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ON THE COMPARABILITY GRAPHS OF LATTICES

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Abstract: The covering graphs and the comparability graphs of ordered sets have received a lot of attention. In this paper, we study the comparability graph of lattices. We discuss the basic properties of the comparability graphs, such as connectedness, diameter, girth, etc. We also use these results to unify many results of the inclusion graph of algebraic structures like vector spaces, modules, rings, semigroups, etc.

1 Introduction

The covering graphs $Cov(P)$ and the comparability graphs $Com(P)$ are widely studied graphs associated with a poset P. Note that the covering graph $Cov(P)$ and the comparability graph $Com(P)$ do not determine the poset P up to isomorphism. However, on the other hand, there are some special classes of posets (e.g., modular lattices) whose covering graph $Cov(P)$ still contains important information about the poset P. In [13], Ward proved that a modular lattice of finite length is distributive if and only if its covering graph contains no subgraph isomorphic to $K_{2,3}$ (the covering graph of M_3). Further, Dilworth [7] proved that the covering graph of a modular lattice of finite length and of breadth n contains a subgraph isomorphic to the n-dimensional hypercube, i.e., $\mathbf{2}^n$.

In general, a *comparability graph* is a simple unoriented graph, and two vertices are adjacent if and only if they are comparable with respect to some partial order on its vertices.

Recently, there has been an ever-growing interest in the graph associated with algebraic structures, for example, zero-divisor graphs of rings and ordered sets, ideal intersection graphs of rings, ideal inclusion graphs of rings (semigroups), etc. A generic definition of an inclusion graph can be done as follows. Let X be an algebraic structure. Then, the inclusion graphs of substructures of X are denoted by $In(X)$, where the vertex set is the collection of all nontrivial substructures of X, and two substructures M and N are adjacent if and only if either $M \subseteq N$ or $N \subseteq M$. The following are examples of the inclusion graphs associated with the various algebraic structures.

(i) The subspace inclusion graph $In(V)$ of a finite-dimensional vector space V (see [5]).

(ii) The submodule inclusion graph $In(M)$ of a module M (see [8]) .

(iii) The ideal inclusion graph $In(R)$ of a ring R (see [1]).

(iv) The ideal inclusion graph $In(S)$ of a semigroup S (see [2]).

(v) The subgroup inclusion graph $In(G)$ of a group G (see [6]).

We have noted that there is a unifying pattern in the results of the inclusion graph of substructures of algebraic structures (cf. [1, Theorem 1], [4, Theorem 4.1], [8, Proposition 2.5]). Furthermore, we observe that the comparability graph of the lattice of substructures of an algebraic structure is a tool for studying these graphs. Hence, it is necessary to know the properties of the lattice of substructures of an algebraic structure. In the Remark 1.1, we have listed the known properties.

Remark 1.1. (i) If V is a finite-dimensional vector space over a field F, then the lattice $L(V)$ of subspaces of V is bounded, modular, atomistic, dual atomistic, complemented and self-dual.

(ii) The lattice $L(M)$ of submodules of a module M is a bounded, modular lattice.

(iii) The lattice $L(R)$ of left ideals of a ring R is a bounded, modular lattice.

(iv) The lattice $L(S)$ of ideals of a semigroup S is a bounded, distributive lattice.

(v) The lattice $L(G)$ of subgroups of a group G is a bounded lattice. If G is abelian, then $L(G)$ is a bounded, modular lattice.

We, now quote the formal definition of the comparability graph.

Definition 1.2. Let L be a bounded lattice. The *comparability graph* of L is an undirected, simple graph denoted by $Com(L)$, where the vertex set is $L \setminus \{0_L, 1_L\}$ and two vertices a and b are adjacent if and only if a and b are comparable, i.e., $a < b$ or $b < a$.

From the definitions of the comparability graph and the inclusion graph, we have the following remark.

Remark 1.3. (i) If V is a finite dimensional vector space over the field F, then the subspace inclusion graph $In(V)$ is the comparability graph of the lattice $L(V)$ of subspaces of V, i.e., $In(V) = Com(L(V))$.

(ii) The submodule inclusion graph $In(M)$ of a module M is the comparability graph of the lattice $L(M)$ of submodules of M, i.e., $In(M) = Com(L(M))$.

(iii) The inclusion ideal graph $In(R)$ of a ring (semigroup) R is the comparability graph of the lattice $L(R)$ of ideals of R, i.e., $In(R) = Com(L(R))$.

(iv) The inclusion graph $I(G)$ of subgroups of a group G is the comparability graph of the lattice $L(G)$ of subgroups of G, i.e., $In(G) = Com(L(G))$.

2 Preliminaries

Let $(L; \leq)$ be a lattice. The least element and greatest element (if they exist) will be denoted by 0_L and 1_L, respectively. A subset L' of L is said to be a *sublattice* of L if it satisfies the property that $a, b \in L'$ implies that $a \vee b, a \wedge b \in L'$ with the $\vee$ and the $\wedge$ of L' are restrictions to L' of the $\vee$ and the $\wedge$ of L. A lattice L is said to be *bounded* if it has both 0_L and 1_L. A *0-1-sublattice* of a bounded lattice L is a sublattice containing the 0_L and 1_L of L. If two elements a, b are incomparable, we denote it by $a \| b$. If any two elements in L are comparable, then L is said to be a *chain*. C_n denotes a chain with n elements, and the length of C_n is $n - 1$. If L has 0_L and every chain in L is finite, then the *height* $h(a)$ of an element $a \in L$ is the maximum length of a maximal chain from 0_L to a. The *length* of L, denoted by $\ell(L)$, is defined as the maximum length of a maximal chain in L. If L has 1_L, then $\ell(L) = h(1_L)$.

For $x, y \in L$ we write $y \prec x$ (y is *covered by* x or x *covers* y) if $y < x$ and $y < z \leq x$ implies that $x = z$. A sublattice L' of L is said to a *cover preserving sublattice* if it satisfies the property that $a \prec b$ in L if and only if $a \prec b$ in L' for any $a, b \in L'$.

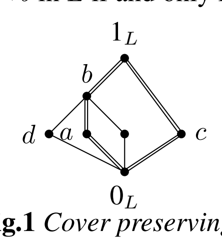


Fig.1 *Cover preserving 0-1-sublattice N_5*

Example 2.1. Consider the lattice L depicted in the adjacent figure. Clearly, $L' = \{0_L, a, b, c, 1_L\}$ is a cover preserving 0-1-sublattice of L whereas $L'' = \{0_L, c, d, 1_L\}$ is a sublattice of L, in fact, a 0-1-sublattice of L but not a cover preserving. The lattice L' is isomorphic to a nonmodular lattice N_5.

An element a of a bounded lattice L is an *atom* if $0_L \prec a$, and it is a *dual atom* if $a \prec 1_L$. A bounded lattice L is called *atomic* if for every non-zero element b, there exists an atom $a \in L$ such that $a \leq b$. A bounded lattice L is called *dual atomic* if for every non-unit element b, there exists a dual atom $a \in L$ such that $b \leq a$. An atomic lattice L is called *atomistic* if every non-zero element is a join of atoms contained in it, and it is called *dual atomistic* if every non-unit element is a meet of dual atoms above it.

