) we have

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^{k-1}F_{3k+r}F_{3k+s}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^{k}L_{6k+r+s}-(-1)^sL_{r-s}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(-5)^k$$

and hence (6.17) using (3.13) and (6.8). □

7 Cubic binomial summation identities

In this section, we derive some binomial identities involving the product of three Fibonacci and/or Lucas numbers. The identities stated in the next lemma are needed for this purpose.

Lemma 7.1. *For any integers k, r, s, and t,*

$$5F_{k+r}F_{k+s}F_{k+t}=F_{3k+r+s+t}-(-1)^{k+r}F_{k+s+t-r}-(-1)^{k+t}L_{s-t}F_{k+r},\tag{7.1}$$

$$5L_{k+r}F_{k+s}F_{k+t}=L_{3k+r+s+t}+(-1)^{k+r}L_{k+s+t-r}-(-1)^{k+t}L_{s-t}L_{k+r},$$

$$L_{k+r}L_{k+s}F_{k+t}=F_{3k+r+s+t}+(-1)^{k+r}F_{k+s+t-r}-(-1)^{k+t}F_{s-t}L_{k+r},$$

$$L_{k+r}L_{k+s}L_{k+t}=L_{3k+r+s+t}+(-1)^{k+r}L_{k+s+t-r}+(-1)^{k+t}L_{s-t}L_{k+r}.\tag{7.2}$$

Proof. These can be derived from the identities stated in Lemma 6.1. Identities (7.1) and (7.2) are also given in [12]. □

Theorem 7.2. *If n is a non-negative integer and r, s, and t are any integers, then*

$$\begin{aligned}
10\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+r}F_{2k+s}F_{2k+t} &= 2^n\big(F_{2n+s+r+t}+(-1)^nF_{n+s+r+t}\big)\\
&\quad-(-1)^r\big(F_{2n+s+t-r}-(-1)^{s+t-r}F_{n-s-t+r}\big)-(-1)^tL_{s-t}\big(F_{2n+r}-(-1)^rF_{n-r}\big),
\end{aligned}\tag{7.3}$$

$$\begin{aligned}
10\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{2k+r}F_{2k+s}F_{2k+t} &= 2^n\big(L_{2n+s+r+t}+(-1)^nL_{n+s+r+t}\big)\\
&\quad+(-1)^r\big(L_{2n+s+t-r}+(-1)^{s+t-r}L_{n-s-t+r}\big)-(-1)^tL_{s-t}\big(L_{2n+r}+(-1)^rL_{n-r}\big),
\end{aligned}$$

$$\begin{aligned}
2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{2k+r}L_{2k+s}F_{2k+t} &= 2^n\big(F_{2n+s+r+t}+(-1)^nF_{n+s+r+t}\big)\\
&\quad+(-1)^r\big(F_{2n+s+t-r}-(-1)^{s+t-r}F_{n-s-t+r}\big)-(-1)^tF_{s-t}\big(L_{2n+r}+(-1)^rL_{n-r}\big),
\end{aligned}$$

$$\begin{aligned}
2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{2k+r}L_{2k+s}L_{2k+t} &= 2^n\big(L_{2n+s+r+t}+(-1)^nL_{n+s+r+t}\big)\\
&\quad+(-1)^r\big(L_{2n+s+t-r}+(-1)^{s+t-r}L_{n-s-t+r}\big)+(-1)^tL_{s-t}\big(L_{2n+r}+(-1)^rL_{n-r}\big).
\end{aligned}$$

Proof. Write $2k$ for k in (7.1) and sum to obtain

$$\begin{aligned}
5\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+r}F_{2k+s}F_{2k+t} &= \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{6k+r+s+t}\\
&\quad-(-1)^r\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+s+t-r}-(-1)^tL_{s-t}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+r},
\end{aligned}$$

whence (7.3) in view of (3.8) and (3.10). □

8 Quartic binomial summation identities

We conclude our study with the derivation of some quartic binomial Fibonacci identities.

The identities stated in the next lemma are required.