Let $(L, \leq)$ be a lattice with a partial order $\leq$. Then the *dual lattice* $(L, \geq)$ of L is the lattice, where the partial order $\geq$ is defined as $x \geq y$ if and only if $x \leq y$ in L for $x, y \in L$. The dual lattice of L is denoted by L^*. A lattice is said to be *self-dual* if $L \cong L^*$.

A lattice is called *upper semimodular* if and only if it satisfies the *upper covering condition*, i.e., $a \prec b$ implies that $a \vee c \prec b \vee c$ or $a \vee c = b \vee c$. The upper semimodular lattices are also known as semimodular lattices. The dual notion of upper semimodular lattice is *lower semimodular* lattice. A lattice L is called *modular*, if for $a, b, c \in L$, $c \vee (a \wedge b) = (c \vee a) \wedge b$ for all $c \leq b$. Clearly, a modular lattice is upper semimodular as well as lower semimodular. A lattice L is said to be *distributive* if it satisfies $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ or $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$. In a bounded lattice L, an element a is a *complement* of an element b if $a \vee b = 1_L$ and $a \wedge b = 0_L$ and the lattice L is said to be *complemented* if every element in L has a complement. A *Boolean lattice* is a lattice that is complemented and distributive. Any undefined concepts related to lattice theory can be found in [3], [9], [12].

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph G is an *edgeless graph* if $E(G)$ is empty. A *simple graph* has no loops and multiple edges. A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If all the vertices of G are pairwise adjacent, then G is said to be *complete*. Two graphs G_1 and G_2 are said to be *isomorphic* if there exists a bijective map, ϕ, from $V(G_1)$ to $V(G_2)$ such that $(u, v) \in E(G_1)$ if and only if $(\phi(u), \phi(v)) \in E(G_2)$, for any $u, v \in V(G_1)$. A finite graph G is a *path* if its vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list and its *length* is the number of edges in it. A path G is a *cycle* if its initial and end vertices coincide. A graph G is said to be *triangulated (hyper-triangulated)* if each vertex (edge) of G is a vertex (edge) of a triangle. A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*. If G is a connected graph, then the *distance* between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is defined as the length of a shortest path joining u and v; otherwise, it is infinity. The *diameter* of G is the maximum number in the set of distances between the pairs of vertices of G. The *girth* of a graph G is the length of a shortest cycle if it exists; otherwise, it is infinity. Any undefined concepts related to graph theory can be found in [14].

Throughout the paper, L is a bounded lattice.

3 Basic results

In this section, we quote a few immediate results from the definitions.

Remark 3.1. (i) $Com(L)$ is a complete graph if and only if L is a chain.

(ii) If L is a lattice, then there is a one-to-one correspondence between (maximal) chains in $L \setminus \{0_L, 1_L\}$ and (maximal) cliques in $Com(L)$.

(iii) If L is an atomistic lattice of length at least 2, then $Com(L)$ is not a complete graph.

The following corollary is immediate from Remark 3.1 and Remark 1.1.

Corollary 3.2 ([4, Corollary 3.3]). *The subspace inclusion graph $In(V)$ of a vector space V is not complete for dim$(V) \geq 2$.*

Lemma 3.3. *If L_1 is a sublattice of a lattice L, then $Com(L_1)$ is an induced subgraph of $Com(L)$.*

Remark 3.4. The converse of Lemma 3.3 is not generally true.

Lemma 3.5. *Let V be a vector space. If W is a subspace of V with a dimension greater than 1, then the lattice $L(W)$ of subspaces of W is a sublattice of $L(V)$.*

Corollary 3.6 ([4, Theorem 3.1]). *If V is a vector space over a field F and W is a subspace of V with a dimension greater than 1, then $In(W)$ is a subgraph of $In(V)$.*

Corollary 3.7 ([8, Proposition 2.1]). *Let M be an R-module. If K is a submodule of M, then $In(K)$ is a subgraph of $In(M)$.*

Corollary 3.8 ([4, Theorem 3.1]). *If G is a group and N is a subgroup of G, then $In(N)$ is a subgraph of $In(G)$.*

Lemma 3.9. *Let L be a lattice and $a, b \in L$ such that $h(a) = h(b) = k < \infty$. Then a and b are not adjacent in $Com(L)$.*

Proof. Let $a, b \in L$ such that $h(a) = h(b) = k < \infty$. Assume that a and b are comparable. Without loss of generality, let $a < b$. Since $h(a) = k$, there exists a maximal chain, $0_L \prec a_1 \prec a_2 \prec \cdots \prec a_k = a$. Also, $a < b$ implies that there is a maximal chain $a \prec a_{k+1} \prec \cdots \prec a_m = b$. Clearly, $0_L \prec a_1 \prec a_2 \prec \cdots \prec a_k (= a) \prec a_{k+1} \prec \cdots \prec a_m = b$ is a chain from 0_L to b. This gives $k = h(b) \geq m > k$, a contradiction. Thus, a and b are incomparable and are not adjacent in $Com(L)$. □

Remark 3.10. In the case of graded lattices, all maximal chains between any two elements are of the same length. Hence, in the above proof, the inequality $h(b) \geq m$ is equality, that is, $h(b) = m$.

Now, we relate the dimension of a subspace and the height of that subspace in the subspace lattice. Note that the subspace lattice is modular; hence, the height function is well-defined.

Lemma 3.11. *Let V be a vector space. If W is a subspace of V, then $dim(W) = h(W)$ in $L(V)$.*

Proof. Let V be a vector space and W be a subspace of V. Suppose that $dim(W) = k$. Let $\{w_1, w_2, \cdots, w_k\}$ be a basis of W. Consider the subspace W_i generated by $\{w_1, w_2, \cdots, w_i\}$ and W_0 is the subspace generated by 0. Clearly, we have a chain of subspaces W_i $(i = 0, 1, \cdots, k)$ of V such that $W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{k-1} \subsetneq W_k$ and $dim(W_i) = i$, for $i = 0, 1, 2, \cdots, k$. Clearly, this is a maximal chain of length k in $L(V)$ from $0_{L(V)}$ to W. Since $L(V)$ is modular and hence graded, we have $h(W) = k$ in $L(V)$. Thus, $dim(W) = h(W)$ in $L(V)$. □

Corollary 3.12 ([4, Lemma 3.1]). *If W_1 and W_2 are two distinct subspaces of V of the same dimension, then W_1 and W_2 are not adjacent in $In(V)$.*

Proof. Follows from Lemma 3.11 and Lemma 3.9. □

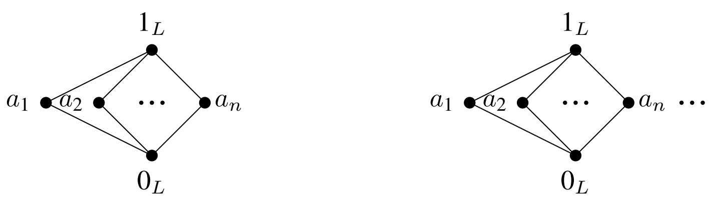
4 Connectedness of the comparability graphs

In this section, we study the connectedness of the comparability graph of lattices. In particular, we study the comparability graph of 0-modular lattices, a general class than the class of modular lattices and atomic lattices with the covering property. We need the following definitions and results.

Definition 4.1 ([12, p. 133]). A lattice L with the least element 0_L is said to be *0-modular* if for $a, b, c \in L$, we have $a \leq c$ and $b \wedge c = 0_L$ implies $(a \vee b) \wedge c = a$.

Theorem 4.2 ([12, p. 134, Theorem 3.4.1]). *A lattice L with 0 is 0-modular if and only if for $a, b, c \in L$ with $a < c$, $b \wedge c = 0_L$, and $b \vee a = b \vee c$ together imply $a = c$. In other words, L is 0-modular if and only if there exists in L no pentagon sublattice N_5 (Fig. 1) containing the least element 0_L of L.*

The Hasse diagrams of M_n and M_∞ are shown below.



M_n M_∞

Theorem 4.3. *Let L be a 0-modular lattice with at least two non-zero, non-unit elements. Then the following statements are equivalent.*

(i) $Com(L)$ *is a disconnected graph.*

(ii) $L \cong M_n$ *or* $L \cong M_\infty$*.*

(iii) $Com(L)$ *is an edgeless graph.*

(iv) L *contains a cover preserving 0-1-sublattice isomorphic to* $C_2 \times C_2$*, where* C_2 *is the two-element chain.*

Proof. $(1) \Rightarrow (2)$**:** Suppose that $Com(L)$ is a disconnected graph. So there exist two vertices a and b that are not connected in $Com(L)$. Clearly, $a, b \in L \setminus \{0_L, 1_L\}$ with $a \wedge b = 0_L$ and $a \vee b = 1_L$. Otherwise, if $a \wedge b \neq 0$ or $a \vee b \neq 1_L$, then we get a path $a \sim a \wedge b \sim b$ or $a \sim a \vee b \sim b$ in $Com(L)$, a contradiction. So $a \| b$ in L.

First, we show that a and b are atoms in L. On the contrary, suppose that a is not an atom in L. Let $a_1 < a$ for some a_1 in $L \setminus \{0_L\}$. If $a_1 \wedge b \neq 0_L$, then we get a path $a \sim a_1 \sim a_1 \wedge b \sim b$, a contradiction. Thus, we have $a_1 \wedge b = 0_L$. Similarly, $a_1 \vee b = 1_L$. Hence $a_1 \| b$ in L. Thus, from the above discussion, we conclude that $\{0_L, a, a_1, b, 1_L\}$ forms a sublattice of L isomorphic to N_5 containing 0_L, a contradiction to the fact that L is 0-modular. Hence a must be an atom. On similar lines, we can show that b is an atom.

We show that a and b are dual atoms. On the contrary, suppose that a is not a dual atom in L. Let $a < a_2$ for some a_2 in L such that $a_2 \neq 1_L$. If $a_2 \vee b \neq 1_L$, then we get a path $a \sim a_2 \sim a_2 \vee b \sim b$, a contradiction. Thus, we have $a_2 \vee b = 1_L$. Similarly, $a_2 \wedge b = 0_L$. Hence $a_2 \| b$ in L. Therefore, from the above discussion, we conclude that $\{0_L, a, a_2, b, 1_L\}$ forms a sublattice of L isomorphic to N_5 containing 0_L, a contradiction to the fact that L is 0-modular. Hence a must be a dual atom. Using similar arguments, we can show that b is a dual atom.

If there is no non-zero, non-unit element other than a, b, then we see that $L \cong M_2$. Thus, in this case, we are through. Now, if c is any other non-zero, non-unit element of $L \setminus \{a, b\}$, then we show that c is an atom that is also a dual atom.

On the contrary, suppose that c is not an atom of L. As a and b, both are atoms as well as dual atoms; we have $a \vee c = 1_L, a \wedge c = 0_L, b \vee c = 1_L$ and $b \wedge c = 0_L$, i.e., $a \| c$ and $b \| c$. Since c is not an atom, there exists c_1 such that $c_1 < c$ and $c_1 \neq 0_L$. Again, we have $c_1 \nsim a$ and $c_1 \nsim b$, as a and b both are atoms as well as dual atoms. Therefore $a \| c_1$ and $b \| c_1$. This gives $a \vee c_1 = 1_L, a \wedge c_1 = 0_L, b \vee c_1 = 1_L$ and $b \wedge c_1 = 0_L$. Thus, we conclude that $\{0_L, c, c_1, a, 1_L\}$ forms a sublattice isomorphic to N_5 containing 0_L, a contradiction to 0-modularity of L. Hence c must be an atom. Using similar arguments, we can also show that c is a dual atom.

Thus, we observe that every non-zero, non-unit element in L is an atom, which is also a dual atom of L. Hence if L is finite, then $L \cong M_n$ and, if L is infinite, then $L \cong M_\infty$.

$(2) \Rightarrow (3)$**:** It is quite straightforward.

$(3) \Rightarrow (4)$**:** Since L has at least two non-zero, non-unit elements, $|V(G(L))| \geq 2$. Let a, b be two vertices of $Com(L)$. As $Com(L)$ is an edgeless graph, $a \nsim b$ and hence $a \vee b = 1_L$ and $a \wedge b = 0_L$. If there exists a non-zero, non-unit element c (say) such that $0_L < c < a$, then $a \sim c$, i.e., there is an edge between a and c, a contradiction. Hence $0_L \prec a$. Using similar arguments, we can show that $0_L \prec b$, $a \prec 1_L$ and $b \prec 1_L$. This gives a cover preserving 0-1-sublattice isomorphic to $C_2 \times C_2$.

$(4) \Rightarrow (1)$**:** Since L contains a cover preserving 0-1-sublattice isomorphic to $C_2 \times C_2$, there are at least two non-zero, non-unit elements a, b (say) in L such that $0_L \prec a \prec 1_L$ and $0_L \prec b \prec 1_L$. Clearly, a and b are not connected in $Com(L)$, i.e., $Com(L)$ is a disconnected graph. □

Theorem 4.4 ([9, p. 59]). *A modular lattice* L *is distributive if and only if it does not contain a sublattice isomorphic to* M_3*.*

Definition 4.5 ([11, p. 31]). In a lattice with 0_L, the following property is called the *covering property*: If p is an atom and $a \wedge p = 0_L$, then $a \prec a \vee p$.

We can prove the following result using similar arguments as in Theorem 4.3. Note that every atomic 0-modular lattice satisfies the covering property.

Theorem 4.6. *Let* L *be an atomic lattice with at least two non-zero, non-unit elements. If* L *satisfies the covering property, then the following statements are equivalent.*

(i) $Com(L)$ *is a disconnected graph.*

(ii) $L \cong M_n$ *or* $L \cong M_\infty$.