Lemma 8.1. *If k, p, q, r, and s are any integers, then*

$$\begin{aligned}
&25F_{k+p}F_{k+q}F_{k+r}F_{k+s}\\
&\quad= L_{4k+p+q+r+s}-(-1)^{s+k}L_{2k+p+q+r-s}-(-1)^{r+k}L_{2k+p+q-r+s}\\
&\qquad-(-1)^{q+k}L_{p-q}L_{2k+r+s}+(-1)^{r+s}L_{p+q-r-s}+(-1)^{q+s}L_{p-q}L_{r-s},
\end{aligned}\tag{8.1}$$

$$\begin{aligned}
5L_{k+p}F_{k+q}F_{k+r}F_{k+s} &= F_{4k+p+q+r+s}-(-1)^{s+k}F_{2k+p+q+r-s}-(-1)^{r+k}F_{2k+p+q-r+s}\\
&\quad-(-1)^{q+k}F_{p-q}L_{2k+r+s}+(-1)^{r+s}F_{p+q-r-s}+(-1)^{q+s}F_{p-q}L_{r-s},
\end{aligned}$$

$$\begin{aligned}
5L_{k+p}L_{k+q}F_{k+r}F_{k+s} &= L_{4k+p+q+r+s}-(-1)^{s+k}L_{2k+p+q+r-s}-(-1)^{r+k}L_{2k+p+q-r+s}\\
&\quad+(-1)^{q+k}L_{p-q}L_{2k+r+s}+(-1)^{r+s}L_{p+q-r-s}-(-1)^{q+s}L_{p-q}L_{r-s},
\end{aligned}$$

$$\begin{aligned}
L_{k+p}L_{k+q}L_{k+r}F_{k+s} &= F_{4k+p+q+r+s}-(-1)^{s+k}F_{2k+p+q+r-s}+(-1)^{r+k}F_{2k+p+q-r+s}\\
&\quad+(-1)^{q+k}L_{p-q}F_{2k+r+s}-(-1)^{r+s}F_{p+q-r-s}-(-1)^{q+s}L_{p-q}F_{r-s},
\end{aligned}$$

$$\begin{aligned}
&L_{k+p}L_{k+q}L_{k+r}L_{k+s}\\
&\quad= L_{4k+p+q+r+s}+(-1)^{s+k}L_{2k+p+q+r-s}+(-1)^{r+k}L_{2k+p+q-r+s}\\
&\qquad+(-1)^{q+k}L_{p-q}L_{2k+r+s}+(-1)^{r+s}L_{p+q-r-s}+(-1)^{q+s}L_{p-q}L_{r-s}.
\end{aligned}\tag{8.2}$$

Proof. These can be derived from the identities stated in Lemma 6.1. Identities (8.1) and (8.2) are also given in [12]. □

Theorem 8.2. *If n is a positive integer and p, q, r, and s are any integers, then*

$$\begin{aligned}
50\sum_{k=0}^{n}\binom{2n}{2k}&F_{k+p}F_{k+q}F_{k+r}F_{k+s}\\
&= (5^n+1)L_{2n+p+q+r+s} - (-1)^s 5^{n/2} 2L_{n+p+q+r-s}\cos(2n\arctan\alpha)\\
&\quad - 5^{n/2}2\cos(2n\arctan\alpha)\big((-1)^r L_{n+p+q-r+s} + (-1)^q L_{p-q}L_{n+r+s}\big)\\
&\quad + (-1)^s 4^n\big((-1)^r L_{p+q-r-s} + (-1)^q L_{p-q}L_{r-s}\big),
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
10\sum_{k=0}^{n}\binom{2n}{2k}&L_{k+p}F_{k+q}F_{k+r}F_{k+s}\\
&= (5^n+1)F_{2n+p+q+r+s} - (-1)^s 5^{n/2} 2F_{n+p+q+r-s}\cos(2n\arctan\alpha)\\
&\quad - 5^{n/2}2\cos(2n\arctan\alpha)\big((-1)^r F_{n+p+q-r+s} + (-1)^q F_{p-q}L_{n+r+s}\big)\\
&\quad + (-1)^s 4^n\big((-1)^r F_{p+q-r-s} + (-1)^q F_{p-q}L_{r-s}\big),
\end{aligned}$$

$$\begin{aligned}
10\sum_{k=0}^{n}\binom{2n}{2k}&L_{k+p}L_{k+q}F_{k+r}F_{k+s}\\
&= (5^n+1)L_{2n+p+q+r+s} - (-1)^s 5^{n/2} 2L_{n+p+q+r-s}\cos(2n\arctan\alpha)\\
&\quad - 5^{n/2}2\cos(2n\arctan\alpha)\big((-1)^r L_{n+p+q-r+s} - (-1)^q L_{p-q}L_{n+r+s}\big)\\
&\quad + (-1)^s 4^n\big((-1)^r L_{p+q-r-s} - (-1)^q L_{p-q}L_{r-s}\big),
\end{aligned}$$

$$\begin{aligned}
2\sum_{k=0}^{n}\binom{2n}{2k}&L_{k+p}L_{k+q}L_{k+r}F_{k+s}\\
&= (5^n+1)F_{2n+p+q+r+s} - (-1)^s 5^{n/2} 2F_{n+p+q+r-s}\cos(2n\arctan\alpha)\\
&\quad + 5^{n/2}2\cos(2n\arctan\alpha)\big((-1)^r F_{n+p+q-r+s} + (-1)^q L_{p-q}F_{n+r+s}\big)\\
&\quad - (-1)^s 4^n\big((-1)^r F_{p+q-r-s} + (-1)^q L_{p-q}F_{r-s}\big),
\end{aligned}$$

$$\begin{aligned}
2\sum_{k=0}^{n}\binom{2n}{2k}&L_{k+p}L_{k+q}L_{k+r}L_{k+s}\\
&= (5^n+1)L_{2n+p+q+r+s} + (-1)^s 5^{n/2} 2L_{n+p+q+r-s}\cos(2n\arctan\alpha)\\
&\quad + 5^{n/2}2\cos(2n\arctan\alpha)\big((-1)^r L_{n+p+q-r+s} + (-1)^q L_{p-q}L_{n+r+s}\big)\\
&\quad + (-1)^s 4^n\big((-1)^r L_{p+q-r-s} + (-1)^q L_{p-q}L_{r-s}\big).
\end{aligned}$$

Proof. From (8.1), we have

$$\begin{aligned}
25&\sum_{k=0}^{n}\binom{2n}{2k}F_{k+p}F_{k+q}F_{k+r}F_{k+s}\\
&= \sum_{k=0}^{n}\binom{2n}{2k}L_{4k+p+q+r+s} - (-1)^s\sum_{k=0}^{n}\binom{2n}{2k}(-1)^k L_{2k+p+q+r-s}\\
&\quad - (-1)^r\sum_{k=0}^{n}\binom{2n}{2k}(-1)^k L_{2k+p+q-r+s} - (-1)^q L_{p-q}\sum_{k=0}^{n}\binom{2n}{2k}(-1)^k L_{2k+r+s}\\
&\quad + (-1)^{r+s}L_{p+q-r-s}\sum_{k=0}^{n}\binom{2n}{2k} + (-1)^{q+s}L_{p-q}L_{r-s}\sum_{k=0}^{n}\binom{2n}{2k},
\end{aligned}$$

which yields (8.3) on account of (3.5), (5.3) and (6.5). □

9 Conclusion

In this paper, we derived summation identities involving Fibonacci (Lucas) numbers and binomial coefficients $\binom{n}{2k}$, $\binom{n}{2k-1}$, $\binom{2n}{2k}$, $\binom{2n}{2k-1}$, $\binom{2n-1}{2k}$, and $\binom{2n-1}{2k-1}$. Also, we presented results containing higher-order (quadratic, cubic, and quartic) binomial Fibonacci, Lucas, and Fibonacci–Lucas summation identities.

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