(iii) $Com(L)$ *is an edgeless graph.*

(iv) L *contains a cover preserving 0-1-sublattice isomorphic to* $C_2 \times C_2$.

Proof. $(1) \Rightarrow (2)$**:** Since $Com(L)$ is disconnected, there exists at least two vertices a and b such that a and b are not connected in $Com(L)$. Clearly, $a \wedge b = 0_L$ and $a \vee b = 1_L$; otherwise, we get a path $a \sim a \wedge b \sim b$ or $a \sim a \vee b \sim b$, a contradiction.

We show that $L \cong M_n$ or $L \cong M_\infty$.

First, we show that a and b are atoms and dual atoms. Since L is atomic, there exists atoms a_1 and b_1 such that $a_1 \leq a$ and $b_1 \leq b$. Since a_1 and b_1 both are two distinct atoms, we have $a_1 \wedge b_1 = 0_L$. If $a_1 \vee b_1 \neq 1_L$, then we get a path $a \sim a_1 \sim a_1 \vee b_1 \sim b_1 \sim b$, a contradiction. Thus, we must have $a_1 \vee b_1 = 1_L$. Now, as a_1 is an atom and $a_1 \wedge b_1 = 0_L$, by using the covering property, we have $b_1 \prec a_1 \vee b_1$, i.e., $b_1 \prec 1_L$. Hence b_1 is a dual atom of L, but $b_1 \leq b$. This implies $b_1 = b$. Thus, b is an atom as well as a dual atom of L. Similarly, we get a as an atom as well as a dual atom of L.

If there is no non-zero, non-unit element other than a, b, then we see that $L \cong M_2$. Thus, in this case, we are through.

Now, if c is any other non-zero, non-unit element of $L \setminus \{a, b\}$, then we show that c is an atom and a dual atom. Clearly, $c \nsim a$ and $c \nsim b$, i.e., $a \vee c = b \vee c = 1_L$ and $a \wedge c = b \wedge c = 0_L$. As L is an atomic lattice, an atom c_1 exists, such as $c_1 \leq c$. Clearly, $c_1 \wedge a = 0_L$, as both are distinct atoms and $c_1 \vee a = 1_L$ as a is a dual atom. Also, since a is an atom and $a \wedge c_1 = 0_L$, by the covering property, we get $c_1 \prec a \vee c_1$, i.e., $c_1 \prec 1_L$. This implies c_1 is a dual atom, but $c_1 \leq c$, i.e., $c_1 = c$. This proves that c is an atom as well as a dual atom of L.

Thus, every non-zero, non-unit element of L is an atom as well as a dual atom of L. So if L is finite, then $L \cong M_n$ and, if L is infinite, then $L \cong M_\infty$.

$(2) \Rightarrow (3)$**:** is straightforward.

$(3) \Rightarrow (4)$**:** Since L has at least two non-zero, non-unit elements, $|V(G(L))| \geq 2$. Let a, b be two vertices of $Com(L)$. As $Com(L)$ is an edgeless graph, $a \nsim b$ and $a \vee b = 1_L$ and $a \wedge b = 0_L$. If there exists a non-zero, non-unit element c (say) such that $0_L < c < a$, then $a \sim c$, i.e., there is an edge between a and c, a contradiction. Hence $0_L \prec a$. Using similar arguments, we can show that $0_L \prec b$, $a \prec 1_L$ and $b \prec 1_L$. This gives a cover preserving sublattice containing 0_L and 1_L isomorphic to $C_2 \times C_2$.

$(4) \Rightarrow (1)$**:** Since L contains a cover preserving sublattice containing 0_L and 1_L isomorphic to $C_2 \times C_2$, there are at least two non-zero, non-unit elements a, b (say) in L such that $0_L \prec a \prec 1_L$ and $0_L \prec b \prec 1_L$. Clearly, a and b are not connected in $Com(L)$, i.e., $Com(L)$ is a disconnected graph. □

The following remark follows from Theorem 4.3, which is also valid if the lattice is an atomic lattice with the covering property.

Remark 4.7. Let L be a 0-modular lattice. Then, the following statements hold.

(i) If L has at least three non-zero, non-unit elements and $Com(L)$ is disconnected, then L is not distributive.

(ii) $Com(L)$ is connected if and only if for any two atoms a, b and for any two dual atoms c, d in L, we have, $a \vee b \neq 1_L$ and $c \wedge d \neq 0_L$.

Corollary 4.8 ([4, Corollary 3.2]). *If* $dim(V) = 2$*, then* $In(V)$ *is an edgeless graph.*

Proof. As the lattice $L(V)$ of subspaces of a vector space V is a 0-modular lattice, and if $dim(V) = 2$, then $L(V)$ is isomorphic to M_n or M_∞. Hence, by Theorem 4.3, $In(V)$ is an edgeless graph. □

Corollary 4.9 ([8, Proposition 2.5]). *Let* M *be a* R*-module. Then* $In(M)$ *is disconnected if and only if* M *is a direct sum of two simple* R*-modules.*

Proof. Suppose that $In(M)$ is disconnected, i.e., $Com(L(M))$ is disconnected. By Theorem 4.3, the lattice $L(M)$ of submodules of M contains a cover preserving sublattice containing 0_L and 1_L isomorphic to $C_2 \times C_2$. Hence there exists two non-trivial submodules M_1 and M_2 of M such that $M_1 \bigcap M_2 = 0_{L(M)}$ and $M_1 + M_2 = M$. Hence, we have $M = M_1 \bigoplus M_2$. The converse follows again by Theorem 4.3 and the fact that M is a direct sum of two simple R-modules. □

Theorem 4.10 ([10, p. 232]). *Let D be a division ring and n be a positive integer number. Then for every left ideal I of a matrix ring $M_n(D)$ over D, there exists an invertible matrix P and integer r, $0 \le r \le n$, such that $I = PH_r(D)P^{-1}$, where $H_r(D)$ denotes the left ideal of $M_n(D)$ containing all matrices whose j^{th} column is zero, for every $r < j \le n$.*

Theorem 4.11. *If a ring R is isomorphic to any one of the following*

(i) $M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k)$, *where* $k \ge 2$*;*

(ii) $D_1 \times D_2 \times \cdots \times D_k$, *where* $k \ge 3$*;*

(iii) $M_n(D)$, *where* $n \ge 3$,

where $D, D_1, \cdots, D_k$ are division rings and $M_{n_i}(D_i)$ is a matrix ring, then $L(R) \ncong M_n$ or $L(R) \ncong M_\infty$.

Proof. To prove the claim, it is enough to show that R has two nontrivial left ideals I_1 and I_2 such that $I_1 \subsetneq I_2$. Let $H_r(D)$ denotes the set of all left ideals of $M_n(D)$ containing all matrices whose j^{th} column is zero, for every $r < j \le n$.

(1) If $R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k)$, for $k \ge 2$, then take $I_1 = H_1(D_1) \times 0 \times \cdots \times 0$ and $I_2 = H_1(D_1) \times H_1(D_2) \times 0 \times \cdots \times 0$.

(2) If $R \cong D_1 \times D_2 \times \cdots \times D_k$, for $k \ge 2$, then take $I_1 = D_1 \times 0 \times \cdots \times 0$ and $I_2 = D_1 \times D_2 \times 0 \times \cdots \times 0$.

(3) If $R \cong M_n(D)$, then take $I_1 = H_1(D)$ and $I_2 = H_2(D)$. □

Theorem 4.12. ***Wedderburn-Artin Theorem:*** *Let R be any left semisimple ring. Then $R = M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ for suitable division rings $D_1, \cdots, D_r$ and positive integers $n_1, \cdots, n_r$. The number r is uniquely determined, as are the pairs $\{(n_1, D_1), \cdots, (n_r, D_r)\}$ (up to a permutation). There are exactly r mutually non-isomorphic left simple modules over R.*

Corollary 4.13 ([1, Theorem 1]). *Let R be a ring. Then $In(R)$ is not connected if and only if $R \cong M_2(D)$ or $D_1 \times D_2$, for some division rings D, D_1, D_2.*

Proof. First, suppose that $In(R)$ is not connected, i.e., $Com(L(R))$ is disconnected. By Theorem 4.3, we have $L(R) \cong M_n$ or $L(R) \cong M_\infty$, i.e., every nontrivial left ideal in R is minimal as well as maximal left ideal of R. Hence for any two nontrivial left ideals I and J of R, we have $I \bigcap J = \{0\}$ and $I + J = R$. Thus, R is a semisimple ring. By Wedderburn–Artin Theorem, $R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k)$, where $D_1, D_2, \cdots, D_k$ are division rings. Thus, by Theorem 4.11 and as $L(R) \cong M_n$ or $L(R) \cong M_\infty$, we must have $R \cong M_2(D)$ or $D_1 \times D_2$. Conversely, Suppose that $R \cong M_2(D)$ or $D_1 \times D_2$, for some division rings D, D_1, D_2. Theorem 4.10 shows that every non-zero left ideal is a minimal ideal of R. Thus, by Theorem 4.3, $Com(L(R))$ is disconnected. □

Corollary 4.14 ([2, Theorem 3.1]). *The graph $In(S)$ of a semigroup S is disconnected if and only if S contains at least two minimal left ideals, and every nontrivial left ideal of S is minimal as well as maximal.*

Corollary 4.15 ([2, Theorem 3.4]). *The graph $In(S)$ of a semigroup S is disconnected if and only if S is the union of exactly two minimal left ideals.*

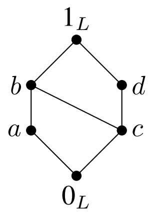
5 Diameter of the comparability graphs

Theorem 5.1. *Let L be a 0-modular lattice. If $Com(L)$ is connected, then $diam(Com(L)) \le 3$.*

Proof. Suppose that $Com(L)$ is connected. Let $a, b \in L \setminus \{0_L, 1_L\}$. We consider the following cases.

Case(1): If a and b are comparable, i.e., $a \sim b$, then $d(a, b) = 1$

Case(2): Suppose that $a \| b$. If either $a \wedge b \neq 0_L$ or $a \vee b \neq 1_L$, then we get a path $a \sim a \wedge b \sim b$ or $a \sim a \vee b \sim b$ that gives $d(a, b) = 2$. Suppose that $a \wedge b = 0_L$ and $a \vee b = 1_L$. Since $Com(L)$ is connected, there exists $c \in L \setminus \{0_L, 1_L\}$ such that $c \sim a$ or $c \sim b$. Without loss of generality, assume that $c \sim a$. If $c \sim b$, then we get $d(a, b) = 2$. Suppose that $c \nsim b$, i.e., $c \| b$. If $c \wedge b = 0_L$ and $c \vee b = 1_L$, then the set $\{0_L, a, c, b, 1_L\}$ will forms a sublattice isomorphic to N_5 containing 0_L, a contradiction to the fact that L is 0-modular. Therefore, either $c \wedge b \neq 0_L$ or $c \vee b \neq 1_L$. Then we have a path $a \sim c \sim c \wedge b \sim b$ or $a \sim c \sim c \vee b \sim b$. Hence $d(a, b) \leq 3$. Thus, from all the above cases, we get $diam(G(L)) \leq 3$. □



Example 5.2. The lattice depicted in the adjacent figure is a 0-modular lattice. From the path, $a \sim b \sim c \sim d$, we have $d(a, b) = 3$. Hence the diameter of the corresponding comparability graph is 3. This shows that the bound in Theorem 5.1 is sharp.

Theorem 5.3. *Let L be an atomic lattice that satisfies the covering property. If $Com(L)$ is connected, then $diam(Com(L)) \leq 4$.*

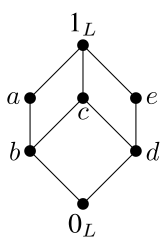
Proof. Let $Com(L)$ be a connected graph. We consider the following cases.

Case(1): If $a \sim b$, then $d(a, b) = 1$.

Case(2): If a and b are not adjacent and $a \wedge b \neq 0_L$ or $a \vee b \neq 1_L$ then we have either $a \sim a \wedge b \sim b$ or $a \sim a \vee b \sim b$. Thus, in either case, $d(a, b) = 2$.

Case(c): If a and b are not adjacent and $a \wedge b = 0_L$ and $a \vee b = 1_L$. Since L is atomic, an atom a_1 exists, such as $a_1 \leq a$. If $a_1 \wedge b \neq 0_L$, then $a_1 \leq b$, as a_1 is an atom. So, $a \sim a_1 \sim b$ and hence $d(a, b) = 2$. If $a_1 \wedge b = 0_L$, then as L satisfies the covering property, $b \prec a_1 \vee b$. If $a_1 \vee b \neq 1_L$, then we get a path $a \sim a_1 \sim a_1 \vee b \sim b$ and $d(a, b) \leq 3$. If $a_1 \vee b = 1_L$, then b will be a dual atom, as $b \prec a_1 \vee b = 1_L$. Since L is atomic, consider an atom b_1 such that $b_1 \leq b$. If $a_1 = b_1$, then $d(a, b) \leq 2$, as we have a path $a \sim a_1 = b_1 \sim b$. Let $a_1 \wedge b_1 = 0_L$. Since a_1 is an atom, using the covering property, $b_1 \prec a_1 \vee b_1$. Now, if $a_1 \vee b_1 \neq 1_L$, then we have a path $a \sim a_1 \sim a_1 \vee b_1 \sim b_1 \sim b$ and $d(a, b) \leq 4$. If $a_1 \vee b_1 = 1_L$, then by the covering property, both a_1 and b_1 are dual atoms. In this case, $a = a_1$ and $b = b_1$. Thus $\{0_L, a, b, 1_L\}$ forms a cover preserving sublattice of L isomorphic to $C_2 \times C_2$. Hence, $Com(L)$ is not connected, a contradiction.

Thus, from the above cases, $d(a, b) \leq 4$, for any $a, b \in L \setminus \{0_L, 1_L\}$. Hence $diam(G(L)) \leq 4$. □



Example 5.4. The lattice depicted in the adjacent figure satisfies the covering property. From the path, $a \sim b \sim c \sim d \sim e$, we have $d(a, e) = 4$. Hence the diameter of the corresponding comparability graph is 4. This shows that the bound in Theorem 5.3 is sharp. Also, it is an example of an atomic lattice that satisfies the covering property but is not 0-modular.

Corollary 5.5 ([4, Lemma 4.1]). *If $dim(V) \geq 3$, then $In(V)$ is connected and $diam(In(V)) \leq 3$.*

Proof. This follows from the fact that $L(V)$ is 0-modular and Theorem 5.1. □

Corollary 5.6. *If L is a complemented, 0-modular lattice of length at least 3 and $Com(L)$ is connected, then $diam(Com(L)) = 3$.*

Proof. By Theorem 5.1, we have $diam(G(L)) \leq 3$. Let $a \in L \setminus \{0_L, 1_L\}$. Note that such a exists as $\ell(L) \geq 3$. Since L is complemented, there exists $b \in L \setminus \{0_L, 1_L\}$ such that $a \wedge b = 0_L$ and $a \vee b = 1_L$. Assume that either a or b is not an atom. Without loss of generality, assume that a is not an atom. Then there exists $c \in L \setminus \{0_L, 1_L\}$ such that $c < a$, then by 0-modularity of L, we get $c = (c \vee b) \wedge a$. Clearly, $c \vee b \neq 1_L$, otherwise $c = a$, a contradiction. Hence, we get a path $a \sim c \sim c \vee b \sim b$ with $c \vee b \neq b$, as $a \wedge b = 0_L$ and $c \neq 0_L$. Thus in this case $d(a,b) = 3$ and hence $diam(Com(L)) = 3$. Now, we can assume that a and b are atoms. Now, without loss of generality, if there exists $d \notin \{0_L, 1_L\}$ such that $d > a$, then $d \vee b = 1_L$, as $a \vee b = 1_L$. We claim that $d \wedge b \neq 0_L$. Suppose $d \wedge b = 0_L$. Then by 0-modularity, $a = (a \vee b) \wedge d = d$, a contradiction. Hence, $d \wedge b \neq 0_L$. Thus, we get a path $a \sim d \sim d \wedge b \sim b$, which is a minimal path between a and b. Hence, $d(a,b) = 3$. Thus $diam(L) = 3$. □

Corollary 5.7. *If L is an atomistic, dual atomic, 0-modular lattice of length at least 3 and $Com(L)$ is connected, then $diam(Com(L)) = 3$.*

Proof. Since $\ell(L) \geq 3$ and L is dual atomic, a dual atom $b \in L$ exists, which is not an atom. Further, as $b < 1_L$ and L is an atomistic lattice, there exists an atom $p \in L$ such that $p < 1_L$ and $p \nleq b$. Since L is atomistic, there exists an atom $q \in L$ such that $q < b$. So we have distinct atoms p and q with $p \prec p \vee q$ and $q \prec p \vee q$. By Remark 4.7, we have $q \vee p \neq 1_L$. So we get a path $b \sim q \sim q \vee p \sim p$. We claim that this path is a minimal path in $Com(L)$. For this, if $b \sim c \sim p$, then we have $p < c$ and, $b < c$ or $c < b$. The possibility $c < b$ is not possible; otherwise, $p < b$, a contradiction. Hence $b < c$. As $c \neq 1_L$, and b is a dual atom, we have $b = c$, a contradiction. Hence, $b \sim q \sim q \vee p \sim p$ is a minimal path. Therefore, $d(b,p) = 3$. Thus, we get the result by Theorem 5.1. □

Corollary 5.8 ([4, Theorem 4.1]). *If $dim(V) \geq 3$, then $diam(In(V)) = 3$.*

Corollary 5.9 ([8, Proposition 2.5]). *If $In(M)$ is connected graph, then $diam(In(M)) \leq 3$.*

Corollary 5.10 ([1, Theorem 1]). *If $In(R)$ is connected graph, then $diam(In(R)) \leq 3$.*

Corollary 5.11 ([2, Theorem 3.5]). *If $In(S)$ is a connected graph, then $diam(In(S)) \leq 3$.*

Corollary 5.12. *If G is an abelian group and the graph $Com(L(G))$ is connected, then $diam(Com(L(G))) \leq 3$.*

Proof. As G is abelian, the subgroup lattice $L(G)$ is modular. By Theorem 5.1, we get the result. □

This shows that our result improves the following result.

Corollary 5.13 ([6, Theorem 2.11]). *If G is a finite abelian group and $I(G)$ be the subgroup inclusion graph, then $diam(I(G)) \in \{1,2,3,4,\infty\}$.*

6 Girth of the comparability graphs

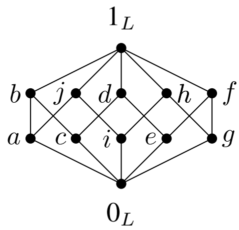
Lemma 6.1. *If L is a lattice of length 3 with at most two atoms or dual atoms, then $Com(L)$ does not contain any cycle.*

Proof. Let $c_1 - c_2 - c_3 - \cdots - c_k - c_1$ be a cycle in $Com(L)$. As $\ell(L) = 3$, every non-zero, non-unit element of L is either an atom or a dual atom of L, i.e., each c_i is either an atom or a dual atom for $i = 1,2,3,\cdots,k$. Without loss of generality, suppose c_1 is an atom. Then we observe that, $k \leq 4$. Suppose $k \geq 5$. Then we will have c_1, c_3, c_5 are three atoms, a contradiction to the fact that L has at most two atoms. Hence, the cycle is $c_1 - c_2 - c_3 - c_4 - c_1$. Further observe that $c_1, c_3 \leq c_2$ and hence $c_1 \vee c_3 \leq c_2$. As $\ell(L) = 3$ and $0_L \prec c_1 < c_1 \vee c_3 \leq c_2 \prec 1_L$, we have $c_1 \vee c_3 = c_2$. Similarly, $c_1 \vee c_3 = c_4$. This implies that $c_2 = c_4$, a contradiction. Thus, $Com(L)$ does not contain a cycle. On similar lines, we can prove that if L has at most two dual atoms, then $Com(L)$ does not contain any cycle. □

Lemma 6.2. *If L is a lattice of length 3 with at least three atoms, then $Com(L)$, does not contain a 4-cycle as well as a cycle of odd length.*

Proof. Suppose that $c_1 - c_2 - c_3 - c_4 - c_1$ is a 4-cycle in $Com(L)$. Using the arguments as in Lemma 6.1, we can say that $Com(L)$ does not contain a 4-cycle.

Now, suppose we have a cycle of odd length $c_1 - c_2 - c_3 - \cdots - c_{2k} - c_{2k+1} - c_1$ in $Com(L)$. Without loss of generality, suppose that c_1 is an atom. Arguing as above, we have c_{2k+1} as an atom. Since c_1 and c_{2k+1} both are atoms in L, a contradiction, as they can not be adjacent in $Com(L)$. Thus, $Com(L)$ does not contain a cycle of odd length. □



Remark 6.3. From Lemma 6.2, it is clear that, if $\ell(L) = 3$, then $girth(G(L)) \geq 2k$ for some $k \geq 3$. Consider a lattice L depicted in the adjacent figure. Clearly, $Com(L)$ contains a 10-cycle. Moreover, there exists a lattice of length 3 containing a $2k$-cycle, where $k \geq 3$.

Lemma 6.4. *If $Com(L)$ contains a cycle of odd length, then it contains a triangle.*

Proof. Suppose that we have a cycle of odd length $c_1 - c_2 - c_3 - \cdots - c_{2k} - c_{2k+1} - c_1$ in $Com(L)$. Since $c_1 \sim c_2$, either $c_1 < c_2$ or $c_1 > c_2$. Without loss of generality, assume that $c_1 < c_2$. If $c_2 < c_3$, then c_1, c_2, c_3 forms a triangle. Let $c_2 > c_3$. Again if $c_3 > c_4$, then c_2, c_3, c_4 forms a triangle. Therefore, on similar arguments, we have $c_3 < c_4$, $c_4 > c_5$ and so on. And lastly we have $c_{2k} > c_{2k+1}$. Now, if $c_1 < c_{2k+1}$, then c_1, c_{2k}, c_{2k+1} forms a triangle and if $c_1 > c_{2k+1}$, then c_1, c_2, c_{2k+1} forms a triangle.

Thus, if $Com(L)$ contains a cycle of odd length, then it must contain a triangle. □

Lemma 6.5. *Let L be a lattice. If $Com(L)$ contains a 4-cycle, then it contains a triangle.*

Proof. Suppose that $c_1 - c_2 - c_3 - c_4 - c_1$ is 4-cycle in $Com(L)$. Since $c_1 \sim c_2$, without loss of generality, assume that $c_1 < c_2$. If $c_2 < c_3$, then $c_1 < c_2 < c_3$, and we get a triangle. Let $c_2 > c_3$. If $c_3 > c_4$, then $c_2 > c_3 > c_4$ and again, we get a triangle. Let $c_3 < c_4$. If $c_4 < c_1$, then $c_3 < c_4 < c_1$ gives a triangle. Let $c_4 > c_1$.

So far we have $c_1 < c_2$, $c_2 > c_3$, $c_3 < c_4$ and $c_4 > c_1$. Clearly, $c_1 \leq c_1 \vee c_3 \leq c_2$ and $c_3 \leq c_1 \vee c_3 \leq c_2$. If $c_1 = c_1 \vee c_3$ or $c_3 = c_1 \vee c_3$, then $c_1 \sim c_3$ and we get a triangle. Let $c_1 \neq c_1 \vee c_3$ and $c_3 \neq c_1 \vee c_3$. This gives $c_1 < c_1 \vee c_3 \leq c_2$ and $c_3 < c_1 \vee c_3 \leq c_2$. Similarly, we get $c_1 < c_1 \vee c_3 \leq c_4$ and $c_1 < c_1 \vee c_3 \leq c_4$. If $c_1 \vee c_3 = c_2$, then $c_1 < c_2 < c_4$ and also if $c_1 \vee c_3 = c_4$, then $c_1 < c_4 < c_2$. Let $c_1 \vee c_3 \neq c_2$ and $c_1 \vee c_3 \neq c_4$. Then $c_1 < c_1 \vee c_3 < c_2$ gives a triangle. Thus, $Com(L)$ contains a triangle. □

Corollary 6.6 ([4, Lemma 4.2]). *If $dim(V) = 3$, then $In(V)$ does not contain any cycle of length 3, 4 or 5.*

Proof. In $L(V)$, the dimension of V is the length of that lattice. As $dim(V) = 3$, $\ell(L(V)) = 3$. Since $L(V)$ is an atomistic lattice and $\ell(L(V)) = 3$, over any field, finite or infinite, the number of atoms in $L(V)$ is at least 3. Hence, the result follows from Lemma 6.2. □

Corollary 6.7 ([1, Lemma 2][8, Proposition 2.6]). *Let M be a ring (R-module). If $In(M)$ has a cycle of length 4 or 5, then $In(M)$ has a triangle.*

Theorem 6.8. *If L is a 0-modular lattice, then $girth(Com(L)) = 3$ if $\ell(L) > 3$; otherwise 6 or ∞.*

Proof. Let L be a 0-modular lattice. We consider the following cases.

Case(1): Assume that, $\ell(L) > 3$. Clearly, there exists a chain of length 3 of non-zero, non-unit elements of L, which yields a cycle of length 3 in $Com(L)$. Thus, in this case $girth(G(L)) = 3$.

Case(2): Assume that, $\ell(L) = 3$. From Lemma 6.1, it is clear that if L has at most two atoms or at most two dual atoms, then $Com(L)$ does not contain a cycle. In such cases, $girth(G(L)) = \infty$. Suppose that L has at least three atoms and at least three dual atoms. By Lemma 6.2, $Com(L)$ does not contain cycles of length 3, 4 and 5. Therefore, it is clear that $girth(G(L)) \geq 6$.

Since $\ell(L) = 3$, each vertex in $Com(L)$ is either an atom or a dual atom. Let a_1, a_2 and a_3 be any three atoms in L.

Suppose that $a_1 \vee a_2, a_1 \vee a_3, a_2 \vee a_3$ all are distinct.

If at least one of them is equal to 1_L, then we get N_5 as a sublattice, a contradiction to 0-modularity.

Therefore, $a_1 \vee a_2, a_1 \vee a_3$ and $a_2 \vee a_3$ are distinct dual atoms in L. Hence $a_1 \sim a_1 \vee a_2 \sim a_2 \sim a_2 \vee a_3 \sim a_3 \sim a_1 \vee a_3 \sim a_1$ is a 6-cycle and we are done.

Now, suppose that $a_1 \vee a_2 = a_1 \vee a_3$. Since $a_2 < a_1 \vee a_2$ and $a_3 < a_1 \vee a_3$, we have $a_2 \vee a_3 \leq a_1 \vee a_3$. As $\ell(L) = 3$, the chain $0_L \prec a_2 < a_2 \vee a_3 \leq a_1 \vee a_3 \prec 1_L$ has the length at most 3. Clearly, $a_2 \vee a_3 = a_1 \vee a_3$. Using similar arguments, we can show that $a_1 \vee a_2 \vee a_3 = a_1 \vee a_2 = a_2 \vee a_3$. As L contains at least three dual atoms, a dual atom, say c, exists, such that $c \neq a_1 \vee a_2 \vee a_3$.

If $c \| a$ for all atoms a contained in $a_1 \vee a_2 \vee a_3$, then $\{0_L, a, a_1 \vee a_2 \vee a_3, c, 1_L\}$ forms a sublattice isomorphic to N_5 containing 0_L, a contradiction. Therefore, without loss of generality, assume that $a_1 < c$.

We claim that if every dual atom other than $a_1 \vee a_2 \vee a_3$ contains exactly one atom, then in such cases, we get a sublattice isomorphic to N_5 consisting of two dual atom and an atom below any one of the dual atom. This contradicts 0-modularity. Thus, this case does not arise.

So, there exists an atom a' (say) below c. Then as $\ell(L) = 3$ and $a_1 < c$, we have $c = a_1 \vee a'$.

Now assume that if $a' \sim a_1 \vee a_2 \vee a_3$, then we can easily get $c = a_1 \vee a_2 \vee a_3$, a contradiction.

Hence, $a' \| a_1 \vee a_2 \vee a_3$. Clearly, $a' \vee a_2 \neq c$ and $a' \vee a_2 \neq a_1 \vee a_2 \vee a_3$, otherwise, we get $c = a_1 \vee a_2 \vee a_3$, a contradiction. Thus, it gives $a' \sim a' \vee a_2 \sim a_2 \sim a_1 \vee a_2 \vee a_3 \sim a_1 \sim c \sim a'$ a 6-cycle. Thus, $girth(G(L)) = 6$, when $\ell(L) = 3$.

Case(3): Assume that, $\ell(L) < 3$. Clearly, $Com(L)$ will be empty or edgeless, as L is of length either 1 or 2. Therefore, we can say $girth(G(L)) = \infty$. □

Theorem 6.9. *Let L be a lattice and $\ell(L) = 3$. Then $girth(Com(L)) = 6$ if and only if L contains a sublattice isomorphic to $\mathbf{2}^3$ (Boolean lattice with three atoms).*

Proof. Since $\ell(L) = 3$, each vertex in $Com(L)$ is either an atom or a dual atom.

Part(A): Suppose that $girth(Com(L)) = 6$. Therefore, there exist a smallest 6-cycle $a_1 - b_1 - a_2 - b_2 - a_3 - b_3 - a_1$ in $Com(L)$. Clearly, all a_i's and b_j's are distinct. Without loss of generality, assume that a_1 is an atom. Hence b_1 is a dual atom. Also, a_2, a_3 are atoms and b_2, b_3 are dual atoms. Clearly, $a_1 < b_1$ and $a_2 < b_1$ and hence $a_1 \vee a_2 \leq b_1$. Then we have a chain $0_L \prec a_1 < a_1 \vee a_2 \leq b_1 \prec 1_L$. However, $\ell(L) = 3$ gives $a_1 \vee a_2 = b_1$. On the similar lines, we get $a_2 \vee a_3 = b_2$ and $a_1 \vee a_3 = b_3$. If $a_1 \vee a_2 = a_1 \vee a_2 \vee a_3$, then we get $a_3 \leq a_1 \vee a_2 = b_1$, a contradiction. Hence, we have $a_1 \vee a_2 < a_1 \vee a_2 \vee a_3$. This together with $\ell(L) = 3$ gives $a_1 \vee a_2 \vee a_3 = 1_L$. Also, we have $a_1 < b_1$ and $a_1 < b_3$ which gives $a_1 \leq b_1 \wedge b_3$. Since $\ell(L) = 3$, the chain $0_L \prec a_1 \leq b_1 \wedge b_3 < b_1 \prec 1_L$ gives $a_1 = b_1 \wedge b_3$. On the similar lines, we get $a_2 = b_1 \wedge b_2$, $a_3 = b_2 \wedge b_3$ and $0_L = b_1 \wedge b_2 \wedge b_3$. Thus, $\{0_L, a_1, a_2, a_3, b_1, b_2, b_1, 1_L\}$ forms a sublattice isomorphic to $\mathbf{2}^3$.

Part(B): Suppose that L contains a sublattice isomorphic to $\mathbf{2}^3$. Since $\ell(\mathbf{2}^3) = 3 = \ell(L)$, the sublattice contains 0_L and 1_L and hence induces a 6-cycle in $Com(L)$. Thus, the result follows from Lemma 6.2. □

Lemma 6.10. *Let V be a vector space. If $dim(V) = 3$, then $L(V)$ contains $\mathbf{2}^3$ as a sublattice.*

Proof. Since $dim(V) = 3$, it has a basis containing three vectors and $\ell(L(V)) = 3$. Let $\{w_1, w_2, w_3\}$ be a basis. By Lemma 3.11, it is clear that $< w_1 >, < w_2 >, < w_3 >$ are atoms and $< w_1, w_2 >, < w_2, w_3 >, < w_1, w_3 >$ are dual atoms in $L(V)$. Thus, $\{\{0_{L(V)}\}, < w_1 >, < w_2 >, < w_3 >, < w_1, w_2 >, < w_2, w_3 >, < w_1, w_3 >, V\}$ forms a sublattice isomorphic to $\mathbf{2}^3$. □

Corollary 6.11 ([4, Theorem 4.2]). *If V be an n-dimensional vector space, then*

$$girth(In(V)) = \begin{cases} 3, & \text{if } n > 3; \\ 6, & \text{if } n = 3; \\ \infty, & \text{if } n < 3. \end{cases}$$

Proof. Follows from Theorem 6.8, Theorem 6.9 and Lemma 6.10. □

Corollary 6.12 ([8, Proposition 2.8]). *Let M be an R-module. Then $girth(In(M)) \in \{3, 6, \infty\}$.*

Corollary 6.13 ([1, Theorem 5][2, Theorem 3.5]). *Let R be a ring (semigroup). Then $girth(In(R)) \in \{3, 6, \infty\}$.*

Theorem 6.14. *Let L be a graded lattice. If $\ell(L) > 3$, then $Com(L)$ is hyper-triangulated and hence triangulated.*

Proof. Let $a - b$ be an edge in $Com(L)$. Clearly, a and b are non-zero, non-unit elements of L. Since L is graded and $\ell(L) \geq 4$, there exists a maximal chain containing $0_L, a, b, 1_L$ of length at least 4. So, there is a non-zero, non-unit element c such that $c \sim a$ and $c \sim b$. Thus, a, b and c forms a triangle in $Com(L)$. □

Corollary 6.15 ([4, Theorem 4.3]). *If $dim(V) \geq 4$, then $In(V)$ is triangulated.*

7 Conclusions

In this paper, we have studied the comparability graph of a lattice and discussed some basic properties such as connectedness, diameter, girth, triangulation, etc. Using this result, we unified many more results of the inclusion graphs of algebraic structures. It is easy to observe that if lattices are isomorphic, then the corresponding comparability graphs are isomorphic. However, the converse need not be true. Hence, finding the class of lattices in which the converse is true will be an interesting problem. Furthermore, some known results are available for other graphs, such as cover graphs, zero-divisor graphs, etc. In the future, we will be interested in this Isomorphism Problem for comparability graphs.

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